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Multipliers of weighted l^p spaces

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Abstract. It is an open question whether polynomials (that is, sequences that are eventually zero) are dense in the multiplier algebra of $l^p(\omega)$ if and only if $l^p(\omega)$ is a convolution algebra. An investigation of compact multipliers leads to an affirmative answer when ω is a regulated weight, that is, when $\lim_{n\to\infty}\omega(n+k)/\omega(n)=0$ for some $k>0$.

Introduction. If ω is a positive sequence on $\mathbf{Z}^+$, we define $l^p(\omega)$ to be the space of all complex-valued sequences $x=\{x(n)\}_{n\in\mathbf{Z}^+}$ for which $\sum_{n=0}^{\infty}|x(n)|^p\omega(n)^p<\infty$. We will assume that $p\in[1,\infty)$ so that $l^p(\omega)$ is a Banach space under the obvious norm. We also will restrict ourselves to weight sequences, ω, for which the right translations are bounded linear operators on $l^p(\omega)$. The main purpose of this paper is to provide a partial answer to a question posed by Shields [4, p. 95] concerning the convergence of partial sums of multipliers on $l^p(\omega)$. To this end we also investigate in some depth the compact multipliers on $l^p(\omega)$.

In Section 1 we define and list some elementary results about the multiplier algebra, $\mathcal{M}^p(\omega)$, of $l^p(\omega)$, consisting of those bounded linear operators on $l^p(\omega)$ which are given by convolution with a sequence. The proofs of these results have been omitted since they can be found in [4] for the case $p=2$ and the generalizations to any $p\in[1,\infty)$ are straightforward.

In Section 2 we study the properties of those multipliers which are compact operators on $l^p(\omega)$ – the compact multipliers. We show that the Cesàro means of compact multipliers converge uniformly and so establish that the compact multipliers are contained in the norm closure of the polynomials in the multiplier algebra. Following Bade, Dales, and Laursen [2, p. 18], we say that ω is *regulated at* $k\in\mathbf{N}$ if $\lim_{n\to\infty}\omega(n+j)/\omega(n)=0$ for all $j\geqslant k$. We then characterize the compact multipliers as precisely those elements, α, in the norm closure of the polynomials which satisfy $\alpha(k)=0$ if ω is not regulated at k.

The main result of Section 3 was motivated by studies of certain special cases. In [2], Bade, Dales, and Laursen assume that ω is submultiplicative, so that $l^1(\omega)$ is an algebra, and investigate the multiplier algebra, $\mathcal{M}(M)$, of that

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ideal of $l^1(\omega)$ consisting of sequences x for which $x(0)=0$. This is a special case of the multiplier algebras we are dealing with since $\mathcal{M}(M)$ is isometrically isomorphic to $\mathcal{M}(l^1(\omega'))$ where ω' is the left shift of ω. They then suppose that ω is regulated at 1 and show that, in this case, the polynomials are dense in $\mathcal{M}(M)$ if and only if $l^1(\omega')$ is an algebra [2, Thm. 1.11]. In [4], Shields surveys results about the multiplier algebra of $l^2(\omega)$ (though his notation is different from ours) and raises the question of whether the partial sums of multipliers necessarily converge if and only if $l^2(\omega)$ is an algebra. In Section 3, we generalize the result in [2] mentioned above and, as a corollary, provide an affirmative answer to Shields' question for those weights which are regulated at some $k\in\mathbf{Z}^+$. Our result, in fact, holds for $1\leqslant p<\infty$.

I would like to take this opportunity to thank Professor William G. Bade for suggesting the problem I have considered and for his guidance and encouragement during its solution.

1. Preliminaries. We will begin by defining the weighted l^p spaces $(1\leqslant p<\infty)$ and their corresponding multiplier algebras which will be the focus of our attention. We will also reformulate in our notation some well-known results about these spaces which will provide the basis for our later work.

A *weight sequence* on $\mathbf{Z}^+$ is a real-valued sequence $\omega=\{\omega(n):\ n\in\mathbf{Z}^+\}$ such that $\omega(n)>0$ $(n\in\mathbf{Z}^+)$, $\omega(0)=1$ and $\sup_{n\in\mathbf{Z}^+}\omega(n+k)/\omega(n)<\infty$ for each $k\in\mathbf{Z}^+$.

If ω is a weight sequence and $1\leqslant p<\infty$, we define $l^p(\omega)$ to be the linear space of complex-valued sequences, $x=\{x(n):\ n\in\mathbf{Z}^+\}$, for which the sum $\sum_{n=0}^{\infty}|x(n)|^p\omega(n)^p$ is finite. Then $l^p(\omega)$ is a Banach space under the norm

$$\|x\|_{p,\omega}:=\Big(\sum_{n=0}^{\infty}|x(n)|^p\omega(n)^p\Big)^{1/p}\qquad(1\leqslant p<\infty).$$

The map $x\mapsto x\cdot\omega$, with $(x\cdot\omega)(n):=x(n)\omega(n)$, establishes an isometry between $l^p(\omega)$ and l^p (where l^p denotes $l^p(\omega)$ when $\omega(n)\equiv 1$). Thus the choice of weight is irrelevant as far as the Banach space structure of $l^p(\omega)$ is concerned.

For $1<p<\infty$, define p' by $1/p+1/p'=1$. We call p' the conjugate index to p. We define the dual of $l^p(\omega)$ to be $l^{p'}(\omega)$ with duality

$$\langle x,y\rangle:=\sum_{n=0}^{\infty}x(n)\overline{y(n)}\omega(n)^2\qquad(x\in l^p(\omega),\ y\in l^{p'}(\omega)).\tag{1}$$

Boundedness is, then, a consequence of Hölder's inequality since

$$\sum_{n=0}^{\infty}x(n)\bar{y}(n)\omega(n)^2=\sum_{n=0}^{\infty}\big(x(n)\omega(n)\big)\big(\bar{y}(n)\omega(n)\big)\leqslant\|x\|_{p,\omega}\|y\|_{p',\omega}.$$

This definition of dual spaces and duality may seem somewhat artificial. It is however "natural" in the sense that, when $p=2$, formula (1) defines an inner product which makes $l^2(\omega)$ into a Hilbert space.

