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The Rockland condition for nondifferential convolution operators II

by

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Abstract. The aim of this paper is to present a slightly refined version of the following theorem: Let K be a singular integral operator on a homogeneous group N. Assume that the kernel of K is C^∞ away from the origin. Then K is left-invertible on $L^2(N)$ if and only if for every nontrivial irreducible representation π of N, π_K is injective on the space of C^∞-vectors for π. In addition, some consequences for the problem of characterization of the Hardy space $H^1(N)$ by means of generalized Riesz transforms are indicated.

Introduction. Let N be a homogeneous group, a nilpotent Lie group endowed with a family of automorphic dilations. Denote by Q the homogeneous dimension of N, and let $x \to |x|$ be a homogeneous norm on N. We consider a singular integral operator

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} f(yx)k(y)\,dy, \quad f \in C_c^1(N),$$

determined by a smooth function $k\colon N\backslash\{0\} \to \mathbf{C}$ which is moreover assumed to be homogeneous of degree $-Q$ and satisfy the mean value zero condition

$$\int_{1/2 \leq |x| \leq 2} k(x)\,dx = 0.$$

It is well known that for every unitary representation π of N on a Hilbert space H_π, π_K is well defined as a bounded linear operator on the dense subspace $C^\infty(\pi)$ of H_π consisting of smooth vectors for π. In particular, $K = \pi_K^1$, where π^1 is the right-regular representation of N on $L^2(N)$, extends to a bounded operator on $L^2(N)$, still denoted by K.

The aim of this paper is to present a slightly refined version of the following theorem:

THEOREM. *K is left-invertible if and only if π_K is injective on $C^\infty(\pi)$ for every nontrivial irreducible unitary representation π of N.*

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Key words and phrases: homogeneous groups, singular integral operators, kernels of order r, stable semigroups of measures, representations of nilpotent groups, Hardy spaces.

Let us remark that in the case when $N = \mathbf{R}^n$ with the usual dilations, our theorem says that K is invertible if and only if the Fourier transform of the principal value distribution

$$\langle f, \mathrm{PV}(k)\rangle = \lim_{\varepsilon\to 0} \int_{|x|\geqslant\varepsilon} f(x)k(x)\,dx$$

does not vanish except at the origin, which is well known and easy to see.

This and other similar problems have been dealt with by Calderón and Zygmund [CZ], Christ [C1] and [C2], Christ and Geller [CG], Moukaddem [Mo], and the author [G2]. Some of these papers were motivated primarily by the study of Hardy spaces. Although our theorem seems to be of interest in its own right, I would like to stress here its actual and possible significance for the theory of Hardy spaces on homogeneous groups, as initiated by Folland and Stein [FS].

In particular, the above theorem applies to the problem of determining which families of singular integral operators on N have the property of characterizing the Hardy space $H^1(N)$. Recall that a finite collection $K_1, \ldots, K_m$ of singular integral operators with smooth kernels $k_1, \ldots, k_m$ (see above) *characterizes* $H^1(N)$ if and only if for every tempered distribution f on N, f belongs to $H^1(N)$ if and only if $K_j f \in L^1(N)$ for all j, and the norm

$$f \to \sum_{j=1}^{m} \|K_j f\|_{L^1(N)}$$

is equivalent to the norm in $H^1(N)$. Classical examples are Riesz transforms on $\mathbf{R}^n$ (see, e.g., Fefferman and Stein [FeS], p. 144).

It is a conjecture of Christ and Geller [CG], p. 573, that the following condition is sufficient for a finite collection of real singular integral operators $K_1, \ldots, K_m$ to characterize $H^1(N)$:

(*) For every nontrivial irreducible unitary representation π of N on a Hilbert space H_π and every nonzero vector $v \in H_\pi$, the subspace of H_π spanned by the vectors $\pi_{K_1}v, \ldots, \pi_{K_m}v$ is of dimension at least 2 over $\mathbf{R}$.

Christ and Geller prove their conjecture only for the Heisenberg group where (*) turns out to be necessary as well. However, as pointed out by Daryl Geller (in a personal communication), in the general case the problem boils down to proving the aforementioned theorem.

A partial result in this direction has been subsequently obtained by N. Moukaddem who proves the inversion theorem in the case when the group N is a graded homogeneous group of step less than or equal to 3 (see Moukaddem [Mo]).

The idea of our proof consists in reducing the problem to certain maximal estimates for convolution operators of positive, though small, order. This, in turn, is accomplished much along the lines of Helffer and Nourrigat's proof of the Rockland conjecture (see Helffer and Nourrigat [HN]). Surprisingly, the theory of semigroups of probability measures also plays an important role.

The methods used here are by no means new. All the ingredients of the proof have already been prepared in [G3], and we merely choose those ingredients and adapt them for our present purpose. Consequently, this paper is far from being self-contained. It should be rather regarded as an integral part of [G3]. However, for the convenience of the reader, all the necessary results have been listed in Section 2.

It is my pleasure to thank D. Geller for bringing the problem considered here to my attention and for his illuminating remarks. I am also grateful to B. Helffer and J. Nourrigat for inspiring conversations on the subject of this paper.

Notation. As usual $\mathbf{R}$ and $\mathbf{C}$ denote the fields of real and complex numbers, respectively.

Let V be a finite-dimensional real vector space of dimension n, and let V^* be its dual. The vector space norms on V and V^* are denoted by $\|\cdot\|$. The duality between $x \in V$ and $\xi \in V^*$ is denoted by $\langle x, \xi\rangle$. If W is a vector subspace of V and $\xi \in V^*$, then the restriction of ξ to W is denoted by $\xi|W$.

A one-parameter family $\{\delta_t\}_{t>0}$ of linear mappings of V into itself is called *dilations* if there exist strictly positive numbers $d_1, \ldots, d_n$ and a basis $\{e_j\}_1^n$ of V such that

$$\delta_t e_j = t^{d_j} e_j, \quad t > 0,\ 1 \leqslant j \leqslant n. \tag{0.1}$$

A *homogeneous norm* on V is defined to be a continuous function $V \ni x \to |x| \in [0, \infty)$ which satisfies

$$|\delta_t x| = t|x|, \quad x \in V,\ t > 0, \tag{0.2}$$

$$|x| = 0 \ \text{ if and only if } \ x = 0. \tag{0.3}$$

For any family of dilations on V, there exists a homogeneous norm. In fact, one needs only take $|0| = 0$ and $|x| = t$, where $\|\delta_{t^{-1}}x\| = 1$, for $x \neq 0$. (The t with this property is, as is easily seen, unique.) Note that the norm thus defined is symmetric and C^∞ away from the origin (see Folland and Stein [FS], p. 8).

