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Boundary estimates for derivatives of harmonic functions

by

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Abstract. Boundary L^p estimates are obtained for maximal functions of gradients of harmonic functions in terms of L^p norms of derivatives with respect to a fixed transverse vector field. As an application, it is shown that, for functions of several complex variables, pluriharmonic conjugation is a continuous operation on H^p for $0 < p < \infty$.

1. Introduction. It is well known that if u is a harmonic function in the unit disk such that a nontangential maximal function of u is an L^p function on the unit circle, then u is the real part of a holomorphic function in the Hardy class H^p on the unit disk. For $1 < p < \infty$, this is equivalent to a classical result of M. Riesz [6], and for $0 < p \leqslant 1$ it is due to Burkholder, Gundy, and Silverstein [2]. Higher dimensional analogues of this result in the setting of a half-space may be found in [7] and [5]. In this paper, we obtain an analogue for domains in (real) Euclidean space of arbitrary dimension.

We begin with a reformulation of the conjugation result in the disk which will be meaningful in the higher dimensional setting. Let X denote the radial vector field in the disk defined by

$$X = x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

Then the above result in the disk is easily seen to be equivalent to the following: If u is a harmonic function in the disk such that a nontangential maximal function of Xu is an L^p function on the unit circle, then any nontangential maximal functions of $\partial u/\partial x$ and $\partial u/\partial y$ are also L^p functions on the unit circle. This formulation has a natural analogue for domains in $\mathbf{R}^N$ in which the radial vector field X is replaced by a smooth vector field which is everywhere transverse to the boundary. Indeed, this higher dimensional analogue is the principal result of this paper (Theorem (3.43)).

The half-space analogue of our main result is well known, and amounts to the boundedness of the Riesz transforms on L^p when $1 < p < \infty$, and on H^p

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when $0 < p \leqslant 1$. (See [5] and [7].) We shall draw heavily on ideas from [5] and [7], but in our setting the arguments are complicated by the fact that differentiation with respect to a transverse field does not preserve harmonicity.

Results analogous to ours with volume L^p norms have been obtained previously by Detraz [4] and by Boas and Straube [1]. The case considered here is more delicate than the volume case.

The paper is organized as follows. In Section 2, we collect some definitions and recall some well-known results concerning maximal functions and area integrals. In Section 3 we prove a series of lemmas, culminating in our main result (Theorem (3.43)). Finally, in Section 4 we give an application to pluriharmonic conjugation, extending a result of Stout [10].

Throughout the paper, we shall use the symbol C to denote various positive constants which are independent of the relevant parameters in the expression in which they occur.

2. Preliminaries. Throughout this paper, D will denote a fixed bounded domain in $\mathbf{R}^N$ with C^2 boundary, and X will denote a C^1 vector field in a neighborhood of ∂D which is everywhere transverse to ∂D. More precisely, D is assumed to be a bounded set in $\mathbf{R}^N$ of the form $D = \{x \in \mathbf{R}^N\colon \varrho(x) < 0\}$ where ϱ is a function on $\mathbf{R}^N$ with $d\varrho \neq 0$ on $\partial D = \{x \in \mathbf{R}^N\colon \varrho(x) = 0\}$. The vector field X is assumed to be of the form $X = \sum a_j \partial_j$ where each coefficient a_j is a C^1 function in a neighborhood of ∂D, and $X\varrho(x) \neq 0$ for every $x \in \partial D$. We shall denote by σ the Euclidean surface measure on ∂D, and we denote the norm in $L^p(\partial D; d\sigma)$ by $\|\ \|_p$. Also, for any function u on a set K, we let $\|u\|_K = \sup\{|u(x)|\colon x \in K\}$.

For $1 \leqslant j \leqslant N$, we shall denote partial differentiation with respect to the jth variable in $\mathbf{R}^N$ by ∂_j. Similarly, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, we let $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_N^{\alpha_N}$. For any nonnegative integer k and any C^k function u, we shall denote by $\mathscr{D}^k u$ the vector $[\partial^\alpha u]$, where α varies over all multi-indices of weight at most k.

We choose and fix a neighborhood U of ∂D and a C^1 diffeomorphism $\Phi\colon \overline{U} \to \partial D \times [-2, 2]$ with the following properties:

(i) $\Phi^{-1}(\partial D \times (0, 2]) = D \cap \overline{U}$ and $\Phi^{-1}(\partial D \times [-2, 0)) = \overline{U} \setminus \overline{D}$;

(ii) $\Phi(x) = (x, 0)$ for any $x \in \partial D$;

(iii) in each connected component of $\overline{U}$ the vector field $\Phi_* X$ is a constant multiple of $\partial/\partial t$, where t denotes the second coordinate of a point in $\partial D \times [-2, 2]$.

For any $x = \partial D$ and any $a > 0$, we denote by $\tilde{\Gamma}_a(x)$ the truncated cone in $\partial D \times (0, 1]$ defined by $\tilde{\Gamma}_a(x) = \{(y, t) \in \partial D \times (0, 1]\colon |x-y| < at\}$, and we denote the preimage of $\tilde{\Gamma}_a(x)$ under Φ by $\Gamma_a(x)$. For any function u on $D \cap \overline{U}$, we shall denote the function $u \circ \Phi^{-1}$ on $\partial D \times (0, 2]$ by $\tilde{u}$, and we shall denote the product measure $d\sigma \times dt$ on $\partial D \times (0, 2]$ by $d\tilde{V}$. Finally, we let $U_0 = \Phi^{-1}[\partial D \times (-1, 1)]$, and $K = D \setminus U_0$.

It will be convenient to have a defining function for D which is harmonic near ∂D. We choose a point $x_0 \in \operatorname{int} K$, and denote by δ the Green function for D with singularity x_0. Thus, δ is harmonic in $D \setminus \{x_0\}$ and $\tilde{\delta}(x, t)$ is comparable with t on $\partial D \times (0, 1]$.

For any function u on $D \cap U_0$ and any $a > 0$, we define the *nontangential maximal function* of u by

$$N_a u(x) = \sup\{|u(y)|\colon y \in \Gamma_a(x)\}$$

whenever $x \in \partial D$. Thus, $N_a u$ is a lower semicontinuous function on ∂D. We define the *radial maximal function* by

$$N_0 u(x) = \sup\{|\tilde{u}(x, t)|\colon 0 < t \leqslant 1\}$$

for $x \in \partial D$. If u is continuously differentiable on $U \cap D$, then we define the *Littlewood–Paley g-function* by

$$gu(x) = \Big[\int_0^1 |\tilde{u}(x, t)|^2\, t\, dt\Big]^{1/2}.$$

For $q > 0$, we denote the *q-area integral* by

$$S_a^{(q)} u(x) = \Big[\int_{\tilde{\Gamma}_a(x)} |\tilde{u}(y, t)|^2\, t^{q-N}\, d\sigma(y)\, dt\Big]^{1/2}$$

where, once again, $x \in \partial D$. In all of the above notations, we shall often drop the subscript a when the aperture is 1. We shall also denote the classical area integral $S_a^{(2)} u$ by the simpler notation $S_a u$. We will use the above operators for both scalar and vector-valued functions. When u is vector-valued, $|u|$ is to be interpreted as the Euclidean norm of u.