We now define the *convolution*, $x*y$, of two sequences, x and y, by

$$(x*y)(n)=\sum_{j=0}^{\infty}x(j)y(n-j)\qquad(n\in\mathbf{Z}^+).$$

Note that if we identify sequences with elements of $\mathbf{C}[[X]]$, the algebra of formal power series in one independent variable over the field $\mathbf{C}$, in the obvious way $(x\mapsto\sum_{n=0}^{\infty}x(n)X^n)$, then $x*y$ is identified with the product $(\sum_{n=0}^{\infty}x(n)X^n)(\sum_{n=0}^{\infty}y(n)X^n)$. We also define the sequences $\{e_n\}_{n\in\mathbf{Z}^+}$ by $e_n\mapsto X^n\in\mathbf{C}[[X]]$. Then, for fixed p and ω, $\{e_n\}\subseteq l^p(\omega)$ and $\|e_n\|_{p,\omega}=\omega(n)$ $(n\in\mathbf{Z}^+)$.

Given a complex sequence, α, we will denote the operator which maps $x\mapsto\alpha*x$ by T_α. Note that then T_{e_1} is the unilateral shift. If $T_\alpha\in\mathscr{B}(l^p(\omega))$, $\|T_\alpha\|_{p,\omega}$ will denote the operator norm of T_α (the subscripts p and ω may be omitted if they are clear from the context). It is then easily seen that our definition of a weight sequence implies that T_{e_k} is a bounded linear operator on $l^p(\omega)$ for each $k\in\mathbf{Z}^+$ with

$$\|T_{e_k}\|_{p,\omega}=\|T_{e_1}^k\|_{p,\omega}=\sup_{n\in\mathbf{Z}^+}\omega(n+k)/\omega(n)\qquad(k\in\mathbf{Z}^+).$$

DEFINITION 1.1. Let $m^p(\omega)$ denote the set of all complex sequences α with the property that $\alpha*x\in l^p(\omega)$ for every $x\in l^p(\omega)$, and let $\mathcal{M}^p(\omega)=\{T_\alpha:\ \alpha\in m^p(\omega)\}$. We call $\mathcal{M}^p(\omega)$ the *multiplier algebra* of $l^p(\omega)$ and elements of $\mathcal{M}^p(\omega)$ *multipliers* of $l^p(\omega)$.

It follows from the closed graph theorem that for each $\alpha\in m^p(\omega)$, the operator T_α of convolution by α is a bounded linear operator on $l^p(\omega)$. Furthermore, the commutativity and associativity of convolution yield that $T_\alpha T_\beta=T_\beta T_\alpha=T_{\alpha*\beta}$. Thus $\mathcal{M}^p(\omega)$ is an abelian subalgebra of $\mathscr{B}(l^p(\omega))$. In fact, it can be shown that $\mathcal{M}^p(\omega)$ is the commutant of T_{e_1} in $\mathscr{B}(l^p(\omega))$ and it follows that $\mathcal{M}^p(\omega)$ is a maximal abelian subalgebra. It can also be shown that $\mathcal{M}^p(\omega)$ is the closed algebra generated by T_{e_1} and the identity operator in the strong operator topology. Hence, in particular, $\mathcal{M}^p(\omega)$ is closed in the uniform operator topology and so is a Banach algebra.

It will greatly simplify notation later on to define the norm $|||\cdot|||_{p,\omega}$ on $m^p(\omega)$ as the one inherited from $\mathcal{M}^p(\omega)$ so that $|||\alpha|||_{p,\omega}=\|T_\alpha\|_{p,\omega}$. The use of triple bars should avoid confusion as to which norm is meant when a sequence, α, is simultaneously in $m^p(\omega)$ and $l^p(\omega)$. This is by no means an uncommon occurrence since the existence of an identity for convolution of norm 1 in $l^p(\omega)$ (namely e_0) ensures that $m^p(\omega)\subseteq l^p(\omega)$ and that $\|x\|\leqslant|||x|||$ for all $x\in l^p(\omega)$. Of course, it is only in the case that $l^p(\omega)$ is an algebra under convolution that we have the reverse inclusion. Hence, in the algebra case, $m^p(\omega)=l^p(\omega)$ as Banach algebras since the open mapping theorem implies that the two norms are equivalent. We refer to [3] for a discussion of which weight sequences do make $l^p(\omega)$ an algebra.

Let $s_n(\alpha):=\sum_{k=0}^{n}\alpha(k)e_k$ be the nth partial sum of α. It follows immediately from the above remarks that if $l^p(\omega)$ is an algebra, then

$$|||\alpha - s_n(\alpha)||| = \|T_\alpha - T_{s_n(\alpha)}\| \to 0 \quad \text{as } n\to\infty \qquad (\alpha\in m^p(\omega)).$$

That is, if $l^p(\omega)$ is an algebra, then the partial sums of multipliers converge. We may now ask whether the converse to this statement is valid: if the partial sums of multipliers converge, is $l^p(\omega)$ necessarily an algebra? In the case $p=1$, this question has been investigated by Bade, Dales, and Laursen [2]. When $p=2$ it has been posed by Shields in [4]. We will return to this question after discussing compact multipliers and regulated weights.

2. Compact multipliers. *Compact multipliers*, as one would expect, are those elements of $\mathcal{M}^p(\omega)$ which are compact operators on $l^p(\omega)$. We will denote the ideal of compact multipliers in $\mathcal{M}^p(\omega)$ by $\mathcal{K}^p(\omega)$. The question of whether a given multiplier is compact turns out to be closely related to a property of the weight which we now define:

DEFINITION 2.1. Given a weight sequence ω and $k\in\mathbf{N}$, we say that ω is *regulated at* k if

$$\lim_{n\to\infty} \omega(n+k)/\omega(n) = 0.$$

(It follows that $\lim_{n\to\infty}\omega(n+j)/\omega(n)=0$ for all $j\geqslant k$.) We say that ω is *regulated* if there exists $k\in\mathbf{N}$ such that ω is regulated at k.

In order to demonstrate the relationship between compact multipliers and regulated weights, we will require the following lemma:

LEMMA 2.2. *Let X be a Banach space, $\{U_\lambda\}\subset\mathcal{B}(X)$ be a bounded net, and $T\in\mathcal{B}(X)$ be compact. If $U_\lambda\to U\in\mathcal{B}(X)$ in the strong operator topology, then $U_\lambda T\to UT$ in the uniform operator topology.*