There is also a homogeneous norm on V^*, still denoted by $|\cdot|$, induced by the dual family of dilations

$$\langle x, \delta_t^* \xi\rangle = \langle \delta_t x, \xi\rangle, \quad x \in V,\ \xi \in V^*.$$

We use the standard notations $C^1(V)$, $C_c^\infty(V)$, and $\mathscr{S}(V)$ for the spaces of continuously differentiable functions, C^∞-functions with compact support, and C^∞-functions rapidly vanishing at infinity. We shall often refer to C^∞-functions as, simply, *smooth* functions.

For $p = 1$ or 2, $L^p(V)$ is the usual L^p-space with respect to Lebesgue measure dx on V. The norm in $L^p(V)$ is denoted by $\|\cdot\|_{L^p(V)}$.

If $f \in C_c^\infty(V)$ and T is a distribution on V, then $\langle f, T\rangle$ denotes the action of T on f. For a distribution T, we define its *reflection* $\check{T}$ by $\langle f, \check{T}\rangle = \langle \check{f}, T\rangle$, where $\check{f}(x) = f(-x)$, $x \in V$. The space of compactly supported distributions on V is denoted by $\mathscr{E}^*(V)$.

If, in addition, V is endowed with a Lie algebra structure, then the Lie bracket of $x, y \in V$ will be denoted by $[x, y]$.

1. General set-up. Let N be a *homogeneous* group, a nilpotent Lie group endowed with a family of automorphic dilations $\{\delta_t\}_{t>0}$. Being homogeneous, N is connected and simply connected, which permits the identification of N as a differentiable manifold with its Lie algebra by means of the exponential map. Therefore, we shall be assuming throughout the paper that N is also equipped with a Lie algebra structure related to its group structure by the Campbell–Hausdorff formula. In this setting Haar measure of N is simply Lebesgue measure of N as a vector space. In addition, the dilations δ_t are Lie algebra automorphisms, and they induce a dual family of vector space dilations on the dual vector space N^*.

Denote by $1 = d_1 < \ldots < d_m$ the exponents of homogeneity of the dilations. We have

$$N = \bigoplus_{j=1}^{m} N_j,$$

where

$$N_j = \{x \in N:\ \delta_t x = t^{d_j} x,\ t > 0\}. \tag{1.1}$$

Also, set $N_{m+1} = \{0\}$. For $1 \leqslant p \leqslant m$, let

$$N(p) = \bigoplus_{j=p}^{m} N_j. \tag{1.2}$$

A continuous complex function f on N is said to be *homogeneous* of degree $r \in \mathbf{R}$ if $f(\delta_t x) = t^r f(x)$ for $x \in N$, $t > 0$. A distribution T on N is said to be a *kernel of order* $r \in \mathbf{R}$ if T coincides with a Radon measure dT on $N \backslash \{0\}$ and satisfies

$$\langle f \circ \delta_t, T\rangle = t^r \langle f, T\rangle$$

for $f \in C_c^\infty(N)$ and $t > 0$. A kernel T of order r is said to be *Lipschitz* if the measure dT has a density F relative to Haar measure which is *Lipschitz*. The latter, by definition, means

$$|F(x) - F(y)| \leqslant C|y^{-1}x|^s$$

for $1/2 \leqslant |x|, |y| \leqslant 2$ and some constants $C > 0$, $0 < s < 1$.

If T is a kernel of order $r > 0$ and π is a unitary representation of N on a Hilbert space H_π, then π_T is well defined as a closable linear operator on the space $C^\infty(\pi) \subseteq H_\pi$ of smooth vectors for π. The closure of π_T is denoted by $\bar{\pi}_T$.

If V and S are vector subspaces of N such that $V \oplus S = N$, then $VS = N$, and there exist polynomial mapings

$$v\colon N \to V, \qquad \sigma\colon N \to S \tag{1.3}$$

such that every $a \in N$ decomposes uniquely as $a = v(a)\sigma(a)$.

We shall write $\xi \in \boldsymbol{H}(V)$ for a linear functional ξ in N^* if V is an isotropic subalgebra of N subordinate to ξ. If S is as above, and $\xi \in \boldsymbol{H}(V)$, then the unitary representation $\pi^{(V,\xi)}$ of N induced from V by the unitary character $x \to e^{i\langle x, \xi\rangle}$ can be realized on $L^2(S)$ in the following way:

$$\pi_a^{(V,\xi)} f(x) = e^{i\langle v(xa), \xi\rangle} f(\sigma(xa)), \tag{1.4}$$

where $f \in L^2(S)$, $a \in N$, and $x \in S$. Recall that a unitary representation π of a Lie group N induced by a unitary character χ from a closed subgroup V of N is called *monomial*.

A subspace S of N is said to be *homogeneous* if it is invariant under dilations. Note that for every subalgebra V of N, there exists a homogeneous linear complement S (cf., e.g., [G3], Lemma (2.7)). In particular, let $\tilde{N}$ be a homogeneous linear complement to $N(m)$. The corresponding projections

$$v\colon N \to N(m), \qquad \sigma\colon N \to \tilde{N} \tag{1.5}$$

are homogeneous and linear, and the multiplication

$$x \circ y = \sigma(xy), \qquad x, y \in \tilde{N}, \tag{1.6}$$

makes $\tilde{N}$ into a homogeneous group isomorphic to $N/N(m)$, σ being the canonical homomorphism. Dilations on $\tilde{N}$ are simply those of N restricted to $\tilde{N}$.