We will need to compare $L^p(\partial D)$ norms of each of the above quantities. These comparisons are facilitated by the fact that finiteness of $\|N_a u\|_p$ and $\|S_a u\|_p$ is independent of the aperture a.

(2.1) Lemma. *Let $0 < p < \infty$, and let $a, b > 0$. There are constants $C = C(a, b, p)$ and $C' = C'(a, b, p)$ such that*

(i) *$\|N_a u\|_p \leqslant C\, \|N_b u\|_p$ for every function u on D;*

(ii) *$\|S_a^{(q)} u\|_p \leqslant C'\, \|S_b^{(q)} u\|_p$ for every measurable function u on D.*

For functions in half-spaces, parts (i) and (ii) of the above lemma appear in [5, p. 166] and [3, p. 309] respectively. The proof in the present context is identical to those appearing in [5] and [3] and will be omitted.

We are now in a position to define the *harmonic Hardy spaces* on the domain D. For $0 < p \leqslant \infty$, we define $H^p = H^p(D)$ to consist of all harmonic functions on D such that $N_a u \in L^p(\partial D)$ for some (and hence every) $a > 0$. With the "norm" defined by

$$\|u\|_{H^p} = \|N_1 u\|_p,$$

H^p is a Banach space when $1 \leqslant p \leqslant \infty$, and a quasi-Banach space when $0 < p < 1$. Of course, in view of Lemma (2.1), the choice of the aperture 1 in the definition of the H^p (quasi-)norm is only a convenience, and any other aperture will give rise to an equivalent (quasi-)norm. In the case $1 < p \leqslant \infty$, H^p is the space of Poisson integrals of functions in $L^p(\partial D)$ (see Chapter 1 of [9] and the references given there). It should be noted before proceeding that, as a consequence of Lemma (2.1), the Hardy spaces $H^p(D)$ are independent of the choice of the transverse vector field X and the diffeomorphism Φ, and that any other choice would lead to an equivalent (quasi-)norm on $H^p(D)$.

The principal interest in the Lusin and Littlewood–Paley functions is that they can be used to characterize $H^p(D)$.

(2.2) Lemma. *Let D be any bounded domain in $\mathbf{R}^N$ with C^2 boundary. Then for u harmonic on D, the following conditions are equivalent:*

(i) $u \in H^p(D)$;
(ii) $g(\mathscr{D}u) \in L^p(\partial D)$;
(iii) $S(\mathscr{D}u) \in L^p(\partial D)$.

Moreover, the (quasi-)norms on $H^p(D)$ defined by $\|Nu\|_p$, $\|g(\mathscr{D}u)\|_p$, and $\|S(\mathscr{D}u)\|_p$ are equivalent.

With D replaced by a half-space, Lemma (2.2) may be found in [7] in the case $1 < p < \infty$ and in [5] in the case $0 < p < 2$. The above formulation does not seem to appear explicitly in the literature, but the techniques of [7] and [5] may be easily adapted to the present situation, so we will omit the proof.

3. Estimates for derivatives of harmonic functions. In this section, we prove our main result, Theorem (3.43), on H^p norms of gradients of harmonic functions. We split the proof into a series of lemmas. The first result provides a pointwise estimate for area integrals of gradients in terms of area integrals of transverse derivatives.

(3.1) Lemma. *Let k be a nonnegative integer, and suppose that the partial derivatives of the coefficients of X up to order $k-1$ satisfy a Lipschitz condition of order $\eta > 0$. For any $0 < a < b$, and any $q > 0$, there is a constant C, depending on a, b, q and k, such that for any harmonic function u on D, and any $x \in \partial D$,*

$$S_a^{(q)}(\mathscr{D}^k u)(x) \leqslant C\,[S_b^{(q)}(X^k u)(x) + \|\mathscr{D}^k u\|_K].$$

Proof. The proof proceeds along the same lines as the proof of the analogous result for differentiation in the half-line direction in a half-space (see [8, pp. 213–216]), but is complicated by the fact that differentiation with respect to X does not preserve harmonicity.

We first prove the inequality

(3.2) $$S_a^{(q)}(X^i \nabla u)(x) \leqslant C_{a,b,q}\,[S_b^{(q)}(X^{i+1} u)(x) + S_b^{(q+2\eta)}(\mathscr{D}^{i+1} u)(x) + \|\mathscr{D}^{i+1} u\|_K]$$

for any nonnegative integer $i < k$ and any $0 < a < b$. The lemma will follow from (3.2) by a simple inductive argument.

For $z = \Phi^{-1}(x,\tau)$ and $w = \Phi^{-1}(y,t)$ in U with $|z-w| < \varepsilon\tau$, we have, since Φ is a diffeomorphism,

(3.3) $$|x-y| + |t-\tau| < c_1\varepsilon\tau,$$

where c_1 is a constant depending only on Φ and thus

(3.4) $$(1-c_1\varepsilon)\tau < t < (1+c_1\varepsilon)\tau.$$

It follows that if $(x,\tau) \in \tilde{\Gamma}_a(x_0)$ for some $x_0 \in \partial D$, and if (y,t) is as above with $0 < \varepsilon < c_1^{-1}$, then

$$|y-x_0| \leqslant |x-x_0| + |x-y| < a\tau + c_1\varepsilon\tau < \frac{a+c_1\varepsilon}{1-c_1\varepsilon}\,t.$$

We choose and fix $0 < \varepsilon < c_1^{-1}$ sufficiently small that $(a+c_1\varepsilon)/(1-c_1\varepsilon) < b$ so that, with (x,τ) and (y,t) as above,

(3.5) $$|y-x_0| < bt.$$

For any $(x,\tau) = \Phi(z) \in \partial D \times (0,1]$, let $B(x,\tau) = \{w \in \mathbf{R}^N\colon |w-z| < \varepsilon\tau\}$, and let $\tilde{B}(x,\tau) = \Phi[B(x,\tau)]$. It thus follows that $B(x,\tau) \subset \Gamma_b(x_0) \cup K$ whenever $(x,\tau) \in \tilde{\Gamma}_a(x_0)$.