Proof. Replacing U_λ by $U_\lambda - U$, we can assume, without loss of generality, that $U_\lambda\to 0$ in the strong operator topology. Let $M:=\sup_\lambda\|U_\lambda\|$. Given $x\in X$ and $\varepsilon>0$, define $B(x,\varepsilon):=\{y\in Y\colon \|y-x\|\leqslant\varepsilon\}$. Since T is compact, $TB(0,1)$ is totally bounded and so, given $\varepsilon>0$, there exists a finite set, $\{x_j\}_{j=1}^m\subseteq B(0,1)$, such that $TB(0,1)\subseteq\bigcup_{j=1}^m B(Tx_j,\varepsilon)$. Choose λ_0 such that $\|U_\lambda Tx_j\|<\varepsilon$ for $\lambda\geqslant\lambda_0$ and $j=1,\ldots,m$. Then, if $x\in B(0,1)$, $Tx\in B(Tx_j,\varepsilon)$ for some j, and so,

$$\|U_\lambda Tx\|\leqslant\|U_\lambda Tx - U_\lambda Tx_j\| + \|U_\lambda Tx_j\| < (M+1)\varepsilon.$$

Hence $U_\lambda T\to 0$ in the uniform operator topology. ∎

DEFINITION 2.3. For $n\in\mathbf{Z}^+$, we define the complementary subspaces E_n and F_n of $l^p(\omega)$ by $E_n=\{x\in l^p(\omega)\colon x(j)=0 \text{ for } j>n\}$ and $F_n=\{x\in l^p(\omega)\colon x(j)=0$ for $j\leqslant n\}$. We also define the projections $P_n, Q_n\in\mathcal{B}(l^p(\omega))$ onto E_n and F_n respectively by

$$(P_n x)(j)=\begin{cases} x(j) & \text{if } j\leqslant n,\\ 0 & \text{if } j>n,\end{cases}$$

for $x\in l^p(\omega)$, and $Q_n = I-P_n$.

Given $\alpha\in m^p(\omega)$, P_nT_α is a finite-rank operator for each $n\in\mathbf{Z}^+$ and it follows that if

$$\lim_{n\to\infty}\|Q_nT_\alpha\|_{p,\omega} = \lim_{n\to\infty}\|T_\alpha - P_nT_\alpha\|_{p,\omega} = 0$$

then $T_\alpha\in\mathcal{K}^p(\omega)$. The converse to this statement is immediate from Lemma 2.2 since $Q_n\to 0$ as $n\to\infty$ in the strong operator topology. Thus we have:

PROPOSITION 2.4. *Given $\alpha\in m^p(\omega)$, $T_\alpha\in\mathcal{K}^p(\omega)$ if and only if* $\lim_{n\to\infty}\|Q_nT_\alpha\|_{p,\omega}=0$.

For $\alpha\in m^p(\omega)$ and $n\in\mathbf{Z}^+$, we denote the operator norm of T_α restricted to F_n by $\|T_\alpha|_{F_n}\|_{p,\omega} = |||\alpha|_{F_n}|||_{p,\omega}$. We then have:

PROPOSITION 2.5. *Given $\alpha\in m^p(\omega)$, $T_\alpha\in\mathcal{K}^p(\omega)$ if and only if $\|T_\alpha|_{F_n}\|_{p,\omega} = |||\alpha|_{F_n}|||_{p,\omega}\to 0$ as $n\to\infty$.*

Proof. If $T_\alpha\in\mathcal{K}^p(\omega)$, then, since $Q_nT_\alpha x = T_\alpha x$ for all $x\in F_n$, it follows from Proposition 2.4 that $\lim_{n\to\infty}\|T_\alpha|_{F_n}\|=0$.

Conversely, suppose that $\lim_{n\to\infty}\|T_\alpha|_{F_n}\|=0$. Then $\|T_\alpha - T_\alpha P_n\| = \|T_\alpha Q_n\| \leqslant \|T_\alpha|_{F_n}\|\to 0$ as $n\to\infty$. But $T_\alpha P_n\in\mathcal{K}^p(\omega)$ for each $n\in\mathbf{Z}^+$ since P_n is a finite-rank operator and so $T_\alpha\in\mathcal{K}^p(\omega)$. ∎

The following corollaries are straightforward but useful.

COROLLARY 2.6. *If $T_\alpha\in\mathcal{K}^p(\omega)$ then*

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|\alpha(k)|^p(\omega(n+k)/\omega(n))^p = 0.$$

Proof. By the proposition, $\|T_\alpha|_{F_n}\|\to 0$ as $n\to\infty$ and so, in particular,

$$\lim_{n\to\infty}\|T_\alpha(e_n/\omega(n))\| = 0,$$

which is equivalent to the conclusion. ∎

COROLLARY 2.7. *If $T_\alpha\in\mathcal{K}^p(\omega)$ then $\alpha(0)=0$.*

Note that when $p=1$ the converse to Corollary 2.6 is also true. Furthermore, in this case, there is a nice expression for the norm of a multiplier, namely,

$$\|T_\alpha\|_{1,\omega} = \sup_{k\in\mathbf{Z}^+}\sum_{n=0}^{\infty}|\alpha(n)|\,\omega(n+k)/\omega(k). \tag{2}$$

These characterizations of compact multipliers and of multipliers in general lead to simplified proofs of some of the results to follow for $p=1$. We refer to [2, Ch. 1] for a discussion of this special case.

Setting α equal to e_k in Corollary 2.6 yields the result that if T_{e_k} is a compact multiplier then $\lim_{n\to\infty}\omega(n+k)/\omega(n)=0$, which is exactly our definition of a weight being regulated at k. The converse of this result does hold for all p:

PROPOSITION 2.8. *Given $k\in\mathbf{Z}^+$, $T_{e_k}\in\mathcal{K}^p(\omega)$ if and only if ω is regulated at k.*

Proof. As we remarked above, it remains only to prove that if ω is regulated at k then $T_{e_k}\in\mathcal{K}^p(\omega)$. Also, since ω can never be regulated at 0, we can assume $k>0$.

We begin by noting that for $n\geqslant k$,

$$\|T_{e_k}-P_nT_{e_k}\| = \sup_{j>n-k}\omega(j+k)/\omega(j). \tag{3}$$

For, in this case,

$$\|T_{e_k}-P_nT_{e_k}x\|^p = \sum_{j=n+1}^{\infty}|x(j-k)|^p\omega(j)^p = \sum_{j=n+1-k}^{\infty}|x(j)|^p\omega(j+k)^p \leqslant \sup_{j>n-k}(\omega(j+k)/\omega(j))^p\|x\|^p,$$

while, if $j>n-k$,

$$\left\|T_{e_k}\left(\frac{e_j}{\omega(j)}\right)-P_nT_{e_k}\left(\frac{e_j}{\omega(j)}\right)\right\| = \left\|T_{e_k}\left(\frac{e_j}{\omega(j)}\right)\right\| = \frac{\omega(j+k)}{\omega(j)}.$$

Now, if ω is regulated at k, then (3) implies that $\|T_{e_k}-P_nT_{e_k}\|\to 0$ as $n\to\infty$. But, for each n, $P_nT_{e_k}$ is a finite-rank operator and, hence, compact. Thus, T_{e_k} is compact. ∎