For $\lambda \in N(m)^*$, let

$$\pi^\lambda = \pi^{(N(m), \lambda)}. \tag{1.7}$$

Then, as is easily seen, the right-regular representation π^1 of N decomposes as

$$\pi_a^1 f(x) = \int_{N(m)^*} e^{2\pi i\langle v(x), \lambda\rangle} \pi_a^\lambda f^\lambda(\sigma(x))\, d\lambda, \tag{1.8}$$

where

$$f^\lambda(\sigma(x)) = \int_{N(m)} f(x) e^{-2\pi i\langle v(x), \lambda\rangle}\, dv(x) \tag{1.9}$$

for $f \in C_c^\infty(N)$.

If T is a kernel of order $r > 0$ on N, then

$$\langle f, \tilde{T}\rangle = \langle f \circ \sigma, T\rangle \tag{1.10}$$

defines a kernel $\tilde{T}$ of order r on $\tilde{N}$ such that

$$\tilde{\pi}_{\tilde{T}}^1 = \pi_T^0, \tag{1.11}$$

where $\tilde{\pi}^1$ is the right-regular representation of $\tilde{N}$.

Let P be a smooth symmetric accretive kernel of order $0 < r \leqslant 1$. Recall that a real distribution P on N is *accretive* if $\langle f, P \rangle \geqslant 0$ for real functions f in $C_c^\infty(N)$ assuming their maximal values at the origin. A kernel P of order r is said to be *smooth* if the measure dP on $N \backslash \{0\}$ has a C^∞-density with respect to Haar measure of N. We may take, for instance,

$$\langle f, P \rangle = \lim_{\varepsilon \searrow 0} \int_{|x| \geqslant \varepsilon} \frac{f(0)-f(x)}{|x|^{Q+r}} dx, \quad f \in C_c^\infty(N), \tag{1.12}$$

where $|x|$ is a homogeneous norm on N. We shall always assume that our homogeneous norm is symmetric and smooth away from the origin. The number

$$Q = \sum_{j=1}^{m} d_j \dim N_j$$

is called the *homogeneous dimension* of N.

Let $\{\mu_t\}$ be the convolution semigroup of probability measures generated by P. It follows, by Lemma (3.28) of [G1], that $\mu_t(dx) = f_t(x)dx$, $t > 0$, where $f_t \in L^1(N)$. Let

$$F(x) = \int_0^\infty e^{-t} f_t(x) dt, \quad x \in N. \tag{1.13}$$

Then $F \in L^1(N)$, and, by elementary properties of semigroups of measures,

$$F * P = \delta - F, \tag{1.14}$$

where $*$ stands for the convolution on N, and δ is the Dirac point mass located at the origin.

(1.15) Lemma. *For every nontrivial monomial unitary representation π of N, $\bar{\pi}_P$ is injective on its domain, and the space of vectors of the form $\pi_P f$, where $f \in C^\infty(\pi)$, is dense in H_π.*

Proof. By Duflo [D], π_P is essentially selfadjoint on $C^\infty(\pi)$ so it is sufficient to show that $\bar{\pi}_P$ is injective.

Let $\bar{\pi}_P f = 0$ for some f in the domain of $\bar{\pi}_P$. Then $\pi_{\mu_t} f = f$ for every $t > 0$, whence

$$\int \langle f, \pi_a f \rangle \mu_t(da) = \|f\|^2$$

for, say, $t = 1$. Since μ_1 is an absolutely continuous probability measure, $\langle f, \pi_a f \rangle = \|f\|^2$ and, consequently,

$$\pi_a f = f \tag{1.16}$$

for every $a \in N$.

Now, since π is monomial, it can be realized as $\pi = \pi^{(V,\xi)}$ on $L^2(S)$, where V is a subalgebra of N, $\xi \in \boldsymbol{H}(V)$, and S is a complementary vector space to V.

By (1.16) and (1.4),

$$|f(\sigma(xa))| = |f(x)|, \quad x \in S,\ a \in N,$$

f being a function in $L^2(S)$, which implies that $|f| = \text{const}$. Since $f \in L^2(S)$, it must be zero, which completes the proof. ∎

2. Main lemmata. The whole of our consideration here is based on the following results which have been proven elsewhere.

(2.1) Lemma. *Let $1 \leqslant p \leqslant m$ be an integer. Let V be a subalgebra of $N(p)$. Let S be a homogeneous linear complement to V in N. Let T be a kernel of order $0 < r < 1$ on N. Then, for any $\xi, \eta \in \boldsymbol{H}(V)$ such that $\xi = \eta$ on $N(p+1)$, $\pi_T^{(V,\xi)} - \pi_T^{(V,\eta)}$ extends to a bounded operator on $L^2(S)$, and*

$$\lim_{\|\xi-\eta\| \to 0} \|\pi_T^{(V,\xi)} - \pi_T^{(V,\eta)}\|_{\text{op}} = 0, \tag{2.2}$$

where $\|\cdot\|_{\text{op}}$ denotes the operator norm on $L^2(S)$.

Proof. See [G3], Proposition (3.1). ∎

To state the second result we need some preparation. We introduce the following notation: If V is a subalgebra of N, and ξ is a functional on N, we set

$$N_V^*(\xi) = \{\eta \in N^*\colon\ \eta | V = \xi | V\},$$

$$V_\xi^\perp = \{x \in N\colon\ \langle [x, y], \xi \rangle = 0 \text{ for all } y \in V\},$$

$$\mathcal{O}(\xi) = \{\mathrm{Ad}_x^* \xi\colon\ x \in V_\xi^\perp\},$$

where Ad^* denotes the co-adjoint action of N on N^*.

Following Helffer and Nourrigat (see [HN], Definition 5.1.6) we say that a pair (V, ξ) is *maximal of order* $1 \leqslant p \leqslant m$ if $V \subseteq N(p)$ and $V(j) = V \cap N(j)$ is a maximal isotropic subalgebra of $N(j)$ for $\xi_j = \xi | N(j)$, where $p \leqslant j \leqslant m$.

If (V, ξ) is maximal of order p, then

$$(V, \eta) \text{ is maximal of order } p \text{ for } \eta \in N_V^*(\xi), \tag{2.3}$$

$$V_\eta^\perp = V_\xi^\perp \quad \text{for } \eta \in N_V^*(\xi), \tag{2.4}$$

$$\mathcal{O}(\xi) \subseteq N_V^*(\xi), \tag{2.5}$$

(2.6) there exists a subalgebra W of $N(p-1)$ such that $W \cap N(p) = V$, $[W, W] \subseteq V$, and (W, η) is maximal of order $p-1$ for $\eta \in N_V^*(\xi)$ $(p \geqslant 2)$.