We next fix a boundary point, which we assume for simplicity to be the origin, and let $(x,t) \in \tilde{\Gamma}_a(0)$. Then for $1 \leqslant j \leqslant N$,

(3.6) $$[X^i \partial_j u]^\sim(x,t) = -\int_t^1 \partial_\tau [X^i \partial_j u]^\sim(x,\tau)\,d\tau + [X^i \partial_j u]^\sim(x,1)$$
$$= C\int_t^1 [X^{i+1}\partial_j u]^\sim(x,\tau)\,d\tau + [X^i \partial_j u]^\sim(x,1).$$

For any fixed $(x,\tau) \in \tilde{\Gamma}_a(0)$, let L denote the constant coefficient differential operator obtained by freezing the coefficients of X^{i+1} at $\Phi^{-1}(x,\tau)$. Then L commutes with each ∂_j, and Lu is harmonic in D, so it follows that

$$|[X^{i+1}\partial_j u]^\sim(x,\tau)|^2 = |L\partial_j u(\Phi^{-1}(x,\tau))|^2 = |\partial_j Lu(\Phi^{-1}(x,\tau))|^2$$
$$\leqslant \frac{C}{\tau^{N+2}} \int_{B(x,\tau)} |Lu|^2\,dV$$
$$\leqslant \frac{C}{\tau^{N+2}} \int_{B(x,\tau)} |X^{i+1}u|^2\,dV + \frac{C}{\tau^{N+2}} \int_{B(x,\tau)} |(L-X^{i+1})u|^2\,dV.$$

But since $i+1$ is at most k, the coefficients of X^{i+1} satisfy a Lipschitz condition of order η, so the integrand in the last integral on the right is bounded by a constant multiple of $\tau^{2\eta}|\mathscr{D}^{i+1}u|^2$, which yields the estimate

$$|[X^{i+1}\partial_j u]^\sim(x,\tau)|^2 \leqslant \frac{C}{\tau^{N+2}}\int_{B(x,\tau)}|X^{i+1}u|^2\,dV+\frac{C}{\tau^{N+2-2\eta}}\int_{B(x,\tau)}|\mathscr{D}^{i+1}u|^2\,dV.$$

Since the Jacobian of Φ is bounded, the integrals on the right hand side may be replaced by the corresponding integrals over the regions $\tilde{B}(x,t)$ in $\partial D\times(0,2)$. But in view of (3.4) and (3.5), $\tilde{B}(x,\tau)$ is contained in the strip

$$S_\tau=\{(y,t)\in\partial D\times(0,2):\ |y|<bt \text{ and } (1-c_1\varepsilon)\tau<t<(1+c_1\varepsilon)\tau\},$$

where c_1 is as in (3.3), and therefore

$$|[X^{i+1}\partial_j u]^\sim(x,\tau)|^2 \leqslant \frac{C}{\tau^{N+2}}\int_{S_\tau}|[X^{i+1}u]^\sim|^2\,d\tilde{V}+\frac{C}{\tau^{N+2-2\eta}}\int_{S_\tau}|[\mathscr{D}^{i+1}u]^\sim|^2\,d\tilde{V}$$
$$=I_1(\tau)+I_2(\tau).$$

Combining this estimate with (3.6) yields

$$|[X^i\partial_j u]^\sim(x,t)|\leqslant C\Big[\int_t^1(I_1(\tau)^{1/2}+I_2(\tau)^{1/2})\,d\tau+\|\mathscr{D}^{i+1}u\|_K\Big].$$

Squaring both sides, multiplying by t^{q-N}, and integrating over $\tilde{\Gamma}_a(0)$ gives

$$(3.7)\qquad (S_a^{(q)}(X^i\partial_j u)(0))^2\leqslant C\Big[\sum_{j=1}^{2}\int_0^1 t^{q-1}\Big(\int_t^1 I_j(\tau)^{1/2}\,d\tau\Big)^2 dt+\|\mathscr{D}^{i+1}u\|_K^2\Big].$$

The two integrals on the right of (3.7) can be estimated by Hardy's inequality. Letting χ_t denote the characteristic function of the strip $S_t\cap\tilde{\Gamma}_b(0)$, we have

$$\int_0^1 t^{q-1}\Big(\int_t^1 I_1(\tau)^{1/2}\,d\tau\Big)^2 dt\leqslant\frac{4}{q^2}\int_0^1 t^{q+1}I_1(t)\,dt=C\int_0^1\frac{1}{t^{N+1-q}}\int_{S_t}|[X^{i+1}u]^\sim|^2\,d\tilde{V}\,dt$$
$$\leqslant C\Big[\int_0^1\frac{1}{t^{N+1-q}}\int_{S_t\cap\tilde{\Gamma}_b(0)}|[X^{i+1}u]^\sim|^2\,d\tilde{V}\,dt+\|\mathscr{D}^{i+1}u\|_K^2\Big]$$
$$=C\Big[\int_0^1\frac{1}{t^{N+1-q}}\int_0^1\int_{\partial D}\chi_t(x,\tau)\,|[X^{i+1}u]^\sim(x,\tau)|^2\,d\sigma(x)\,d\tau\,dt+\|\mathscr{D}^{i+1}u\|_K^2\Big]$$
$$\leqslant C\Big[\int_{\tilde{\Gamma}_b(0)}\int_{c_2\tau}^{c_3\tau}\frac{dt}{t^{N+1-q}}|[X^{i+1}u]^\sim(x,\tau)|^2\,d\tilde{V}(x,\tau)+\|\mathscr{D}^{i+1}u\|_K^2\Big]$$
$$\leqslant C\Big[\int_{\tilde{\Gamma}_b(0)}\tau^{q-N}|[X^{i+1}u]^\sim(x,\tau)|^2\,d\tilde{V}(x,\tau)+\|\mathscr{D}^{i+1}u\|_K^2\Big]$$
$$=C\big[(S_b^{(q)}(X^{i+1}u)(0))^2+\|\mathscr{D}^{i+1}u\|_K^2\big].$$

Here $c_2=(1+c_1\varepsilon)^{-1}$ and $c_3=(1-c_1\varepsilon)^{-1}$ with c_1 as in (3.3). The same technique may be used to estimate the second integral on the right in (3.7). The result is

$$\int_0^1 t^{q-1}\Big(\int_t^1 I_2(\tau)^{1/2}\,d\tau\Big)^2 dt\leqslant C\big[(S_b^{(q+2\eta)}(\mathscr{D}^{i+1}u)(0))^2+\|\mathscr{D}^{i+1}u\|_K^2\big].$$

Inserting these last two estimates into (3.7) gives (3.2).

Applying (3.2) inductively, with a and b replaced by suitable intermediate values, gives

$$S_a^{(q)}(X^i\mathscr{D}^j u)(0)\leqslant C_{a,b,q}\big[S_b^{(q)}(X^{i+j}u)(0)+S_b^{(q+2\eta)}(\mathscr{D}^{i+j}u)(0)+\|\mathscr{D}^{i+j}u\|_K\big]$$

whenever $0<a<b$ and $i+j\leqslant k$, and, in particular,

$$(3.8)\qquad S_a^{(q)}(\mathscr{D}^k u)(0)\leqslant C_{a,b,q}\big[S_b^{(q)}(X^k u)(0)+S_b^{(q+2\eta)}(\mathscr{D}^k u)(0)+\|\mathscr{D}^k u\|_K\big].$$