We noted in Section 1 that $\mathcal{M}^p(\omega)$ is the closed algebra generated by T_{e_1} and the identity operator in the strong operator topology, or, equivalently, that $\mathcal{M}^p(\omega)$ is the closure in the strong operator topology of the polynomial multipliers, i.e. those multipliers given by convolution with sequences which are eventually zero. This can be proven as in [4, Thm. 12] by showing that if $\alpha\in m^p(\omega)$ then $\alpha-\sigma_n(\alpha)\to 0$ in the strong operator topology as $n\to\infty$, where $\sigma_n(\alpha):=(n+1)^{-1}\sum_{k=0}^{n}s_k(\alpha)$ is the nth Cesàro mean of α. We will now show that the Cesàro means of compact multipliers converge uniformly and so $\mathcal{K}^p(\omega)$ is a subset of the closure in the uniform operator topology of the polynomial multipliers.

Following Shields [4, p. 88], for $\alpha\in l^p(\omega)$ and $t\in\mathbf{R}$, we define α_t by

$$\alpha_t(n):=\exp(int)\alpha(n)\qquad(n\in\mathbf{Z}^+).$$

It is easily verified that $(\alpha*x)_t=\alpha_t*x_t$ and it is then straightforward to show that if $\alpha\in m^p(\omega)$ then $\alpha_t\in m^p(\omega)$ with $|||\alpha_t|||=|||\alpha|||$. For fixed α, the map $t\mapsto\alpha_t$ is continuous from $\mathbf{R}$ into $\mathcal{M}^p(\omega)$ with the strong operator topology and $T_{\sigma_n(\alpha)}$ is then given by the Bochner integral

$$(2\pi)^{-1}\int_0^{2\pi}K_n(-t)T_{\alpha_t}dt$$

where $\{K_n\}$ denotes the Fejér kernel. Thus,

$$K_n(t)=\sum_{-n}^{n}\left(1-\frac{|k|}{n+1}\right)\exp(ikt)=\frac{1}{n+1}\left(\frac{\sin(n+1)t/2}{\sin t/2}\right)^2\qquad(t\in\mathbf{R}).$$

We will begin by proving the following lemma:

LEMMA 2.9. *If $T_\alpha\in\mathcal{K}^p(\omega)$ then $\lim_{t\to 0}|||\alpha-\alpha_t|||=0$.*

Proof. If $p=1$ then the result follows easily from Corollary 2.6 and (2). Hence we will assume $1<p<\infty$.

If $T_\alpha\in\mathcal{K}^p(\omega)$ then, by Lemma 2.2,

$$\|Q_n(\alpha*x)\|\to 0\quad\text{ as } n\to\infty \tag{4}$$

uniformly for $x\in B(l^p(\omega))$, the closed unit ball of $l^p(\omega)$. Furthermore, for all $t\in\mathbf{R}$,

$$\|Q_n(\alpha*x)\|=\|(Q_n(\alpha*x))_t\|=\|Q_n((\alpha*x)_t)\|=\|Q_n(\alpha_t*x_t)\|.$$

For fixed $t\in\mathbf{R}$, $x\mapsto x_t$ is an isometry of $B(l^p(\omega))$ onto itself, and so we conclude that

$$\sup_{x\in B(l^p(\omega))}\|Q_n(\alpha_t*x)\|=\sup_{x\in B(l^p(\omega))}\|Q_n(\alpha*x)\|\to 0\quad\text{ as } n\to\infty.$$

Thus, $\|Q_nT_{\alpha_t}\|=\|Q_nT_\alpha\|\to 0$ as $n\to\infty$. Given $\varepsilon>0$, choose $N\in\mathbf{Z}^+$ such that $\|Q_N(T_\alpha-T_{\alpha_t})\|<\varepsilon$ $(t\in\mathbf{R})$. Then

$$\begin{aligned}\|T_\alpha-T_{\alpha_t}\|^p &= \sup_{x\in B(l^p(\omega))}\|(\alpha-\alpha_t)*x\|^p\\ &\leqslant \sup_{x\in B(l^p(\omega))}\sum_{n=0}^{N}\Big|\sum_{k=0}^{n}(\alpha(k)-\alpha_t(k))x(n-k)\Big|^p\omega(n)^p+\varepsilon^p\\ &= \sup_{x\in B(l^p(\omega))}\sum_{n=0}^{N}\Big|\sum_{k=0}^{n}(1-\exp(ikt))\alpha(k)x(n-k)\Big|^p\omega(n)^p+\varepsilon^p\\ &\leqslant \sum_{n=0}^{N}\left(\sum_{k=0}^{n}|1-\exp(ikt)|^{p'}|\alpha(k)|^{p'}\frac{\omega(n)^{p'}}{\omega(n-k)^{p'}}\right)^{p/p'}+\varepsilon^p\leqslant 2\varepsilon^p\end{aligned}$$

for t sufficiently close to 0. ∎

PROPOSITION 2.10. *If $T_\alpha\in\mathcal{K}^p(\omega)$ then $\|T_\alpha-T_{\sigma_n(\alpha)}\|\to 0$ as $n\to\infty$.*

Proof. As noted above, we have

$$T_{\sigma_n(\alpha)}=(2\pi)^{-1}\int_0^{2\pi}K_n(-t)T_{\alpha_t}dt$$

where $\{K_n\}$ is the Fejér kernel. Since $K_n(t) \geqslant 0$ and $(2\pi)^{-1}\int_0^{2\pi} K_n(-t)dt = 1$, we have

$$\|T_\alpha - T_{\sigma_n(\alpha)}\| = \Big\|(2\pi)^{-1}\int_{-\pi}^{\pi}(T_\alpha - T_{\alpha_t})K_n(-t)dt\Big\| \leqslant (2\pi)^{-1}\int_{-\pi}^{\pi}\|T_\alpha - T_{\alpha_t}\|K_n(-t)dt.$$

Let $\varepsilon > 0$ be given. By Lemma 2.9, there exists $\delta > 0$ such that $\|T_\alpha - T_{\alpha_t}\| < \varepsilon$ if $|t| < \delta$. Since $\lim_{n\to\infty}\int_{|t|\in[\delta,\pi]} K_n(t)dt = 0$, we can choose $N \in \mathbf{Z}^+$ such that

$$(2\pi)^{-1}\int_{|t|\in[\delta,\pi]} K_n(t)dt < \varepsilon/(2\|T_\alpha\|) \quad \text{for all } n \geqslant N.$$

Then, for $n \geqslant N$,

$$\|T_\alpha - T_{\sigma_n(\alpha)}\| \leqslant (2\pi)^{-1}\int_{|t|\leqslant\delta}\|T_\alpha - T_{\alpha_t}\|K_n(-t)dt + \|T_\alpha\|\pi^{-1}\int_{|t|\in[\delta,\pi]} K_n(-t)dt < 2\varepsilon$$

and the result follows. ∎

Defining $\mathscr{A}^p(\omega)$ to be the closure in the uniform operator topology of the polynomial multipliers, we have the following immediate corollary:

COROLLARY 2.11. $\mathscr{K}^p(\omega) \subseteq \mathscr{A}^p(\omega)$.