(2.3) follows by general considerations concerning polarizations on nilpotent Lie algebras (see Pukanszky [P]), while (2.4)–(2.6) are proved by Helffer and Nourrigat (see [HN], Sections 5.1 and 5.5).

Let

$$\mathrm{dist}_W(\xi, \mathcal{O}(\eta)) = \inf_{\zeta \in \mathcal{O}(\eta)} \|\xi | W - \zeta | W\|.$$

(2.7) Lemma. *Let (V, ξ) be maximal of order $2 \leqslant p \leqslant m$, and let W be as in* (2.6) *above. Then, for every F in $L^1(N)$,*

$$\lim \|\pi_F^{(W,\eta)}\|_{\mathrm{op}} = 0 \quad as \quad \mathrm{dist}_W(\eta, \mathcal{O}(\xi)) \to \infty \ with\ \eta \in N_V^*(\xi).$$

Proof. This is a straightforward consequence of Proposition 5.5.1 of Helffer and Nourrigat [HN] as explained in Melin [Me], Proposition 5.9. ■

Let V and W be subalgebras of N such that $[W, W] \subseteq V \subseteq W$, and let $S_W \subseteq S_V$ be linear complements in N to W and V, respectively. Let T be a linear complement to S_W in S_V. For $f \in L^2(S_V)$ and $\eta \in W^*$, let

$$f^\eta(y) = \int_T f(ty) e^{-2\pi i \langle t, \eta | T\rangle} dt, \tag{2.8}$$

where $a = ty$ is the unique decomposition of $a \in S_V$ as a product of $t \in T$ and $y \in S_W$.

(2.9) Remark. If $f \in C_c^\infty(S_V)$, then $f^\eta \in C_c^\infty(S_W)$ for every $\eta \in W^*$. Conversely, for every $g \in C_c^\infty(S_W)$ and every $\eta \in W^*$, there exists f in $C_c^\infty(S_V)$ such that $f^\eta = g$.

(2.10) Lemma. *Let $\xi \in \boldsymbol{H}(W)$. Then the representation $\pi^{(V,\xi)}$ acting on $L^2(S_V)$ is unitarily equivalent to the direct integral*

$$\bigoplus \int_{N_V^*(\xi)|W} \pi^{(W,\eta)} d\eta$$

with $f \to \bigoplus \int_{N_V^*(\xi)|W} f^\eta d\eta$ being the intertwining operator from $L^2(S_V)$ to $\bigoplus \int_{N_V^*(\xi)|W} L^2(S_W) d\eta$, where $d\eta$ is Lebesgue measure on the linear variety $N_V^*(\xi)|W \subseteq W^*$.

Proof. This is well known. See, e.g., Helffer and Nourrigat [HN], Sect. 2. ■

(2.11) Corollary. *Let P be a kernel of order $0 < r < 1$ and let $f \in C_c^\infty(S_V)$. Then the mappings*

$$N_V^*(\xi)|W \ni \eta \to f^\eta \in L^2(S_W) \quad and \quad N_V^*(\xi)|W \ni \eta \to \pi_P^{(W,\eta)} f^\eta \in L^2(S_W)$$

are continuous.

Proof. This is a direct consequence of Lemma (2.1) and (2.8). ■

3. Maximal estimates. In this section we prove maximal estimates for kernels of order $0 < r < 1$ satisfying the Rockland condition, which is the essence of the proof of our inversion theorem (see Section 4 below).

(3.1) Theorem. *Let R be a kernel of order $0 < r < 1$ on a homogeneous group N. Suppose that R satisfies the Rockland condition:*

(3.2) *For every nontrivial irreducible unitary representation π of N, the operator $\bar{\pi}_R$ is injective on its domain.*

Then, for every kernel T of order r,

$$\|\pi_T^1 f\| \leqslant C \|\pi_R^1 f\|, \quad f \in C_c^\infty(N).$$

(Cf. Remarks (3.14) and (3.15) at the end of this section.)

Proof. Suppose first that N is abelian. Then the Fourier transform $\hat{R}$ of the tempered distribution R is a continuous function on N^* homogeneous of degree r, and by (3.2), $\hat{R}$ does not vanish outside the origin. Therefore, there exists a constant $C > 0$ such that

$$|\hat{R}(\xi)| \geqslant C|\xi|^r, \quad \xi \in N^*,$$

which immediately implies the assertion of our theorem.

Now we proceed by induction. Assume that Theorem (3.1) holds true for $\tilde{N} = N/N(m)$. Once we prove that this implies that the theorem is valid for N as well, the proof will be completed.

By the induction hypothesis and (1.11),

$$\|\pi_T^0 f\| \leqslant C \|\pi_R^0 f\|, \quad f \in C_c^\infty(N),$$

whence, by Lemma (2.1) and the homogeneity of both T and R,

$$\|\pi_T^\lambda f\| \leqslant C_0 (\|\pi_R^\lambda f\| + \|f\|), \quad f \in C_c^\infty(\tilde{N}), \tag{3.3}$$

for $|\lambda| = 1$ and some $C_0 > 0$. Here and below we set $\|\cdot\| = \|\cdot\|_{L^2(S)}$ for an appropriate $S \subseteq N$.

Our next step will be proving the following claim:

(3.4) *For every subalgebra V of N containing $N(m)$ and every $\xi \in \boldsymbol{H}(V)$ such that $|\xi_m| = 1$,*

$$\|\pi_T^{(V,\xi)} f\| \leqslant C_0 (\|\pi_R^{(V,\xi)} f\| + \|f\|),$$

where $f \in C_c^\infty(S_V)$, and the constant C_0 is that of (3.3).

Firstly, note that $V \subseteq N(p)$ and $V \nsubseteq N(p+1)$ for some $1 \leqslant p \leqslant m$. We shall make an induction on p starting with $p = m$. In this case $V = N(m)$, and our claim is reduced to (3.3).

