[DM$_1$] W. Dahmen and C. A. Micchelli, *Translates of multivariate splines*, Linear Algebra Appl. 52/3 (1983), 217–234.

[DM$_2$] —, —, *On the approximation order from certain multivariate spline spaces*, J. Austral. Math. Soc. Ser. B 26 (1984), 233–246.

[DR] N. Dyn and A. Ron, *Local approximation by certain spaces of multivariate exponential-polynomials, approximation order of exponential box splines and related interpolation problems*, Trans. Amer. Math. Soc. 319 (1990), 381–404.

[J] R.-Q. Jia, *A counterexample to a result concerning controlled approximation*, Proc. Amer. Math. Soc. 97 (1986), 647–654.

[R$_1$] A. Ron, *Relations between the support of a compactly supported function and the exponential-polynomials spanned by its integer translates*, Constr. Approx. 6 (1990), 139–155.

[R$_2$] —, *Factorization theorems for univariate splines on regular grids*, Israel J. Math. 70 (1990), 48–68.

[SF] G. Strang and G. Fix, *A Fourier analysis of the finite element variational method*, in: Constructive Aspects of Functional Analysis, G. Geymonat (ed.), C.I.M.E. II Ciclo 1971, Ed. Cremonese, Roma 1973, 793–840.

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Received November 23, 1989 (2624)
Revised version February 26, 1990

STUDIA MATHEMATICA 98 (1) (1991)

A note on Olech's Lemma

by

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Abstract. A variant of Olech's Lemma in multifunctions integration is presented; it covers conditions for weak implies strong L_1-convergence.

We provide a version of the Olech Lemma concerning convergence to extreme points in set-valued integration. Terminology and notations are recalled after the result is stated. We then compare our observation with the original Olech Lemma. After the proof is presented, we show how the new version covers, and somewhat generalizes, some results in the compensated compactness theory, of how weak convergence in L_1 may imply strong convergence.

The main result is as follows.

PROPOSITION. *Let $(\Omega, \mathscr{A}, \nu)$ be a measure space with ν an atomless, positive σ-additive measure. Let $F(\cdot)$ be a measurable $\mathbf{R}^n$ set-valued map with closed values. Let e be an extreme point of $\int F(\omega)\,d\nu$. If $f_k(\cdot)$, $k = 1, 2, \ldots$, is a uniformly integrable sequence of selections of $F(\cdot)$, and $\int f_k(\omega)\,d\nu$ converges to e, then the $f_k(\cdot)$ form a Cauchy sequence in $L_1(\Omega, \mathbf{R}^n)$. In particular, there exists a unique selection $e(\cdot)$ of $F(\cdot)$ such that $\int e(\omega)\,d\nu = e$, and the $f_k(\cdot)$ converge to $e(\cdot)$ in the $L_1(\Omega, \mathbf{R}^n)$ norm.*

The terminology we use is standard, a good source is Castaing and Valadier [4]. For completeness we recall that $\int F(\omega)\,d\nu$ is defined as the set $\{\int f(\omega)\,d\nu : f(\cdot)$ is integrable, and $f(\omega) \in F(\omega)$ for ν-almost every $\omega\}$. The set $\int F(\omega)\,d\nu$ is convex, since ν is atomless (see e.g. [4, Section IV.4]). A point e is an extreme point of the convex set C if $e = \frac{1}{2}a + \frac{1}{2}b$ with a and b in C implies $e = a = b$. An extreme point of C may not be an extreme point of cl$C$, the closure of C, and this may be the case in the proposition, as $\int F(\omega)\,d\nu$ may not be a closed set.

The Olech Lemma is an extremely useful tool in the theory of existence and robustness of solutions to optimal control and variational problems; it was verified in Olech [5], see also Olech [6]. In the original version of the lemma,

1985 *Mathematics Subject Classification*: Primary 28B30.

* Incumbent of The Hettie H. Heinemann Professorial Chair in Mathematics; this work was supported by a grant from the Israel Academy of Science and Humanities.

the sequence $f_k(\cdot)$ is not required to be uniformly integrable; in turn e is required to be an extreme point of the closure of $\int F(\omega)\,dv$. It is easy to see that in our Proposition the uniform integrability cannot be dropped. Our proof is based on the verification of the Olech Lemma contained in a proof of the related Theorem in [1, Appendix], combined with the Fatou Lemma (or rather, the Fatou–Lebesgue Lemma) in [2]; the latter was given a different proof by Balder [3], see also Olech [7]. The proof goes as follows.