We can actually specify which elements of $\mathscr{A}^p(\omega)$ are compact. Given a weight sequence, ω, define

$$k_\omega := \inf\{k\colon\ \omega \text{ is regulated at } k\}.$$

(If ω is not regulated, we set $k_\omega = \infty$.) We then have:

PROPOSITION 2.12. $\mathscr{K}^p(\omega) = \{T_\alpha \in \mathscr{A}^p(\omega)\colon\ \alpha(k) = 0$ *for* $k < k_\omega\}$.

Proof. Let $T_\alpha \in \mathscr{K}^p(\omega)$. Assume α is not zero and let $j = \inf\{k\colon\ \alpha(k) \neq 0\}$. By Corollary 2.7,

$$\lim_{n\to\infty}\sum_{k=j}^{\infty}|\alpha(k)|^p\big(\omega(n+k)/\omega(n)\big)^p = 0,$$

and it follows that $\lim_{n\to\infty}\omega(n+j)/\omega(n) = 0$. Thus ω is regulated at j. But then $j \geqslant k_\omega$ and so

$$\mathscr{K}^p(\omega) \subseteq \{T_\alpha \in \mathscr{A}^p(\omega)\colon\ \alpha(k) = 0 \text{ for } k < k_\omega\}.$$

Now, by Proposition 2.8, $e_{k_\omega} \in \mathscr{K}^p(\omega)$. Clearly, $\{T_\alpha \in \mathscr{A}^p(\omega)\colon\ \alpha(k) = 0$ for $k < k_\omega\}$ is just the uniformly closed ideal of $\mathscr{M}^p(\omega)$ generated by e_{k_ω}. Hence we have the reverse inclusion. ∎

3. The multipliers of $l^p(\omega)$ for a regulated weight. We will now return to the question posed at the end of Section 1. Recall that when $l^p(\omega)$ is an algebra, the partial sums of multipliers converge, so certainly $\mathscr{M}^p(\omega) = \mathscr{A}^p(\omega)$ in this case. We can now prove the converse of this statement if we assume that ω is regulated. That is, given ω regulated, we will show that if $\mathscr{M}^p(\omega) = \mathscr{A}^p(\omega)$, then $l^p(\omega)$ is an algebra. This was proved in [2, Thm. 1.11] in the case $p = 1$ under

the additional assumptions that ω be regulated at 1 and that $l^1(\omega')$ be an algebra, where ω' is the weight sequence defined by $\omega'(0) = 1$ and $\omega'(n) = \omega(n-1)$ $(n \geqslant 1)$. That proof suggested the method used below. We will firstly require a couple of lemmas.

Note that if $l^p(\omega)$ is an algebra, then it follows from our remarks in Section 1 that there exists a constant $M > 0$ such that $|||x||| \leqslant M\|x\|$ for all $x \in l^p(\omega)$ and so $\|x*y\| \leqslant M$ whenever $x, y \in B(l^p(\omega))$. If $l^p(\omega)$ is not an algebra, then this is of course no longer true, however the next lemma shows that boundedness still holds as long as y is restricted to polynomials of a fixed degree.

LEMMA 3.1. *Given* $j \in \mathbf{Z}^+$, *there exists a constant* $C_j > 0$ *such that* $\|x*y\|_{p,\omega} \leqslant C_j$ *whenever* $x \in B(l^p(\omega))$ *and* $y \in B(l^p(\omega)) \cap E_j$.

Proof. Let $x \in l^p(\omega)$ and $y \in E_j$. Then we have

$$\begin{aligned}
(5)\qquad \|x*y\|_{p,\omega} &= \sup_{z\in B(l^{p'}(\omega))}|\langle x*y, z\rangle| \\
&\leqslant \sup_{z\in B(l^{p'}(\omega))}\sum_{k=0}^{j}|y(k)|\sum_{n=0}^{\infty}|x(n)||z(n+k)|\omega(n+k)^2 \\
&\leqslant \sum_{k=0}^{j}|y(k)|\Big(\sum_{n=0}^{\infty}|x(n)|^p\omega(n+k)^p\Big)^{1/p} \\
&\leqslant \Big(\sup_{\substack{k\in[0,j]\\ n\in\mathbf{Z}^+}}\omega(n+k)/\omega(n)\Big)\|x\|_{p,\omega}\sum_{k=0}^{j}|y(k)|.
\end{aligned}$$

But then, by Hölder's inequality (seeting $j^{1/p'} = 1$ in case $p = 1$), we have

$$\sum_{k=0}^{j}|y(k)| \leqslant \Big(\sup_{k\in[0,j]}1/\omega(k)\Big)\sum_{k=0}^{j}|y(k)|\omega(k) \leqslant j^{1/p'}\Big(\sup_{k\in[0,j]}1/\omega(k)\Big)\|y\|_{p,\omega}.$$

Substituting into (5) and letting

$$C_j = j^{1/p'}\Big(\sup_{k\in[0,j]}1/\omega(k)\Big)\Big(\sup_{\substack{k\in[0,j]\\ n\in\mathbf{Z}^+}}\omega(n+k)/\omega(n)\Big)$$

yields $\|x*y\|_{p,\omega} \leqslant C_j\|x\|_{p,\omega}\|y\|_{p,\omega}$ as was required. ∎

We noted prior to Lemma 3.1 that if $l^p(\omega)$ is an algebra then there exists $M > 0$ such that

$$(6)\qquad |||x||| \leqslant M\|x\|$$

for all $x \in l^p(\omega)$. Of course, the converse of this remark is also true. In fact, it is even true that if (6) holds for all polynomials x then $l^p(\omega)$ is an algebra. This is because if $z \in l^p(\omega)$, we can choose a sequence of polynomials $\{x_n\}$ with $\|z - x_n\| \to 0$. If (6) holds for polynomials, then $\{x_n\}$ is a Cauchy sequence in $m^p(\omega)$ and so converges in $|||\cdot|||$ as well, necessarily to z. Hence we have $m^p(\omega) = l^p(\omega)$ and so $l^p(\omega)$ is an algebra. Thus, if $l^p(\omega)$ were not an algebra, we could certainly find polynomials of norm one in $l^p(\omega)$ whose norms in $m^p(\omega)$ were arbitrarily large. The next lemma shows that a much stronger statement is true.