Let $0<a<b$, and let r be a positive integer. Choose $a<a_0<a_1<\ldots<a_{r-1}=b$. By applying (3.8) with a and b replaced by a_i and a_{i+1} for $0\leqslant i\leqslant r-1$, we obtain

$$S_a^{(q)}(\mathscr{D}^k u)(0)\leqslant C_{a,b,q,r}\Big[\sum_{j=0}^{r-1}S_{a_j}^{(q+2j\eta)}(X^k u)(0)+S_b^{(q+2r\eta)}(\mathscr{D}^k u)(0)+\|\mathscr{D}^k u\|_K\Big].$$

But each term in the sum on the right is dominated by $S_b^{(q)}(X^k u)(0)$, so, since r is at our disposal, it follows that for any $M>0$,

$$(3.9)\qquad S_a^{(q)}(\mathscr{D}^k u)(0)\leqslant C_{a,b,q,M}\big[S_b^{(q)}(X^k u)(0)+S_b^{(q+M)}(\mathscr{D}^k u)(0)+\|\mathscr{D}^k u\|_K\big].$$

For any $(x,t)\in\tilde{\Gamma}_a$, the radius of the ball $B(x,t)$ is comparable with t, so we have

$$|\mathscr{D}^k u(\Phi^{-1}(x,t))|^2\leqslant Ct^{-2k-N}\int_{B(x,t)}|u|^2\,dV\leqslant Ct^{-2k-N}\int_{\tilde{B}(x,t)}|\tilde{u}(y,\tau)|^2\,d\tilde{V}(y,\tau).$$

Since, by (3.4), τ is comparable with t in $\tilde{B}(x,t)$, it follows that

$$|\mathscr{D}^k u(\Phi^{-1}(x,t))|^2\leqslant Ct^{-2k-q}\int_{\tilde{B}(x,t)}|\tilde{u}(y,\tau)|^2\tau^{-N+q}\,d\tilde{V}(y,\tau)$$
$$\leqslant C_{a,b}\big[t^{-2k-q}(S_b^{(q)}u(0))^2+t^{-2k}\|u\|_K^2\big].$$

Multiplying by $t^{-N+2k+q+1}$ and integrating over $\tilde{\Gamma}_a(0)$ gives

$$(3.10)\qquad S_a^{(q+2k+1)}(\mathscr{D}^k u)(0)\leqslant C_{a,b,q}\big(S_b^{(q)}u(0)+\|u\|_K\big)$$

whenever $0<a<b$. For any $0<a<b$, choose b' between a and b. Applying (3.9) with b replaced by b' and $M=2k+1$, and (3.10) with a replaced by b' gives

$$(3.11)\qquad S_a^{(q)}(\mathscr{D}^k u)(0)\leqslant C_{a,b,q}\big[S_b^{(q)}(X^k u)(0)+S_b^{(q)}u(0)+\|\mathscr{D}^k u\|_K\big].$$

But since u can be recovered from $X^k u$ by k-fold integration, the second term on the right of (3.11) can be absorbed by the first and the last, and the lemma is proved.

(3.12) COROLLARY. *With X and k as above, there is a constant C such that for any harmonic function u on D*

$$g(\mathscr{D}^k u)\leqslant C\big[S(X^k u)+\|\mathscr{D}^k u\|_K\big].$$

Proof. The case $k = 0$ is proved exactly as in the case of a half-space. (See [8, p. 90].) But applying the case $k = 0$ to the components of $\mathscr{D}^k u$ gives

$$g(\mathscr{D}^k u) \leqslant C(S(\mathscr{D}^k u) + \|\mathscr{D}^k u\|_K)$$

and the result follows from Lemma (3.1).

(3.13) Lemma. *Let X and k be as in Lemma* (3.1). *For any $0 < a < b$ there is a constant C such that for any harmonic function u on D and any $x \in \partial D$*

$$N_a(\delta \mathscr{D}^{k+1} u)(x) \leqslant C[N_b(X^k u)(x) + \|\mathscr{D}^{2k} u\|_K].$$

Proof. We fix a point ∂D, which we assume is the origin. For $(x, t) \in \tilde{\Gamma}_a(0)$ we have

(3.14) $$[\mathscr{D}^{k+1} u]^\sim (x, t) = C \int_t^1 \int_{\tau_1}^1 \ldots \int_{\tau_{k-1}}^1 [X^k \mathscr{D}^{k+1} u]^\sim (x, \tau_k)\, d\tau_k \ldots d\tau_1 + \mathscr{E}$$

where $\mathscr{E} = \mathscr{E}(x, t)$ satisfies $|\mathscr{E}| \leqslant C\|\mathscr{D}^{2k} u\|_K$. For any fixed x and τ, let $L_{(x,\tau)}$ denote the constant coefficient operator obtained by freezing the coefficients of X^k at $\Phi^{-1}(x, \tau)$. Then, letting $B(x, \tau)$ be as in the proof of Lemma (3.1),

$$|[X^k \mathscr{D}^{k+1} u]^\sim (x, \tau)| = |[\mathscr{D}^{k+1} L_{(x,\tau)} u]^\sim (x, \tau)| \leqslant \frac{C}{\tau^{k+1}} \|L_{(x,\tau)} u\|_{B(x,\tau)}$$

$$\leqslant \frac{C}{\tau^{k+1}} [\|X^k u\|_{B(x,\tau)} + \|(L_{(x,\tau)} - X^k) u\|_{B(x,\tau)}].$$

But the coefficients of X^k are Lipschitz of order η, so we obtain

(3.15) $$|[X^k \mathscr{D}^{k+1} u]^\sim (x, \tau)| \leqslant C[\tau^{-k-1} \|X^k u\|_{B(x,\tau)} + \tau^{-k-1+\eta} \|\mathscr{D}^k u\|_{B(x,\tau)}].$$

For $0 \leqslant \alpha < 1$ and $a > 0$, let

$$\tilde{\Gamma}_a^{(\alpha)}(x) = \{(y, t) \in \tilde{\Gamma}_a(x) \colon t > \alpha\}, \qquad \Gamma_a^{(\alpha)}(x) = \Phi^{-1}[\tilde{\Gamma}_a^{(\alpha)}(x)].$$

Then for $(x, t) \in \tilde{\Gamma}_a(0)$ and $t \leqslant \tau \leqslant 1$, we have $B(x, \tau) \subset \Gamma_b^{(\alpha t)}(0) \cup K$, where $0 < \alpha < 1$ and α depends only on a and b. Thus, letting

$$N_a^{(\alpha)}(v)(x) = \sup\{|v(z)| \colon z \in \Gamma_a^{(\alpha)}(x)\},$$

it follows from (3.15) that

$$|[X^k \mathscr{D}^{k+1} u]^\sim (x, \tau)| \leqslant C_{a,b}[\tau^{-k-1} N_b(X^k u)(0) + \tau^{-k-1+\eta} N_b^{(\alpha t)}(\mathscr{D}^k u)(0)].$$