Assume now that (3.4) holds true for some $2 \leqslant p \leqslant m$, and let $V \subseteq N(p-1)$, $V \nsubseteq N(p)$, $\xi \in \boldsymbol{H}(V)$. Then (3.4) holds for $W = V \cap N(p)$ and ξ. Moreover, $W \subseteq V$ and $[V, V] \subseteq W$, which permits an application of Lemma (2.10) (with the roles of V and W interchanged). Therefore, if S_V is a linear complement to V in N, there exists a constant C_0 such that

$$\|\pi_T^{(V,\eta)} f^\eta\| \leqslant C_0 (\|\pi_R^{(V,\eta)} f^\eta\| + \|f^\eta\|)$$

for $f \in C_c^\infty(S_V)$ and almost all $\eta \in N_V^*(\xi)|V$. Actually, since both sides of the estimate are continuous in η (see Corollary (2.11)), it remains true for all $\eta \in N_V^*(\xi)|V$.

Our next claim is the following:

(3.5) *Let V be a maximal isotropic subalgebra of N for a functional $\xi \in N^*$ such that $|\xi_m| = 1$. Then*

$$\|f\| \leqslant C\|\pi_R^{(V,\xi)} f\|, \quad f \in C_c^\infty(S_V),$$

where the constant C depends only on V and ξ.

For the sake of simplicity, let $\pi = \pi^{(V,\xi)}$. By Kirillov [K], π is irreducible. Let P be an accretive kernel of order r with dP being absolutely continuous relative to Haar measure on $N\backslash\{0\}$ (see (1.12)).

We are going to show that the assumption that (3.5) is false leads to a contradiction. Indeed, under this assumption, there exists a sequence $f_j \in C_c^\infty(S_V)$ with $\|f_j\| = 1$ such that $\pi_R f_j$ tends to zero in $L^2(S_V)$. It follows from (3.4) that the sequence $\pi_P f_j$ is bounded in $L^2(S_V)$. Let F be the resolvent for P as defined in (1.13). By (1.14), we have

$$\pi_F \pi_P f_j = f_j - \pi_F f_j \tag{3.6}$$

for every j. Since π is irreducible and $F \in L^1(N)$, π_F is compact (see Kirillov [K]). Consequently, by (3.6), there exists a subsequence of f_j which converges to an f in $L^2(S_V)$. Consequently, f is in the domain of $\bar{\pi}_R$, and $\bar{\pi}_R f = 0$, which is a contradiction as $\bar{\pi}_R$ satisfies the Rockland condition (3.2) and $\|f\| = 1$.

Having established the estimate (3.5) for irreducible representations, we now want to "lift" it to π^λ's. More precisely, we aim at proving the following:

(3.7) *Let $\xi \in N^*$ with $|\xi_m| = 1$. Then, for every $1 \leqslant p \leqslant m$ and every subalgebra V of N such that the pair (V, ξ) is maximal of order p, there is a constant $C(\xi_p)$ depending only on $\xi_p = \xi | N(p)$ such that*

$$\|f\| \leqslant C(\xi_p)\|\pi_R^{(V,\xi)} f\|, \quad f \in C_c^\infty(S_V).$$

Note first that the case $p = 1$ has just been proven. Indeed, in order to see this one only has to know that for every $\xi \in N^*$, there exists a subalgebra V of N such that (V, ξ) is maximal of order 1. This, in turn, can be proven by using (2.6) and a simple induction.

Suppose, then, that (3.7) has been proven for some $1 \leqslant p \leqslant m-1$, and let (V, ξ) be maximal od order $p+1$ with $|\xi_m| = 1$. By (2.6), there exists a subalgebra W of $N(p)$ such that $W \cap N(p+1) = V$, $[W, W] \subseteq V$, and (W, η) is maximal of order p for every $\eta \in N_V^*(\xi)$. Therefore,

$$\|f\| \leqslant C(\eta)\|\pi_R^{(W,\eta)} f\|, \quad f \in C_c^\infty(S_W), \tag{3.8}$$

for $\eta \in N_V^*(\xi)$.

Once we show that the function $\eta \to C(\eta)$ is bounded, we shall be able to complete the proof by integrating both sides of (3.8) over $N_V^*(\xi)|W$ and invoking Lemma (2.10). Thus, we want to show that $C(\eta)$ is bounded for $\eta \in N_V^*(\xi)$. Firstly, we may consider only those η which coincide with ξ on $N(p+1)$. It follows from Lemma (2.1) that $C(\eta)$ is locally bounded. In addition, it is invariant under the co-adjoint action of N on N^*. In particular, it is

constant along the $V^\perp$-orbit $\mathcal{O}_\xi$ of ξ. Therefore, to complete the proof that $C(\eta)$ is bounded on $N_V^*(\xi)$ for a given $\xi \in \boldsymbol{H}(V)$, it is sufficient to know that $C(\eta)$ is bounded for $\mathrm{dist}_W(\eta, \mathcal{O}(\xi))$ being large enough.

For simplicity of notation, let $\pi^\eta = \pi^{(W,\eta)}$. Let P and F be as above. Then, by (1.14),

$$\|f\| \leqslant \|\pi_F^\eta\|_{\mathrm{op}}(\|\pi_P f\| + \|f\|) \tag{3.9}$$

for $\eta \in N_V^*(\xi)$. Since $F \in L^1(N)$,

$$\|\pi_F^\eta\|_{\mathrm{op}} \to 0 \tag{3.10}$$

as $\mathrm{dist}_W(\eta, \mathcal{O}(\xi)) \to \infty$ (Lemma (2.7)); therefore, for every $\varepsilon > 0$,

$$\|f\| \leqslant \varepsilon\|\pi_F^\eta f\| \tag{3.11}$$

if $\mathrm{dist}_W(\eta, \mathcal{O}(\xi))$ is sufficiently large. Consequently, by (3.4),

$$\|f\| \leqslant C_0 \varepsilon(\|\pi_R^\eta f\| + \|f\|) \leqslant \|\pi_R^\eta f\| + \tfrac{1}{2}\|f\|$$

with $\varepsilon = 1/(2C_0)$, whence $\|f\| \leqslant \|\pi_R^\eta f\|$ for $\mathrm{dist}_W(\eta, \mathcal{O}(\xi))$ sufficiently large, which concludes the proof of (3.7).