LEMMA 3.2. *Suppose $l^p(\omega)$ is not an algebra. Then given $j, n, N \in \mathbf{Z}^+$ there exists a polynomial $\beta \in B(l^p(\omega)) \cap F_j$ such that*

$$|||\beta|_{F_n}|||_{p,\omega} > N|||\beta|_{E_n}|||_{p,\omega}.$$

Proof. Choose $K > (N+1)C_n + C_j$, where C_n and C_j are as in Lemma 3.1. Since $l^p(\omega)$ is not an algebra, we can find polynomials x and y in $B(l^p(\omega))$ with $\|x*y\| > K$. Let $\beta = Q_j x$ and $y_0 = Q_n y$. Then

$$\begin{aligned}\|\beta*y_0\| &= \|(x-P_jx)*(y-P_ny)\| \geq \|x*y\| - \|x*P_ny\| - \|P_jx*y_0\| \\ &> K - C_j - C_n > NC_n,\end{aligned}$$

by our choice of K. The conclusion now follows since $C_n \geq |||\beta|_{E_n}|||$. ∎

We now state the main theorem of this section.

THEOREM 3.3. *Let ω be a regulated weight sequence. Then $\mathscr{M}^p(\omega) = \mathscr{A}^p(\omega)$ if and only if $l^p(\omega)$ is an algebra.*

As noted above, it remains to show that if $l^p(\omega)$ is not an algebra then $\mathscr{A}^p(\omega) \subsetneq \mathscr{M}^p(\omega)$. By Proposition 2.12, it will be sufficient to show that if $l^p(\omega)$ is not an algebra, then there exists $\alpha \in m^p(\omega)$ with $\alpha(j) = 0$ for $j < k_\omega$, but $T_\alpha \notin \mathscr{K}^p(\omega)$.

The proof will consist of an inductive construction of α satisfying the properties listed above. To this end we will rely heavily on Lemma 3.2 to guarantee the existence of polynomials such that the effect of convolving by them is concentrated farther and farther out. More precisely, given $n \in \mathbf{Z}^+$, we can find a polynomial β such that $|||\beta|_{F_n}|||$ is very much larger than $|||\beta|_{E_n}|||$. The idea will be to construct α by adding up such β's, keeping the operator norms of the sums bounded in order to ensure that $\alpha \in \mathscr{M}^p(\omega)$. However, we will use the fact that the effect of the β's is concentrated farther and farther out to show that $|||\alpha|_{F_n}|||$ does not converge to zero. Proposition 2.5 then implies that $T_\alpha \notin \mathscr{K}^p(\omega)$. It also follows from Lemma 3.2 that we can choose the β's supported as far out as we like. That is, we can choose $\beta \in F_j$ for j arbitrarily large. Also, since ω is regulated, we will be able to choose each β so that T_β is a compact multiplier. Proposition 2.5 then ensures that the effect of convolving by each of the β's is concentrated on a finite interval of $\mathbf{Z}^+$. These last two conditions will enable us to choose the β's nicely separated in order to ensure that no unwanted cancellation takes place. We will now proceed with the details.

DEFINITION 3.4. We define $\underline{\gamma}, \bar{\gamma}$: $l^p(\omega) \to \mathbf{Z}^+ \cup \{\infty\}$ by

$$\underline{\gamma}(\beta) = \inf\{n:\ \beta(n) \neq 0\}, \qquad \bar{\gamma}(\beta) = \sup\{n:\ \beta(n) \neq 0\},$$

for $\beta \in l^p(\omega)$. If $A \subseteq l^p(\omega)$, we let $\underline{\gamma}(A) = \inf\{\underline{\gamma}(\beta):\ \beta \in A\}$ and $\bar{\gamma}(A) = \sup\{\bar{\gamma}(\beta):\ \beta \in A\}$.

Proof of Theorem 3.3. Let $\beta_0 = e_{k_\omega}$. Choose $a_0 > 0$ so that $|||a_0\beta_0||| = 1$. By Proposition 2.12, $T_{a_0\beta_0} \in \mathscr{K}^p(\omega)$ and so, by Proposition 2.5, we can find $j_1 \in \mathbf{Z}^+$ such that

$$|||a_0\beta_0|_{F_{j_1}}||| < 10^{-4}/c \tag{7}$$

where $c = p2^{2p}$. Here c has been chosen so that

$$(a+b)^p \leq a^p + cb \quad \text{ if } 0 \leq a \leq 2 \text{ and } 0 \leq b \leq 1. \tag{8}$$

Note also that, if $0 \leq b \leq 1$ and $0 \leq a-b \leq 2$, then replacing a by $a-b$ in (8) yields

$$(a-b)^p \geq a^p - cb. \tag{9}$$

Now, by Lemma 3.2, we can find a polynomial β_1 with $\beta_1(k) \geq 0$ $(k \in \mathbf{Z}^+)$ such that

$$|||\beta_1|_{F_{j_1}}||| > 10^4 c|||\beta_1|_{E_{j_1}}|||. \tag{10}$$

Furthermore, by choosing $\beta_1 \in F_k$ with $k > k_\omega$ sufficiently large, we can ensure that

$$\underline{\gamma}(\beta_1 * F_{j_1}) > \bar{\gamma}(\beta_0 * E_{j_1}). \tag{11}$$

It follows again from Proposition 2.12 and our choice of β_1 that T_{β_1} is a compact multiplier. Hence, using Proposition 2.5, we can choose $n_1 > j_1$ so that $|||\beta_1|_{F_{n_1}}|||$ is arbitrarily small. In particular, we can choose n_1 sufficiently large so that

$$|||\beta_1|_{E_{n_1}\cap F_{j_1}}||| > 10^4 c|||\beta_1|_{F_{n_1}}|||, \tag{12}$$

and, by increasing n_1 yet further if necessary, and using (10), we can ensure that

$$|||\beta_1|_{E_{n_1}\cap F_{j_1}}||| > 10^4 c|||\beta_1|_{E_{j_1}}|||. \tag{13}$$

Combining (12) and (13) yields

$$|||\beta_1|_{E_{n_1}\cap F_{j_1}}||| > 10^4 c \max[|||\beta_1|_{E_{j_1}}|||,\ |||\beta_1|_{F_{n_1}}|||]. \tag{14}$$