Proof of the Proposition. We start as in [1, p. 413]. For convenience we assume $e = 0$. If the $\mathbf{R}^n$-valued functions $f_k(\cdot) = (f_k^1(\cdot), \ldots, f_k^n(\cdot))$ do not converge in L_1, then for one of the coordinates, say 1, the sequence $f_k^1(\cdot)$ is not Cauchy in L_1. Therefore one can find an $\varepsilon > 0$ and two increasing sequences of integers n_k and m_k such that

$$\int |f_{n_k}^1(\omega) - f_{m_k}^1(\omega)|\,dv \geqslant \varepsilon$$

for all k. Define

$$h_k(\omega) = f_{n_k}(\omega) \quad \text{if } f_{n_k}^1(\omega) \geqslant f_{m_k}^1(\omega), \qquad g_k(\omega) = f_{n_k}(\omega) \quad \text{if } f_{n_k}^1(\omega) < f_{m_k}^1(\omega),$$
$$= f_{m_k}(\omega) \quad \text{otherwise}, \qquad\qquad = f_{m_k}(\omega) \quad \text{otherwise}.$$

Then $h_k(\cdot)$ and $g_k(\cdot)$ are sequences of measurable selections of $F(\cdot)$ and are clearly uniformly integrable. Therefore $\int h_k(\omega)\,dv$ and $\int g_k(\omega)\,dv$ are bounded sequences in $\mathbf{R}^n$, and, without loss of generality, we may assume that they converge, say to a and b respectively. By the three displayed formulas

$$\|a-b\| \geqslant \varepsilon$$

where $\|\cdot\|$ is the sup norm in $\mathbf{R}^n$. On the other hand,

$$a+b = 0$$

since $\int (h_k(\omega)+g_k(\omega))\,dv$ converges to $e = 0$. The uniform integrability of the selections $h_k(\cdot)$ and $g_k(\cdot)$ allows us to use Theorem A of [2], and deduce the existence of pointwise cluster points (hence selections of $F(\cdot)$), say $h(\cdot)$ and $g(\cdot)$, such that

$$\int h(\omega)\,dv = a \quad \text{and} \quad \int g(\omega)\,dv = b.$$

Hence a and b belong to $\int F(\omega)\,dv$, which together with $a+b = 0$ and $a \neq b$ contradicts the extremality of 0 in $\int F(\omega)\,dv$. This proves that $f_k(\cdot)$ converges in L_1. Its limit is a selection of F (since a subsequence converges pointwise, and $F(\omega)$ is closed), and the integral of the limit is equal to e. This verifies the existence of the promised selection $e(\cdot)$; it is unique since alternating between two distinct selections, $e_1(\cdot)$, $e_2(\cdot)$, would give rise to a sequence $f_k(\cdot)$ which contradicts the first statement of the result. This completes the proof.

In the context of variational problems, in particular for the compensated compactness methods Visintin [9] proved a result that under the conditions of our Proposition reads as follows:

Let $f_k(\cdot)$ be a sequence of selections of F which converges weakly in L_1 to $e(\cdot)$. If for all ω the point $e(\omega)$ is an extreme point of cl co $F(\omega)$, *the closure of the convex hull of $F(\omega)$, then $f_k(\cdot)$ converges in norm to $e(\cdot)$.*

This result was generalized by Rzeżuchowski [8] to the case where the extreme point $e(\omega)$ is replaced by an extremal face of $F(\omega)$. We show here how Visintin's result is contained in, and somewhat generalized by, our version of the Olech Lemma.

An extreme point e of a convex set C in $\mathbf{R}^n$ can be characterized as the lexicographic minimum of the vectors

$$\{(v_1\cdot c, \ldots, v_n\cdot c)\colon\ c \in C\}$$

where $v_1, \ldots, v_n$ is an orthonormal basis of $\mathbf{R}^n$ and $v\cdot c$ is the scalar product. We then set $e = C_{v_1,\ldots,v_n}$. In particular,

$$e(\omega) = (\mathrm{cl\,co}\, F(\omega))_{v_1(\omega),\ldots,v_n(\omega)}$$

and standard selection techniques (see Castaing and Valadier [4]) would yield $v_1(\cdot), \ldots, v_n(\cdot)$ measurable. The measurable change of coordinates $T(\omega)$ which maps $v_1(\omega)$ to $u_1 = (1, 0, \ldots, 0)$, $v_2(\omega)$ to $u_2 = (0, 1, \ldots, 0)$, etc. transforms the set-valued map $F(\omega)$ to the set-valued map $G(\omega) = \{T(\omega)x\colon\ x \in F(\omega)\}$ and then $T(\omega)e(\omega) = G(\omega)_{u_1,\ldots,u_n}$. Clearly $\int T(\omega)e(\omega)\,dv$ is an extreme point of $\int G(\omega)\,dv$, characterized by the orthonormal basis $u_1, \ldots, u_n$. It is also clear that the $T(\omega)f_k(\omega)$ are uniformly integrable, and $\int T(\omega)f_k(\omega)\,dv$ converges to $\int T(\omega)e(\omega)\,dv$. By the Proposition, $T(\omega)f_k(\omega)$ converges in the L_1 norm to $T(\omega)e(\omega)$; hence, by inversing the coordinates, $f_k(\cdot)$ converges in the L_1 norm to $e(\cdot)$, which is Visintin's result. What our result adds is that in case $e(\omega) = (\mathrm{cl\,co}\,F(\omega))_{v_1,\ldots,v_n}$ for $v_1, \ldots, v_n$ the same for all ω, there is no need to assume weak convergence of $f_k(\cdot)$ to $e(\cdot)$; uniform integrability of $f_k(\cdot)$ and convergence of $\int f_k(\omega)\,dv$ to $\int e(\omega)\,dv$ suffice.