In the case $p = m$, (3.7) reads

$$\|f\| \leqslant C(\lambda)\|\pi_R^\lambda f\|, \quad f \in C_c^\infty(\tilde{N}), \tag{3.12}$$

for $|\lambda| = 1$. But, by the same argument as above, $C(\lambda)$ is locally bounded, hence bounded, on the compact sphere $|\lambda| = 1$, which finally implies $\|f\| \leqslant C\|\pi_R^\lambda f\|$ for $|\lambda| = 1$ and $C = \sup_{|\lambda|=1} C(\lambda)$. This, combined with (3.3), yields

$$\|\pi_T^\lambda f\| \leqslant C\|\pi_R^\lambda f\|, \quad f \in C_c^\infty(\tilde{N}), \tag{3.13}$$

for $|\lambda| = 1$, and, by homogeneity, for *all* $\lambda \in N(m)^*$.

To end the proof of Theorem (3.1), it is sufficient to integrate both sides of (3.13) over $N(m)^*$ with f replaced by f^λ, f being now a C_c^∞-function on N. ■

(3.14) Remark. The restriction $r < 1$ in Theorem (3.1) can be removed if we assume that dR has a density on $N\backslash\{0\}$ which is, for example, a Lipschitz function. As a matter of fact, if $r \leqslant 2$, it is sufficient to assume that R is symmetric. This, however, will not be discussed in this paper.

(3.15) Remark. If R is smooth away from the origin, then the kernel of $\bar{\pi}_R$ is contained in $C^\infty(\pi)$ so that in the Rockland condition (3.2) the domain of $\bar{\pi}_R$ can be replaced by the space $C^\infty(\pi)$ independent of R. (Cf. Lemma (4.9) below.)

4. The inversion theorem. Let φ be a C_c^∞-function on N equal to 1 in a neighbourhood of the origin. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Let K be a Lipschitz kernel of order 0, and let

$$\langle f, K_\varepsilon \rangle = \langle f\varphi_\varepsilon, K \rangle, \quad f \in C_c^\infty(N). \tag{4.1}$$

It has been proved by Goodman [Go] that if π is a unitary representation of N, then there exists a dense subspace H_0 of H_π contained in $C^\infty(\pi)$ such that

$$\lim_{\varepsilon\to 0}\pi_{K_\varepsilon}f = \pi_K f \tag{4.2}$$

exists for $f\in H_0$ and defines a bounded operator $\bar{\pi}_K$ on H_π. The choice of the cut-off function φ is immaterial.

We recall that π^1 is the right-regular representation of N on $L^2(N)$.

(4.3) THEOREM. *Let K be a Lipschitz kernel of order 0 on N. Then the operator $\bar{\pi}^1_K$ is left-invertible on $L^2(N)$ if and only if the operator $\bar{\pi}_K$ is injective for every nontrivial irreducible unitary representation π of N.*

Proof. First, we shall prove the "if" part.

Let $0<s<1$ be the exponent in the Lipschitz condition for K on $N\backslash\{0\}$. Let $0<r<s$ and let P be the smooth kernel of order r as defined by (1.12).

(4.4) LEMMA. *There exists a Lipschitz kernel T of order r such that*

$$\bar{\pi}_K\bar{\pi}_P = \bar{\pi}_T \tag{4.5}$$

for every unitary representation π of N.

Proof of the lemma. As $K\in\mathscr{E}^*(N)+L^2(N)$, and $P\in\mathscr{E}^*(N)+(L^1(N)\cap C^\infty(N))$, the convolution $T=K*P$ is well defined. It is also obvious that $\langle f\circ\delta_t, T\rangle = t^r\langle f, T\rangle$, and if π is a unitary representation of N, (4.5) holds true. It remains to show that T is Lipschitz away from the origin.

To this end, let $\varepsilon>0$, and let K_ε, P_ε be defined as in (4.1). Then

$$T = K_\varepsilon * P_\varepsilon + K_\varepsilon *(P-P_\varepsilon)+(K-K_\varepsilon)*P.$$

As the support of $K_\varepsilon * P_\varepsilon$ can be as small as we wish, it is sufficient to show that the remaining two terms are Lipschitz. But $K_\varepsilon*(P-P_\varepsilon)\in C^\infty$; therefore, we only have to concern ourselves with $(K-K_\varepsilon)*P = k*P$, where k is a Lipschitz function with Lipschitz exponent s.

Indeed, for any $x, y\in N$,

$$\begin{aligned}
&|k*P(x)-k*P(y)|\\
&\leqslant \int_{|z|\leqslant|y^{-1}x|}\frac{|k(x)-k(xz^{-1})|}{|z|^{Q+r}}dz+\int_{|z|\leqslant|y^{-1}x|}\frac{|k(y)-k(xz^{-1})|}{|z|^{Q+r}}dz\\
&\quad+\int_{|z|\geqslant|y^{-1}x|}\frac{|k(x)-k(y)|}{|z|^{Q+r}}dz+\int_{|z|\geqslant|y^{-1}x|}\frac{|k(yz^{-1})-k(xz^{-1})|}{|z|^{Q+r}}dz\\
&\leqslant 2C\Big(\int_{|z|\leqslant|y^{-1}x|}|z|^{-Q-r+s}dz+|y^{-1}x|^s\int_{|z|\geqslant|y^{-1}x|}|z|^{-Q-r}dz\Big)= C'|y^{-1}x|^{s-r},
\end{aligned}$$

which shows that $k*P$ is Lipschitz with exponent $s'=s-r$. ■

(4.6) LEMMA. *Let T be as above. For every nontrivial irreducible unitary representation π of N, $\bar{\pi}_T$ is injective.*

Proof. Let v be a vector in the domain of $\bar{\pi}_T$ such that $\bar{\pi}_T v=0$. Since $\bar{\pi}_K$ is bounded and injective and $\bar{\pi}_K\bar{\pi}_P=\bar{\pi}_T$ (see (4.5) above), and $\bar{\pi}_P$ is closed, v belongs to the domain of $\bar{\pi}_P$, too. Therefore, by the assumption on K again, $\bar{\pi}_P v=0$. As π is monomial, it follows by Lemma (1.15) that $v=0$. ■

We return to the proof of Theorem (4.3). By Theorem (3.1),

$$\|\pi^1_P f\|\leqslant C\|\pi^1_T f\| = C\|\bar{\pi}^1_K\pi^1_P f\|$$

for $f\in C_c^\infty(N)$. Consequently, by Lemma (1.15), $\|f\|\leqslant C\|\bar{\pi}^1_K f\|$ for $f\in L^2(N)$, which shows that $\bar{\pi}^1_K$ is left-invertible. Thus, the first part of the proof is completed.

