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Weak vs. norm compactness in L_1 : the Bocce criterion

by

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Abstract. We present a new simple proof that if a relatively weakly compact subset of L_1 satisfies the Bocce criterion (an oscillation condition), then it is relatively norm compact. The converse of this fact is easy to verify. A direct consequence is that, for a bounded linear operator T from L_1 into a Banach space \mathfrak{X} , T is Dunford–Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion.

A relatively weakly compact subset of L_1 is relatively norm compact if and only if it satisfies the Bocce criterion (an oscillation condition) [G1]. We shall present a new simple proof that if a relatively weakly compact subset of L_1 satisfies the Bocce criterion, then it is relatively norm compact. The converse is easy to verify.

Recall that a Banach space \mathfrak{X} has the *complete continuity property* (CCP) if each bounded linear operator from L_1 into \mathfrak{X} is *Dunford–Pettis* (i.e. maps weakly convergent sequences to norm convergent ones). The CCP is a weakening of the Radon–Nikodým property and of strong regularity. Since a bounded linear operator T from L_1 into \mathfrak{X} is Dunford–Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 is relatively norm compact, the above fact gives that T is Dunford–Pettis if and only if $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion. This oscillation characterization of Dunford–Pettis operators leads to dentability and tree characterizations of the CCP [G2]. Namely, \mathfrak{X} has the CCP if and only if all bounded subsets of \mathfrak{X} are weak-norm-one dentable. Also, \mathfrak{X} has the CCP if and only if no bounded separated δ -trees grow in \mathfrak{X} , or equivalently, no bounded δ -Rademacher trees grow in \mathfrak{X} .

Throughout this note, \mathfrak{X} denotes an arbitrary Banach space. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All unexplained notation and terminology is as in [DU].

[G1] introduces the following definitions.

DEFINITIONS. For f in L_1 and A in Σ^+ , the *Bocce oscillation of f on A* is given by

$$\text{Bocce-osc } f|_A \equiv \frac{1}{\mu(A)} \int_A \left| f - \frac{1}{\mu(A)} \int_A f d\mu \right| d\mu.$$

A subset K of L_1 satisfies the *Bocce criterion* if for each $\varepsilon > 0$ and B in Σ^+ there is a finite collection \mathcal{F} of subsets of B each with positive measure such that for each f in K there is an A in \mathcal{F} satisfying $\text{Bocce-osc } f|_A < \varepsilon$.

This note's main purpose is to present a new proof to the theorem below. The author is grateful to Michel Talagrand for his helpful discussions concerning this theorem and proof.

THEOREM. *If a relatively weakly compact subset of L_1 satisfies the Bocce criterion, then it is relatively L_1 -norm compact.*

We need the following lemma which we shall verify after the proof of the Theorem.

LEMMA. *If a subset of L_1 satisfies the Bocce criterion, then so does its translate by any L_1 -function.*

Proof of Theorem. Assume that the relatively weakly compact subset K of L_1 is not relatively norm compact. We shall show that K does not satisfy the Bocce criterion.

Since K is not relatively norm compact but is relatively weakly compact, there is a sequence $\{f_n\}$ in a translate \tilde{K} of K satisfying

- (1) $\{f_n\}$ has no L_1 -convergent subsequence,
- (2) $\{f_n\}$ converges weakly in L_1 to 0,
- (3) $\{|f_n|\}$ converges weakly in L_1 , say to f ,
- (4) $\int f d\mu \geq 4\varepsilon$ for some $\varepsilon > 0$.

Set $B = [f \geq 3\varepsilon]$. Condition (4) guarantees that $B \in \Sigma^+$.

Let \mathcal{F} be a finite collection of subsets of B , each with positive measure. Choose N such that for each $A \in \mathcal{F}$

- (5) $|\int_A f_N d\mu| < \varepsilon\mu(A)$ (possible by (2)),
- (6) $|\int_A f d\mu - \int_A |f_N| d\mu| < \varepsilon\mu(A)$ (possible by (3)).

Then for each $A \in \mathcal{F}$ we have

$$\begin{aligned} \text{Bocce-osc } f_N|_A &\equiv \frac{1}{\mu(A)} \int_A \left| f_N - \frac{1}{\mu(A)} \int_A f_N d\mu \right| d\mu \geq \frac{\int |f_N| d\mu}{\mu(A)} - \frac{|\int f_N d\mu|}{\mu(A)} \\ &\geq \frac{\int f d\mu - \varepsilon\mu(A)}{\mu(A)} - \frac{\varepsilon\mu(A)}{\mu(A)} \geq \frac{3\varepsilon\mu(A)}{\mu(A)} - \varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

Thus \tilde{K} , and hence also K , does not satisfy the Bocce criterion. ■

Proof of Lemma. Let the subset K of L_1 satisfy the Bocce criterion and $f \in L_1$. We need to show that the set $K+f \equiv \{g+f: g \in K\}$ satisfies the Bocce

criterion. Towards this end, fix $\varepsilon > 0$ and $B \in \Sigma^+$. Find $B_0 \subset B$ with $B_0 \in \Sigma^+$ such that f is bounded on B_0 .

Approximate $f\chi_{B_0}$ in L_∞ -norm within $\varepsilon/4$ by a simple function \tilde{f} . Find $C \subset B_0$ with $C \in \Sigma^+$ such that \tilde{f} is constant on C . Since K satisfies the Bocce criterion, we can find a finite collection \mathcal{F} of subsets corresponding to $\varepsilon/2$ and C .

Fix $g+f \in K+f$. Find $A \in \mathcal{F}$ such that $\text{Bocce-osc } g|_A < \varepsilon/2$. Note that since \tilde{f} is constant on A , $\text{Bocce-osc } g|_A = \text{Bocce-osc } (g+\tilde{f})|_A$. Now,

$$\begin{aligned} \text{Bocce-osc } (g+f)|_A &\leq \text{Bocce-osc } (g+\tilde{f})|_A + \text{Bocce-osc } (\tilde{f}-f)|_A \\ &\leq \text{Bocce-osc } g|_A + 2 \|(\tilde{f}-f)\chi_A\|_{L_\infty} < \varepsilon. \end{aligned}$$

Thus $K+f$ satisfies the Bocce criterion. ■

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