Inserting this estimate in (3.14) and evaluating the resulting integral gives

(3.16) $$|[\mathscr{D}^{k+1} u]^\sim (x, t)| \leqslant \frac{C}{t}[N_b(X^k u)(0) + t^\eta N_b^{(\alpha t)}(\mathscr{D}^k u)(0) + \|\mathscr{D}^{2k} u\|_K]$$

and hence, since t is comparable with $\tilde{\delta}$,

(3.17) $$N_a(\delta \mathscr{D}^{k+1} u)(0) \leqslant C_{a,b}[N_b(X^k u)(0) + N_b(\delta^\eta \mathscr{D}^k u)(0) + \|\mathscr{D}^{2k} u\|_K].$$

To complete the proof, we must argue that the second term on the right of (3.17) can be absorbed by the first and the last. The argument which led to (3.16) gives, for any $(x, t) \in \tilde{\Gamma}_a(0)$ and any $b > a$, the estimate

$$|[\mathscr{D}^k u]^\sim (x, t)| \leqslant C_{a,b}[\log(1/t) N_b(X^k u)(0) + t^\eta N_b^{(\alpha t)}(\mathscr{D}^k u)(0) + \|\mathscr{D}^{2k-1} u\|_K],$$

from which it follows that, for any $\beta > 0$,

$$N_a(\delta^\beta \mathscr{D}^k u)(0) \leqslant C_{a,b,\beta}[N_b(X^k u)(0) + N_b(\delta^{\beta+\eta} \mathscr{D}^k u)(0) + \|\mathscr{D}^{2k-1} u\|_K].$$

Iteration of this last estimate gives, for any $M > 0$,

$$N_a(\delta^\beta \mathscr{D}^k u)(0) \leqslant C_{a,b,\beta,M}[N_b(X^k u)(0) + N_b(\delta^M \mathscr{D}^k u)(0) + \|\mathscr{D}^{2k-1} u\|_K].$$

On the other hand, we also have the elementary inequalities

$$N_a(\delta^k \mathscr{D}^k u)(0) \leqslant C_{a,b}(N_b u(0) + \|u\|_K)$$

for any $0 < a < b$, and

$$N_a u(0) \leqslant N_a(X^k u)(0) + C\|\mathscr{D}^{k-1} u\|_K.$$

Combining these last three inequalities gives

$$N_a(\delta^\beta \mathscr{D}^k u)(0) \leqslant C_{a,b,\beta}[N_b(X^k u)(0) + \|\mathscr{D}^{2k-1} u\|_K],$$

which, in view of (3.17), completes the proof.

The remainder of this section will be devoted to demonstrating that the L^p norms of area integrals of gradients are dominated by L^p norms of maximal functions of transverse derivatives. We will consider separately the cases $0 < p < 2$, $p = 2$, and $2 < p < \infty$. We begin with the simplest case, $p = 2$.

(3.18) Lemma. *Let k be a nonnegative integer, and assume that the vector field X is of class C^{k+1}. There is a constant C such that $\|S(\mathscr{D}^{k+1} u)\|_2 \leqslant C(\|N(X^k u)\|_2 + \|\mathscr{D}^{2k} u\|_K)$ for any harmonic function u on D.*

Proof. There is no loss of generality in assuming that u is real-valued and it will be convenient to do so. By Lemma (3.1), it suffices to estimate $\|S(\nabla X^k u)\|_2$. We have

$$\int_{\partial D} (S(\nabla X^k u))^2\, d\sigma = \int_{\partial D} \int_{\tilde{\Gamma}(x)} |[\nabla X^k u(y, t)]^\sim|^2 t^{2-N}\, d\tilde{V}(y, t)\, d\sigma(x)$$

$$= \int_{\partial D} \int_{\partial D} \int_0^1 \chi_t(|x - y|)\, |[\nabla X^k u]^\sim (y, t)|^2 t^{2-N}\, dt\, d\sigma(y)\, d\sigma(x)$$

where χ_t denotes the characteristic function of the interval $(0, t]$. Integrating out the variable x yields

$$\int_{\partial D} (S(\nabla X^k u))^2\, d\sigma \leqslant C \int_{\partial D} \int_0^1 |[\nabla X^k u]^\sim (y, t)|^2\, t\, dt\, d\sigma(y).$$

It will be convenient to extend X to be a C^{k+1} vector field in a neighborhood of $\bar{D}$ which vanishes in a neighborhood of the singularity of δ. (Recall that δ was chosen to be a Green function for D.) Thus

$$\int_{\partial D} (S(\nabla X^k u))^2 \, d\sigma \leqslant C \int_D |\nabla X^k u|^2 \delta \, dV = C \lim_{\varepsilon \to 0^+} \int_{D_\varepsilon} |\nabla X^k u|^2 \delta \, dV$$

where $D_\varepsilon = K \cup \Phi^{-1}[\partial D \times (\varepsilon, 1]]$. Using the identity

$$\Delta(U^2) = 2|\nabla U|^2 + 2U \Delta U, \tag{3.19}$$

we obtain

$$\int_{\partial D} (S(\nabla X^k u))^2 \, d\sigma \tag{3.20}$$

$$\leqslant C \big[\limsup_{\varepsilon \to 0^+} \big| \int_{D_\varepsilon} \Delta((X^k u)^2) \delta \, dV \big| + \int_D |X^k u| \, |\Delta X^k u| \, \delta \, dV \big].$$

But by Green's Formula,

$$\int_{D_\varepsilon} \Delta((X^k u)^2) \delta \, dV = \int_{\partial D_\varepsilon} \partial_\nu ((X^k u)^2) \delta \, d\sigma_\varepsilon + \int_{\partial D_\varepsilon} (X^k u)^2 \partial_\nu \delta \, d\sigma_\varepsilon,$$

where $d\sigma_\varepsilon$ is the Euclidean volume element on ∂D_ε, and ∂_ν denotes outward normal differentiation along ∂D_ε. Thus

$$\big| \int_{D_\varepsilon} \Delta((X^k u)^2) \delta \, dV \big| \leqslant C \big[\int_{\partial D} N_0(X^k u) N_0((\mathscr{D}^{k+1} u) \delta) \, d\sigma + \int_{\partial D} (N_0(X^k u))^2 \, d\sigma \big].$$

It follows from Lemma (3.13) that

$$\big| \int_{D_\varepsilon} \Delta((X^k u)^2) \delta \, dV \big| \leqslant C \big[\int_{\partial D} (N(X^k u))^2 \, d\sigma + \|\mathscr{D}^{2k} u\|_K^2 \big]. \tag{3.21}$$

As for the second term on the right of (3.20), note that, since u is harmonic, $\Delta X^k u = [\Delta, X^k] u$, where $[\Delta, X^k]$ denotes the commutator of the operators Δ and X^k, which is of order $k+1$. It thus follows that

$$\int_D |X^k u| \, |\Delta X^k u| \, \delta \, dV \tag{3.22}$$

$$\leqslant C \big[\int_{\partial D} \int_0^1 |[X^k u]^\sim (x, t)| \, |[\mathscr{D}^{k+1} u]^\sim (x, t)| \, t \, dt \, d\sigma(x) + \|\mathscr{D}^{k+1} u\|_K^2 \big]$$

$$\leqslant C \big[\int_{\partial D} N_0(X^k u) N_0(\delta \mathscr{D}^{k+1} u) \, d\sigma + \|\mathscr{D}^{k+1} u\|_K^2 \big]$$

$$\leqslant C \big[\int_{\partial D} (N(X^k u))^2 \, d\sigma + \|\mathscr{D}^{2k} u\|_K^2 + \|\mathscr{D}^{k+1} u\|_K^2 \big].$$

Inserting (3.21) and (3.22) into (3.20) gives

$$\|S(\mathscr{D}^{k+1} u)\|_2 \leqslant C [\|N(X^k u)\|_2 + \|\mathscr{D}^J u\|_K],$$

with $J = \max\{1, 2k\}$. This is the required estimate if $k > 0$. On the other hand, if $k = 0$, and if $\|Nu\|_2 < \infty$, then u is the Poisson integral of a function in $L^2(\partial D)$, and therefore the last term on the right of the above inequality can be absorbed into the first, and again the required estimate obtains.