The "only if" part will follow immediately as soon as we prove the following lemma:

(4.7) LEMMA. *Let K be a Lipschitz kernel of order 0 on N such that $\bar{\pi}^1_K$ is left-invertible on $L^2(N)$. Then, for every nontrivial monomial unitary representation π of N, $\bar{\pi}_K$ is left-invertible on H_π.*

Proof. Let $\pi=\pi^{(V,\xi)}$, where V is a subalgebra of N and $\xi\in\boldsymbol{H}(V)$. Let $1\leqslant p\leqslant m+1$ be the smallest integer such that $V\subseteq N(p)$.

We shall proceed by induction on p. If $p=m+1$, then $V=\{0\}$, and $\pi^{(V,\xi)}_K=\pi^1_K$ so that our assertion follows by hypothesis. Assume now that it holds true for some $2\leqslant p+1\leqslant m+1$ and let $W=V\cap N(p+1)$. Since $\bar{\pi}^{(W,\xi)}_K$ is left-invertible, we have

$$\|\pi^{(W,\xi)}_P f\|\leqslant C\|\pi^{(W,\xi)}_T f\|$$

for $f\in C_c^\infty(S_W)$, where T is as in Lemma (4.4), and S_W is a linear complement to W in N. It follows now, by Lemma (2.10) and Corollary (2.11) (with the roles of V and W interchanged) that

$$\|\pi^{(V,\eta)}_P f\|\leqslant C\|\pi^{(V,\eta)}_T f^\eta\| \tag{4.8}$$

for $f\in C_c^\infty(S_W)$ and $\eta\in N^*_V(\xi)|W$. In particular, (4.8) holds for $\eta=\xi|W$.

But, for every $g\in C_c^\infty(S_V)$, where $V\oplus S_V=N$, there exists $f\in C_c^\infty(S_W)$ such that $f^\xi=g$. Therefore, by Lemma (4.4) again,

$$\|\pi^{(V,\xi)}_P g\|\leqslant C\|\pi^{(V,\xi)}_K\pi^{(V,\xi)}_P g\|$$

for $g\in C_c^\infty(S_V)$. Since, by Lemma (1.15), the functions of the form $\pi^{(V,\xi)}_P g$, where g ranges over $C_c^\infty(S_V)$, are dense in $L^2(S_V)$, we conclude that $\bar{\pi}^{(V,\xi)}_K$ is left-invertible on $L^2(S_V)$. ■

(4.9) COROLLARY. *Let K be a smooth kernel of order 0 on N. The operator $\bar{\pi}^1_K$ is left-invertible on $L^2(N)$ if and only if π_K is injective on $C^\infty(\pi)$ for every nontrivial irreducible unitary representation π of N.*

Proof. In virtue of Theorem (4.3), it is sufficient to show that if π is an irreducible unitary representation of N and $f \in H_\pi$, then $\bar{\pi}_K f = 0$ implies $f \in C^\infty(\pi)$.

Let $\pi = \pi^{(V,\xi)}$ and $H_\pi = L^2(S)$, where V is a subalgebra of N and S is a complementary subspace to V in N. Then, by Kirillov [K], $C^\infty(\pi) = \mathscr{S}(S)$ and for every differential operator ∂ on S with polynomial coefficients, there exists a distribution $D \in \mathscr{E}^*(N)$ supported at the origin such that $\pi_D = \partial$. Let P be a smooth accretive kernel of order $r > 0$ on N (cf. (1.12)). By Theorem (2.2) of [G2], there exist a constant $C > 0$ and a positive integer k such that

$$\|\pi_D^1 F\| \leqslant C(\|(\pi_P^1)^k \pi_K^1 F\| + \|F\|) \tag{4.10}$$

for $F \in C_c^\infty(N)$. (The assumption in [G2], p. 58, that $\bar{\pi}_K^1$ has a two-sided inverse is superfluous. It is sufficient to assume that it is left-invertible, as we do here.) Now, by the "decomposition" of the estimate (4.10), as was explained in the course of the proofs of (3.4) and Lemma (4.7), we get

$$\|\partial f\| \leqslant C(\|(\pi_P)^k \pi_K f\| + \|f\|) \tag{4.11}$$

for $f \in C_c^\infty(S)$. As $C_c^\infty(S)$ is dense in $C^\infty(\pi)$ (cf. [G3], Lemma (2.9)), (4.11) holds for all f in $C^\infty(\pi)$, which immediately implies that the kernel of $\bar{\pi}_K$ is contained in $C^\infty(\pi)$. ∎

5. Applications. We denote by $H^1(N)$ the Hardy space on N, as defined by Folland and Stein [FS], p. 75. We say that smooth kernels $K_1, \ldots, K_m$ of order 0 are *generalized Riesz transforms* if they characterize $H^1(N)$ in the sense described in the introduction.

(5.1) PROPOSITION. *Let $K_1, \ldots, K_m$ be smooth real kernels of order* 0 *on N. If for every nontrivial irreducible unitary representation π of N on a Hilbert space H_π and every nonzero vector v in H_π, the vectors $\pi_{K_j} v$ are not collinear over* $\mathbf{R}$, *then the kernels K_j are generalized Riesz transforms.*

Proof. This is a consequence of Theorem A of Christ and Geller [CG], p. 548, and our Corollary (4.9), as explained in [CG], Section 6, and Moukaddem [Mo], Section 6. ∎

Let $\{e_j\}_1^n$ be a homogeneous linear basis of the Lie algebra N, as defined by (0.1). Let

$$\langle f, D_j \rangle = d/dt|_{t=0} f(te_j), \quad f \in C_c^\infty(N).$$

Let P be a smooth symmetric accretive kernel of order 1 on N (cf., e.g., (1.12)). For every $d > -Q$, the fractional power P^d, as defined in [G2], (3.9) and (3.10), is a smooth kernel of order d.