Now choose $a_1 > 0$ such that

$$|||(a_0\beta_0 + a_1\beta_1)|_{E_{n_1}}||| = 1 + 1/2. \tag{15}$$

The proof will now be completed via the following steps:

(a) There exist sequences of positive integers j_k and n_k (with $n_{k-1} < j_k < n_k$), positive real numbers a_k, and polynomials $\beta_k \in F_{k_\omega}$ with positive coefficients satisfying the analogues of (7), (11), (14), and (15) for each integer $k \geq 1$. That is (setting $\alpha_k := \sum_{i=0}^{k} a_i\beta_i$ to simplify notation):

$$|||\alpha_{k-1}|_{F_{j_k}}||| < 10^{-2(k+1)}/c, \tag{16}$$

$$(17)\qquad \underline{\gamma}(\beta_k * F_{j_k}) > \bar{\gamma}(\beta_{k-1} * E_{j_k}),$$

$$(18)\qquad |||\beta_k|_{E_{n_k}\cap F_{j_k}}||| > 10^{2(k+1)} c \max[|||\beta_k|_{E_{j_k}}|||, |||\beta_k|_{F_{n_k}}|||],$$

$$(19)\qquad |||\alpha_k|_{E_{n_k}}||| = \sum_{i=0}^{k} 2^{-i}.$$

(b) If $\alpha := \sum_{i=0}^{\infty} a_i \beta_i$ then $\alpha \in m^p(\omega)$.

(c) $T_\alpha \notin \mathscr{K}^p(\omega)$.

We now proceed with the proofs of these statements.

(a) We will use induction and assume that j_i, n_i, a_i, and β_i have been chosen for $1 \leqslant i \leqslant k$ with (16)–(19) satisfied. As above, using the compactness of the β_i's and Lemma 3.2, we can find $j_{k+1} > n_k$, $n_{k+1} > j_{k+1}$, and a polynomial $\beta_{k+1} \in F_{k_\omega}$ with positive coefficients such that

$$|||\alpha_k|_{F_{j_{k+1}}}||| < 10^{-2(k+2)}/c, \quad \underline{\gamma}(\beta_{k+1} * F_{j_{k+1}}) > \bar{\gamma}(\beta_k * E_{j_{k+1}}),$$

$$|||\beta_{k+1}|_{E_{n_{k+1}}\cap F_{j_{k+1}}}||| > 10^{2(k+2)} c \max[|||\beta_{k+1}|_{E_{j_{k+1}}}|||, |||\beta_{k+1}|_{F_{n_{k+1}}}|||].$$

Next, we show that we can find $a_{k+1} > 0$ so that (19) is satisfied when k is replaced by $k+1$. Clearly, this is possible if we can show that

$$|||\alpha_k|_{E_{n_{k+1}}}||| < \sum_{i=0}^{k+1} 2^{-i}.$$

Let $x \in E_{n_{k+1}} \cap B(l^p(\omega))$ and let $x_1 = P_{n_k} x \in E_{n_k}$, $x_2 = Q_{n_k} x \in F_{n_k}$ so that $x = x_1 + x_2$. Then, by (16) and (19) and the fact that $F_{n_k} \subseteq F_{j_k}$,

$$(21)\qquad \|\alpha_k * x\| \leqslant \|\alpha_k * x_1\| + \|\alpha_k * x_2\| \leqslant \sum_{i=0}^{k} 2^{-i} + \|\alpha_{k-1} * x_2\| + \|a_k \beta_k * x_2\|$$

$$< \sum_{i=0}^{k} 2^{-i} + 10^{-2(k+1)}/c + \|a_k \beta_k * x_2\|.$$

Now, by (18),

$$|||a_k \beta_k|_{F_{n_k}}||| < \frac{10^{-2(k+1)}}{c} |||a_k \beta_k|_{E_{n_k}\cap F_{j_k}}||| \leqslant \frac{10^{-2(k+1)}}{c} |||\alpha_k|_{E_{n_k}}|||.$$

By (19),

$$(22)\qquad |||a_k \beta_k|_{F_{n_k}}||| < \frac{10^{-2(k+1)}}{c} \sum_{i=0}^{k} 2^{-i} < \frac{2 \cdot 10^{-2(k+1)}}{c}.$$

Finally, by (22) and (21),

$$(23)\qquad \|\alpha_k * x\| < \sum_{i=0}^{k} 2^{-i} + \frac{3 \cdot 10^{-2(k+1)}}{c} \qquad (x \in E_{n_{k+1}} \cap B(l^p(\omega))),$$

and so,

$$|||\alpha_k|_{E_{n_{k+1}}}||| < \sum_{i=0}^{k+1} 2^{-i}.$$

Hence we can choose $a_{k+1} > 0$ such that

$$|||\alpha_{k+1}|_{E_{n_{k+1}}}||| = |||(\sum_{i=0}^{k+1} a_i \beta_i)|_{E_{n_{k+1}}}||| = \sum_{i=0}^{k+1} 2^{-i}.$$

This completes the proof of statement (a).

(b) Let $\alpha = \sum_{l=0}^{\infty} a_l \beta_l$ and let $x \in B(l^p(\omega))$ with $x(l) \geqslant 0$ for all $l \in \mathbf{Z}^+$. Then, for $i \in \mathbf{Z}^+$,

$$(24)\qquad \|P_i(\alpha * x)\| \leqslant \|P_{i_1} \alpha * P_{i_2} x\| \quad \text{if } i_1, i_2 \geqslant i.$$

Given $i \in \mathbf{Z}^+$, choose k so large that $n_k > i$ and $\underline{\gamma}(\beta_l) > i$ for all $l > k$. This will always be possible since (17) implies that $\underline{\gamma}(\bar{\beta}_l)$ is an increasing sequence. Hence, $P_i \alpha = P_i(\sum_{l=0}^{k} a_l \beta_l) = P_i \alpha_k$ and so, by (24) and (19),

$$\|P_i(\alpha * x)\| \leqslant \|\alpha_k * P_{n_k} x\| \leqslant \sum_{l=0}^{k} 2^{-l} \leqslant 2$$

for all $i \in \mathbf{Z}^+$. Therefore, $\|\alpha * x\| \leqslant 2$ for all $x \in B(l^p(\omega))$, and it follows that $\alpha \in m^p(\omega)$ with $|||\alpha||| \leqslant 2$.