(3.23) Lemma. *Assume that X is of class C^{k+1}. If u is harmonic in D, and if $N(X^k u) \in L^p(\partial D)$ for some $0 < p < 2$ then $S(\mathscr{D}^{k+1} u) \in L^p(\partial D)$ and*

$$\|S(\mathscr{D}^{k+1} u)\|_p \leqslant C_p [\|N(X^k u)\|_p + \|\mathscr{D}^{2k} u\|_K].$$

Proof. As in the proof of Lemma (3.18), it will be convenient to assume that u is real-valued. By Lemma (3.1), it suffices to estimate $\|S(\nabla X^k u)\|_p$. We will estimate the distribution function of $S(\nabla X^k u)$ using a technique of Fefferman and Stein [5, pp. 162–163]. Let $a > 1$. Then according to Lemma (2.1), $N_a(X^k u) \in L^p(\partial D)$, and $\|N_a(X^k u)\|_p \leqslant C_a \|N(X^k u)\|_p$. Fix $\alpha > 0$, and let $E = E_\alpha = \{x \in \partial D: N_a(X^k u) \leqslant \alpha\}$ and $F = F_\alpha = \partial D \setminus E_\alpha = \{x \in \partial D: N_a(X^k u) > \alpha\}$. Then E is closed by the lower semicontinuity of $N_a(X^k u)$. Let $R = R_a = \bigcup \{\Gamma(x): x \in E\}$ and $\tilde{R} = \Phi[R]$. It follows from Fubini's Theorem that

$$\int_E (S(\nabla X^k u))^2 \, d\sigma \leqslant C \int_{\tilde{R}} |[\nabla X^k u]^\sim|^2 \, t \, d\tilde{V} \leqslant C \int_R |\nabla X^k u|^2 \delta \, dV. \tag{3.24}$$

We will estimate the integral on the right by applying Green's Formula in a sequence of piecewise smooth domains which approximate R from within. Choose a sequence φ_j of C^2 functions on ∂D such that the tangential gradients of the φ_j's are bounded uniformly in j and α, and such that φ_j decreases to $\varphi(x) = \operatorname{dist}(x, E)$ as $j \to \infty$. (The existence of such a sequence follows from the fact that φ is Lipschitz of order 1.) Let $\tilde{R}_j = \{(x, t) \in \tilde{R}: t > \varphi_j(x)\}$. By (3.24) and the identity (3.19),

$$\int_E (S(\nabla X^k u))^2 \, d\sigma \leqslant C \lim_{j \to \infty} \int_{R_j} |\nabla X^k u|^2 \delta \, dV \tag{3.25}$$

$$= C \lim_{j \to \infty} \big[\int_{R_j} \Delta[(X^k u)^2] \delta \, dV - 2 \int_{R_j} (X^k u) \Delta(X^k u) \delta \, dV \big]$$

$$\leqslant C \big[\limsup_{j \to \infty} \big| \int_{R_j} \Delta[(X^k u)^2] \delta \, dV \big| + \int_R |X^k u| \, |\Delta(X^k u)| \, \delta \, dV \big].$$

But Green's Formula gives

$$\int_{R_j} \Delta[(X^k u)^2] \delta \, dV = \int_{\partial R_j} \partial_\nu [(X^k u)^2] \delta \, d\sigma_j - \int_{\partial R_j} (X^k u)^2 \partial_\nu \delta \, d\sigma_j \tag{3.26}$$

where σ_j denotes the surface measure on ∂R_j and ∂_ν denotes the outer normal derivative on ∂R_j. To estimate these integrals, we split ∂R_j into three pieces

$$\partial R_j = B_j^1 \cup B_j^2 \cup B_j^3$$

where $\Phi[B_j^1] \subset \partial D \times \{1\}$, $\Phi[B_j^2] \subset E \times (0, 1)$, and $\Phi[B_j^3] \subset F \times (0, 1)$. We will denote the image of B_j^i under Φ by $\tilde{B}_j^i$.

We will now estimate the first term on the right of (3.26). For the integrand, we have the pointwise estimate

$$|\partial_\nu [(X^k u)^2]| \, \delta \leqslant C |X^k u| \, |\delta \mathscr{D}^{k+1} u|, \tag{3.27}$$

from which it follows trivially that

$$\big| \int_{B_j^1} \partial_\nu [(X^k u)^2] \delta \, d\sigma_j \big| \leqslant C \|\mathscr{D}^{k+1} u\|_K^2. \tag{3.28}$$

For the integral over B_j^2, note that, in view of the fact that the functions φ_j which define the regions R_j have uniformly bounded derivatives, we have $d\tilde{\sigma}_j(x,t) \leqslant C d\sigma(x)$ along $\tilde{B}_j^2 \cup \tilde{B}_j^3$, where $d\tilde{\sigma}$ is obtained by pulling back the measure $d\sigma_j$ by Φ^{-1}. Moreover, by Lemma (3.13) w have

$$N(\delta \mathscr{D}^{k+1} u) \leqslant C_a [N_a(X^k u) + \|\mathscr{D}^{2k} u\|_K], \tag{3.29}$$

so (3.27) implies

$$\begin{aligned} |[\partial_\nu (X^k u)^2 \delta]^\sim (x,t)| &\leqslant C[N(X^k u)(x)]\, N(\delta \mathscr{D}^{k+1} u)(x) \\ &\leqslant C[[N_a(X^k u)(x)]^2 + \|\mathscr{D}^{2k} u\|_K^2]. \end{aligned} \tag{3.30}$$

Thus we have the estimate

$$\begin{aligned} \Big|\int_{B_j^2} \partial_\nu [(X^k u)^2]\, \delta\, d\sigma_j\Big| &\leqslant C[\int_E [N_a(X^k u)]^2 d\sigma + \|\mathscr{D}^{2k} u\|_K^2] \\ &= C[\int_0^\alpha t\, \sigma\{x \in E:\ N_a(X^k u) > t\}\, dt + \|\mathscr{D}^{2k} u\|_K^2]. \end{aligned} \tag{3.31}$$