Let π be a unitary representation of N. Recall from Section 1 that π_P is essentially selfadjoint on $C^\infty(\pi)$ which is its invariant domain. Moreover, π_P is positive and, by Duflo [D], $\bar{\pi}_{P^d} = \bar{\pi}_P^d$ for every $d > 0$. If π is irreducible, then π_P is injective on $C^\infty(\pi)$ (see Lemma (1.15)).

In [G2], it was claimed that the kernels $K_0, K_1, \ldots, K_n$, where $K_0 = \delta$ and $K_j = D_j * P^{-d_j}$ for $j = 1, \ldots, n$, are generalized Riesz transforms ([G2], Theorem (3.13)). As a matter of fact, the argument given there is defective since the kernels K_j are not odd. We take this opportunity to remark that the proof presented in [G2] can be corrected if the kernels K_j, $j \geqslant 1$, are replaced by

$$R_j = P^{-d_j/2} * D_j * P^{-d_j/2}.$$

Another possibility is to take

$$S_j = P^{-d_j} * D_j + D_j * P^{-d_j}.$$

Obviously, the kernels thus defined are odd.

Now we are going to present another, essentially different, construction of generalized Riesz transforms. In this construction, as distinct from those above, only derivatives in generating directions play a role. In addition, it is no longer required that the kernels under consideration are odd.

(5.2) PROPOSITION. *Let $\{e_j\}_1^k$ be homogeneous generators of the Lie algebra N corresponding to the eigenvalues $\{d_j\}_1^k$. Then the kernels $L_0, L_1, \ldots, L_k$, where $L_0 = \delta$ and*

$$L_j = P^{-d_j} * D_j, \quad 1 \leqslant j \leqslant k,$$

are generalized Riesz transforms.

Proof. Let π be a nontrivial irreducible unitary representation of N. Suppose that $\pi_{L_j} v = 0$ for a vector $v \in C^\infty(\pi)$ and all $1 \leqslant j \leqslant k$. It is not hard to see that $C^\infty(\pi)$ is invariant under π_{L_j} and

$$\pi_{P^{d_j}} \pi_{L_j} = \pi_{D_j}$$

on $C^\infty(\pi)$ (cf. [G2], (1.9) and Lemma (3.15)). Therefore, $\pi_{D_j} v = 0$ for all $1 \leqslant j \leqslant k$ and since e_j's are generators of N, v is a fixed vector for π. As π is irreducible v must be zero.

Therefore, if v is a nonzero vector in $C^\infty(\pi)$, there exists $1 \leqslant s \leqslant k$ such that $\pi_{L_s} v = 0$. Thus, by Proposition (5.1), it is sufficient to show that the vectors v and $\pi_{L_s} v$ are linearly independent over $\mathbf{R}$.

Suppose they are not. Then there exists a real number $t \neq 0$ such that $\pi_{L_s} v = tv$, whence (cf. above) $\pi_{D_s} v = t\pi_{P^{d_s}} v$ and, consequently,

$$\langle \pi_{D_s} v, v \rangle = t\|\pi_{P^{d_s/2}} v\|^2. \tag{5.3}$$

Now, the right-hand side of (5.3) is real, while the real part of the left-hand side is zero so either $t = 0$ or $\pi_P v = 0$. Both conclusions are false, which concludes the proof. ∎

(5.4) EXAMPLE. It follows immediately from the proof of Proposition (5.2) and Corollary (4.9) that for every real k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$, the kernel

$$L_\lambda = \sum_{j=1}^{k} (L_j - \lambda_j \delta)^* * (L_j - \lambda_j \delta)$$

gives rise to a left-invertible singular integral operator $\pi^1_{L_\lambda}$ on $L^2(N)$.

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The Mackey completions of some interpolation F-spaces

by

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Abstract. We characterize the Mackey completions of locally concave F-spaces which are interpolation spaces with respect to a special couple of Banach lattices. The results are applied to the interpolation spaces generated by the K method of interpolation.

1. Introduction. An *F-quasinorm* on a vector space X is a nonnegative function $\|\cdot\|$ on X which vanishes only at zero and has the following properties for every $x, y \in X$ and scalar t with $|t| \leqslant 1$:

(i) $\|tx\| \leqslant \|x\|$,
(ii) $\|x+y\| \leqslant C(\|x\|+\|y\|)$ for some $C > 0$,
(iii) $\|tx\| \to 0$ as $t \to 0$.

An F-quasinorm for which $C = 1$ is called an *F-norm*, and an F-norm which is *p-homogeneous* for some $0 < p \leqslant 1$,

(iv) $\|\lambda x\| = |\lambda|^p \|x\|$ whenever λ is scalar,

is called a *p-norm* (a *norm* if $p = 1$). An F-quasinorm which is 1-homogeneous is called a *quasinorm*.

A linear space equipped with a Hausdorff vector topology determined by an F-norm (p-norm, quasinorm) is called an *F^*-space* (*p-normed space*, *quasinormed space*, respectively). A topologically complete p-normed space (quasinormed space) X is called a *p-Banach space* (*quasi-Banach space*).

Two topological vector spaces (tvs) X and Y are considered as equal ($X = Y$) whenever $X = Y$ as sets and their topologies are equivalent. If τ is a topology on X and Z is a subspace of X, then $\tau_{|Z}$ is the topology induced on Z by τ.

A pair $\bar{A} = (A_0, A_1)$ of normed (Banach) spaces is called a *normed* (*Banach*) *couple* if A_0 and A_1 are both algebraically and topologically imbedded in some Hausdorff tvs.

For a normed (Banach) couple $\bar{A} = (A_0, A_1)$ we can form the *sum* $\Sigma(\bar{A}) = A_0 + A_1$ and the *intersection* $\Delta(\bar{A}) = A_0 \cap A_1$. They are both normed (Banach) spaces, in the natural norms $\|a\|_\Sigma = K(1, a; \bar{A})$ and $\|a\|_\Delta = J_1(1, a; \bar{A})$,

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