Acknowledgement. The observation in this note was conceived during the authors' participation in the Mini-Semester on Differential Inclusions held under the kind hospitality of the Stefan Banach International Mathematical Center, Warsaw. We wish to thank Czesław Olech for very helpful conversations on the problem.

References

[1] Z. Artstein, *On a variational problem*, J. Math. Anal. Appl. 45 (1974), 404–415.
[2] —, *A note on Fatou's lemma in several dimensions*, J. Math. Econom. 6 (1979), 277–282.
[3] E. J. Balder, *A unifying note on Fatou's lemma in several dimensions*, Math. Oper. Res. 9 (1984), 267–275.
[4] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Math. 580, Springer, Berlin 1977.

[5] C. Olech, *Integrals of set-valued functions and linear optimal control problems*, in: Colloque sur la Théorie Mathématique du Contrôle Optimal, C.B.R.M., Vander Louvain 1970, 109–125.
[6] —, *Existence theory in optimal control*, in: Control Theory and Topics in Functional Analysis, Vol. I, International Atomic Energy Agency, 1976, 291–328.
[7] —, *On n-dimensional extensions of Fatou's lemma*, J. Appl. Math. Phys. (ZAMP) 38 (1987), 266–272.
[8] T. Rzeżuchowski, *Strong convergence of selections implied by weak*, Bull. Austral. Math. Soc. 39 (1989), 201–214.
[9] A. Visintin, *Strong convergence results related to strict convexity*, Comm. Partial Differential Equations 9 (1984), 439–466.

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Received January 4, 1990
Revised version February 20, 1990
(2631)

STUDIA MATHEMATICA 98 (1) (1991)

Weak vs. norm compactness in L_1: the Bocce criterion

by

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Abstract. We present a new simple proof that if a relatively weakly compact subset of L_1 satisfies the Bocce criterion (an oscillation condition), then it is relatively norm compact. The converse of this fact is easy to verify. A direct consequence is that, for a bounded linear operator T from L_1 into a Banach space $\mathfrak{X}$, T is Dunford–Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 satisfies the Bocce criterion.

A relatively weakly compact subset of L_1 is relatively norm compact if and only if it satisfies the Bocce criterion (an oscillation condition) [G1]. We shall present a new simple proof that if a relatively weakly compact subset of L_1 satisfies the Bocce criterion, then it is relatively norm compact. The converse is easy to verify.

Recall that a Banach space $\mathfrak{X}$ has the *complete continuity property* (CCP) if each bounded linear operator from L_1 into $\mathfrak{X}$ is *Dunford–Pettis* (i.e. maps weakly convergent sequences to norm convergent ones). The CCP is a weakening of the Radon–Nikodým property and of strong regularity. Since a bounded linear operator T from L_1 into $\mathfrak{X}$ is Dunford–Pettis if and only if the subset $T^*(B(\mathfrak{X}^*))$ of L_1 is relatively norm compact, the above fact gives that T is Dunford–Pettis if and only if $T^*(B(\mathfrak{X}^*))$ satisfies the Bocce criterion. This oscillation characterization of Dunford–Pettis operators leads to dentability and tree characterizations of the CCP [G2]. Namely, $\mathfrak{X}$ has the CCP if and only if all bounded subsets of $\mathfrak{X}$ are weak-norm-one dentable. Also, $\mathfrak{X}$ has the CCP if and only if no bounded separated δ-trees grow in $\mathfrak{X}$, or equivalently, no bounded δ-Rademacher trees grow in $\mathfrak{X}$.

Throughout this note, $\mathfrak{X}$ denotes an arbitrary Banach space. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All unexplained notation and terminology is as in [DU].

[G1] introduces the following definitions.

DEFINITIONS. For f in L_1 and A in Σ^+, the *Bocce oscillation of f on A* is given by

1980 *Mathematics Subject Classification*: 47B38, 46B20, 28B99.