For the integral over B_j^3, note that for any $x \in R$ we have, by (3.27) and (3.29),

$$|\partial_\nu [(X^k u)^2](x)|\, \delta(x) \leqslant C_a [\sup_E (N_a(X^k u))^2 + \|\mathscr{D}^{2k} u\|_K^2] \leqslant C_a [\alpha^2 + \|\mathscr{D}^{2k} u\|_K^2]$$

so

$$\Big|\int_{B_j^3} \partial_\nu [(X^k u)^2\, \delta\, d\sigma_j\Big| \leqslant C_a [\alpha^2 \sigma(F) + \|\mathscr{D}^{2k} u\|_K^2]. \tag{3.32}$$

Combining the three estimates (3.28), (3.31), and (3.32) gives an estimate for the first integral on the right of (3.26):

$$\begin{aligned} &\Big|\int_{\partial R_j} \partial_\nu [(X^k u)^2]\, \delta\, d\sigma_j\Big| \\ &\quad \leqslant C_a [\int_0^\alpha t\, \sigma\{x \in E:\ N_a(X^k u) > t\}\, dt + \alpha^2 \sigma(F) + \|\mathscr{D}^J u\|_K^2], \end{aligned} \tag{3.33}$$

where $J = \max\{2k, 1\}$. A similar argument shows that the right hand side of (3.33) also dominates the remaining integral on the right of (3.26), and we thus have

$$\begin{aligned} &\limsup_{j \to \infty} \Big|\int_{R_j} \Delta [(X^k u)^2]\, \delta\, dV\Big| \\ &\quad \leqslant C_a [\int_0^\alpha t\, \sigma\{x \in E:\ N_a(X^k u) > t\}\, dt + \alpha^2 \sigma(F) + \|\mathscr{D}^J u\|_K^2]. \end{aligned} \tag{3.34}$$

We now turn our attention to the last term on the right of (3.25). Since u is harmonic, $|\Delta X^k u| = |[\Delta, X^k] u| \leqslant C|\mathscr{D}^{k+1} u|$. (Here $[\Delta, X^k]$ denotes the commutator of Δ and X^k.) Thus, letting χ denote the characteristic function of $\tilde{R}$, we have

$$\begin{aligned} \int_R |X^k u|\, |\Delta(X^k u)|\, \delta\, dV &\leqslant C \int_R |X^k u|\, |\mathscr{D}^{k+1} u|\, \delta\, dV \\ &\leqslant C \int_{\partial D} \int_0^1 \chi(x,t)\, |[X^k u]^\sim (x,t)|\, |[\mathscr{D}^{k+1} u]^\sim (x,t)|\, \tilde{\delta}(x,t)\, dt\, d\sigma(x) \\ &\leqslant C[\int_E (N_0 X^k u)\, N_0(\delta \mathscr{D}^{k+1} u)\, d\sigma + \int_F \alpha^2\, d\sigma + \|\mathscr{D}^{2k} u\|_K^2] \\ &\leqslant C[\int_E (N X^k u)^2\, d\sigma + \alpha^2 \sigma(F) + \|\mathscr{D}^{2k} u\|_K^2] \\ &= C[\int_0^\alpha t\, \sigma\{x \in E:\ N_a(X^k u)(x) > t\}\, dt + \alpha^2 \sigma(F) + \|\mathscr{D}^{2k} u\|_K^2]. \end{aligned} \tag{3.35}$$

Inserting (3.34) and (3.35) into (3.25) gives

$$\int_E (S(\nabla X^k u))^2\, d\sigma \leqslant C_a [\int_0^\alpha t\, \sigma\{x \in E:\ N_a(X^k u) > t\}\, dt + \alpha^2 \sigma(F) + \|\mathscr{D}^J u\|_K^2].$$

By Chebyshev's Inequality, it follows that

$$\begin{aligned} &\sigma\{x \in E:\ S(\nabla X^k u) > \alpha\} \\ &\quad \leqslant C\Big[\sigma(F) + \frac{1}{\alpha^2} \int_0^\alpha t\, \sigma\{x \in E:\ N_a(X^k u) > t\}\, dt + \frac{1}{\alpha^2} \|\mathscr{D}^J u\|_K^2\Big]. \end{aligned}$$

But since the right hand side trivially dominates $\sigma(F)$, it in fact bounds $\sigma\{x \in \partial D:\ S(\nabla X^k u) > \alpha\}$. On the other hand, we also have the trivial estimate $\sigma\{x \in \partial D:\ S(\nabla X^k u) > \alpha\} \leqslant \sigma(\partial D) < \infty$, and so letting $M = \|\mathscr{D}^J u\|_K$, we have

$$\begin{aligned} \int (S(\nabla X^k u))^p\, d\sigma &= \int_0^\infty \alpha^{p-1} \sigma\{x \in \partial D:\ S(\nabla X^k u) > \alpha\}\, d\alpha = \int_0^M + \int_M^\infty \\ &\leqslant \frac{\sigma(\partial D)}{p} M^p + C \int_0^\infty \alpha^{p-1} \sigma\{N_a X^k u > \alpha\}\, d\alpha \\ &\quad + C \int_0^\infty \alpha^{p-3} \int_0^\alpha t\, \sigma\{N_a X^k u > t\}\, dt\, d\alpha + C\|\mathscr{D}^J u\|_K^2 \int_M^\infty \alpha^{p-3}\, d\alpha \\ &\leqslant C[\|\mathscr{D}^J u\|_K^p + \|N_a X^k u\|_{L^p}^p + \int_0^\infty t\, \sigma\{N_a X^k u > t\} \int_t^\infty \alpha^{p-3}\, d\alpha\, dt] \\ &= C\Big[\|\mathscr{D}^J u\|_K^p + \|N_a X^k u\|_{L^p}^p + \frac{1}{2-p} \int_0^\infty t^{p-1} \sigma\{N_a X^k u > t\}\, dt\Big] \\ &= C\Big[\|\mathscr{D}^J u\|_K^p + \frac{3-p}{2-p} \|N_a X^k u\|_{L^p}^p\Big]. \end{aligned}$$

This proves the lemma in the case $k > 0$. But when $k = 0$ the first term on the right of the above inequality can be absorbed by the second, since u is harmonic, and so the proof is complete.