(c) To show that $T_\alpha \notin \mathscr{K}^p(\omega)$, it is sufficient by Proposition 2.5 to show that $|||\alpha|_{F_i}||| \not\to 0$ as $i \to \infty$. Hence, to conclude the proof, we will show that given $i \in \mathbf{Z}^+$, there exists $x \in B(l^p(\omega)) \cap F_i$ with $\|\alpha * x\| \geqslant 1$.

Given $i \in \mathbf{Z}^+$, choose $k \geqslant p$ so large that $j_k \geqslant i$. By (19) we can find $x \in E_{n_k}$ with $\|x\| = 1$, $x(l) \geqslant 0$ $(l \in \mathbf{Z}^+)$, and

$$(25)\qquad \|\alpha_k * x\| \geqslant \sum_{l=0}^{k} 2^{-l} - 10^{-2k}/c.$$

Let $x_1 = P_{j_k} x \in E_{j_k}$ and $x_2 = Q_{j_k} x \in F_{j_k}$. Then

$$(26)\qquad \|\alpha_k * x\| \leqslant \|\alpha_{k-1} * x_1 + a_k \beta_k * x_2\| + \|\alpha_{k-1} * x_2\| + \|a_k \beta_k * x_1\|.$$

Now, by (16),

$$(27)\qquad \|\alpha_{k-1} * x_2\| < 10^{-2(k+1)}/c$$

and, by (18) and (19),

$$(28)\qquad \|a_k \beta_k * x_1\| < \frac{10^{-2(k+1)}}{c} |||a_k \beta_k|_{E_{n_k}\cap F_{j_k}}||| \leqslant \frac{10^{-2(k+1)}}{c} \sum_{l=0}^{k} 2^{-l}$$

$$< \frac{2 \cdot 10^{-2(k+1)}}{c}.$$

Hence, substituting (27) and (28) into (26) yields

$$(29)\qquad \|\alpha_k * x\| < \|\alpha_{k-1} * x_1 + a_k \beta_k * x_2\| + 3 \cdot 10^{-2(k+1)}/c.$$

Also, since $|||\alpha||| \leqslant 2$, we have

(30) $$\|\alpha_{k-1} * x_1 + a_k \beta_k * x_2\| \leqslant 2.$$

From (29), (30), and (8), we get

(31) $$\|\alpha_k * x\|^p < \|\alpha_{k-1} * x_1 + a_k \beta_k * x_2\|^p + 3 \cdot 10^{-2(k+1)}.$$

Then, if we use (17), (31) becomes

(32) $$\|\alpha_k * x\|^p < \|\alpha_{k-1} * x_1\|^p + \|a_k \beta_k * x_2\|^p + 3 \cdot 10^{-2(k+1)}.$$

Now by (23),

(33) $$\begin{aligned}\|\alpha_{k-1} * x_1\|^p &< [(\sum_{l=0}^{k-1} 2^{-l}) \|x_1\| + 3 \cdot 10^{-2k}/c]^p \\ &< (\sum_{l=0}^{k-1} 2^{-l})^p \|x_1\|^p + 3 \cdot 10^{-2k}\end{aligned}$$

where the second inequality follows again from (8). By (19),

(34) $$\|\alpha_k \beta_k * x_2\| \leqslant (\sum_{l=0}^{k} 2^{-l}) \|x_2\|.$$

Substituting (33) and (34) into (32), we get

(35) $$\|\alpha_k * x\|^p < (\sum_{l=0}^{k-1} 2^{-l})^p \|x_1\|^p + (\sum_{l=0}^{k} 2^{-l})^p \|x_2\|^p + 4 \cdot 10^{-2k}.$$

Then by (35) and (25),

$$[\sum_{l=0}^{k} 2^{-l} - 10^{-2k}/c]^p < (\sum_{l=0}^{k-1} 2^{-l})^p \|x_1\|^p + (\sum_{l=0}^{k} 2^{-l})^p \|x_2\|^p + 4 \cdot 10^{-2k}.$$

Therefore, using (9),

(36) $$(\sum_{l=0}^{k} 2^{-l})^p (1 - \|x_2\|^p) - (\sum_{l=0}^{k-1} 2^{-l})^p \|x_1\|^p < 5 \cdot 10^{-2k}.$$

But $\|x_1\|^p + \|x_2\|^p = \|x\|^p = 1$. Hence, $1 - \|x_2\|^p = \|x_1\|^p$ and (36) yields

$$\|x_1\|^p < \frac{5 \cdot 10^{-2k}}{(\sum_{l=0}^{k} 2^{-l})^p - (\sum_{l=0}^{k-1} 2^{-l})^p} < 2^k \cdot 5 \cdot 10^{-2k} < 10^{-k}.$$

Therefore,

(37) $$\|x_1\| < 10^{-k/p}.$$

Again by (25),

$$\sum_{l=0}^{k} 2^{-l} - 10^{-2k}/c \leqslant \|\alpha_k * x\| \leqslant \|\alpha_k * x_1\| + \|\alpha_k * x_2\| \leqslant 2\|x_1\| + \|\alpha_k * x_2\|.$$

Thus, using (37),

$$\|\alpha_k * x_2\| > \sum_{l=0}^{k} 2^{-l} - 10^{-2k}/c - 2 \cdot 10^{-k/p} > 1$$

since we chose $k \geqslant p$. But $\|\alpha * x_2\| > \|\alpha_k * x_2\|$, so we can conclude that $\|\alpha * x_2\| > 1$ where $x_2 \in B(l^p(\omega)) \cap F_{J_k} \subseteq F_i$. This completes the proof of (c) and hence the proof of Theorem 3.3. ∎

The following corollary is immediate:

Corollary 3.5. *If ω is a regulated weight sequence and $s_n(\alpha) = \sum_{k=0}^{n} \alpha(k) e_k$ converges to α in $|||\cdot|||$ for all $\alpha \in m^p(\omega)$, then $l^p(\omega)$ is an algebra.*

When $p = 2$, Corollary 3.5 provides an affirmative answer to Question 12 in [4] in the case of a regulated weight. (Note that Shields denotes $l^2(\omega)$ by $H^2(\omega)$ and $\mathcal{M}^2(\omega)$ by $H^\infty(\omega)$. This notation comes from the unweighted case when l^2 can be identified with the Hardy space H^2 on the unit disk and it can be shown that its multiplier algebra is isometrically isomorphic to H^∞, the bounded analytic functions on the unit disk.)

The question remains whether the hypothesis of a regulated weight can be dispensed with in Theorem 3.3. The answer is in the negative when $p = 1$—see [2] or [1] for examples where $\mathcal{M}^1(\omega) = \mathcal{A}^1(\omega)$ but $l^1(\omega)$ is not an algebra. However, the question is still open for $p > 1$.

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