(3.36) LEMMA. *Let k be a nonnegative integer, and assume that the vector field X is of class C^{k+1}. For any $2 < p < \infty$ there is a constant C_p such that for any harmonic function u on D*

$$\|S\mathcal{D}^{k+1} u\|_p \leqslant C_p [\|NX^k u\|_p + \|\mathcal{D}^{2k} u\|_K].$$

Proof. When $k = 0$, the result is contained in Lemma (2.2). We shall show that the case $k \geqslant 1$ also follows from Lemma (2.2) via a duality argument (cf. [7, pp. 455–458]). As in the proof of Lemma (3.18), it will be convenient to assume that u is real-valued and that the vector field X is defined in a full neighborhood of $\overline{D}$ and vanishes in a neighborhood of the singularity of δ. By Lemma (3.1), it suffices to prove

$$\|S(X^{k+1} u)\|_{L^p} \leqslant C_p \|N(X^k u)\|_{L^p} + \|\mathcal{D}^{2k} u\|_K.$$

We have

$$\|S(X^{k+1} u)\|_p^2 = \|(SX^{k+1} u)^2\|_{p/2} = \sup \int_{\partial D} (S(X^{k+1} u))^2 \varphi \, d\sigma$$

where the supremum is taken over all nonnegative functions φ in $L^q(\partial D)$ satisfying $\|\varphi\|_q \leqslant 1$ with $1/q + 2/p = 1$. We begin with the elementary estimate

$$\int_{\partial D} (SX^{k+1} u)^2 \varphi \, d\sigma = \int_{\partial D} \int_{\Gamma(x)} |[X^{k+1} u]^\sim|^2 t^{2-N} d\tilde{V} \varphi(x) \, d\sigma(x)$$

$$= \int_{\partial D} \int_{\partial D} \int_0^1 |[X^{k+1} u]^\sim (y, t)|^2 \varphi(x) \chi_t(|x-y|) t^{2-N} dt \, d\sigma(y) \, d\sigma(x)$$

where χ_t denotes the characteristic function of the interval $[0, t]$. But, letting $P(z, x)$ denote the Poisson kernel of D on $D \times \partial D$, we have $t^{1-N} \chi_t(|x-y|) \leqslant CP(\Phi^{-1}(y, t), x)$. Letting v denote the Poisson integral of φ, it thus follows that

$$\int_{\partial D} (SX^{k+1} u)^2 \varphi \, d\sigma \tag{3.37}$$

$$\leqslant C \int_{\partial D} \int_0^1 |[X^{k+1} u]^\sim (y, t)|^2 \int_{\partial D} \varphi(x) P(\Phi^{-1}(y, t), x) \, d\sigma(x) \, t dt \, d\sigma(y)$$

$$\leqslant C \int_D |X^{k+1} u|^2 v \delta \, dV \leqslant C \int_D |\nabla X^k u|^2 v \delta \, dV$$

$$= C \int_D \left(\tfrac{1}{2} \Delta ((X^k u)^2 v) - \nabla ((X^k u)^2) \cdot \nabla v - v (X^k u) \Delta (X^k u)\right) \delta \, dV$$

$$\leqslant C \left|\int_D \Delta ((X^k u)^2 v) \delta \, dV\right| + C \int_D |X^k u| |\nabla X^k u| |\nabla v| \delta \, dV$$

$$+ C \int_D |X^k u| v |[\Delta, X^k] u| \delta \, dV.$$

Here we have used the identity

$$|\nabla U|^2 V = \tfrac{1}{2} \Delta (U^2 V) - \tfrac{1}{2} U^2 \Delta V - \nabla (U^2) \cdot \nabla V - UV \Delta U.$$

The first term on the right of (3.37) may be estimated by applying Green's Formula in the approximating domains $D_\varepsilon = \{\Phi^{-1}(x, t) \colon t > \varepsilon\} \cup K$, and letting $\varepsilon \to 0^+$ to obtain

$$\left|\int_D \Delta ((X^k u)^2 v) \delta \, dV\right| \leqslant C \int_{\partial D} [N_0 (X^k u)]^2 (N_0 v + N_0 (\delta \nabla v)) \, d\sigma \tag{3.38}$$

$$+ \int_{\partial D} N_0 (X^k u) N_0 (\delta \nabla X^k u) N_0 v \, d\sigma.$$

Since $N_0 (\delta \nabla v) \leqslant C(Nv + \|v\|_K)$, it follows from Hölder's Inequality that the first integral on the right is majorized by $\|N_0 (X^k u)\|_p^2 \|Nv\|_q$. For the last term in (3.38), note that by Lemma (3.13), $N_0 (\delta \nabla X^k u) \leqslant C(N(X^k u) + \|\mathcal{D}^{2k} u\|_K)$, so, again by Hölder's Inequality, the last integral is majorized by $(\|N(X^k u)\|_p^2 + \|\mathcal{D}^{2k} u\|_K^2) \|N_0 v\|_q$. Since v is the Poisson integral of φ, it follows that $\|Nv\|_q \leqslant C \|\varphi\|_q \leqslant C$, so it follows from (3.38) that

$$\left|\int_D \Delta ((X^k u)^2 v) \delta \, dV\right| \leqslant C [\|N(X^k u)\|_p^2 + \|\mathcal{D}^{2k} u\|_K^2]. \tag{3.39}$$

The second term on the right of (3.37) is more delicate. We have

$$\int_D |X^k u| |\nabla X^k u| |\nabla v| \delta \, dV$$

$$\leqslant C \Big[\int_{\partial D \times (0,1]} |[X^k u]^\sim| |[\mathcal{D}^{k+1} u]^\sim| |[\nabla v]^\sim| \, t \, d\tilde{V} + \|\mathcal{D}^{k+1} u\|_K^2 \Big]$$

$$\leqslant C \int_{\partial D} N_0 (X^k u) \Big(\int_0^1 |[\mathcal{D}^{k+1} u]^\sim (x, t)|^2 t \, dt\Big)^{1/2} \Big(\int_0^1 |[\nabla v]^\sim (x, t)|^2 t \, dt\Big)^{1/2} d\sigma(x)$$

$$+ C \|\mathcal{D}^{k+1} u\|_K^2$$

$$= C \Big[\int_{\partial D} N_0 (X^k u) g(\mathcal{D}^{k+1} u) g(\nabla v) \, d\sigma + \|\mathcal{D}^{k+1} u\|_K^2 \Big].$$

By Hölder's Inequality, the integral on the right is dominated by $\|N_0 (X^k u)\|_p \times \|g(\mathcal{D}^{k+1} u)\|_p \|g(\nabla v)\|_q$, and by Lemma (2.2), $\|g(\nabla v)\|_q \leqslant C \|Nv\|_q \leqslant C \|\varphi\|_q \leqslant C$. In addition, by Corollary (3.12), $\|g(\mathcal{D}^{k+1} u)\|_p$ is dominated by $\|S(X^{k+1} u)\|_p + \|\mathcal{D}^{k+1} u\|_K$, and, trivially, the radial maximal function is dominated by any nontangential maximal function, so we obtain

$$\int_D |X^k u| |\nabla X^k u| |\nabla v| \delta \, dV \tag{3.40}$$

$$\leqslant C [\|N(X^k u)\|_p \|S(X^{k+1} u)\|_p + \|N(X^k u)\|_p \|\mathcal{D}^{k+1} u\| + \|\mathcal{D}^{k+1} u\|_K^2].$$

Estimation of the last integral in (3.37) is straightforward: passing to the radial maximal function and integrating over ∂D gives

$$\int_D |X^k u| v |[\Delta, X^k] u| \delta \, dV \tag{3.41}$$

$$\leqslant \int_{\partial D} N_0(X^k u)\, N_0 v\, N_0(\delta\, \mathscr{D}^{k+1} u)\, d\sigma + C\,\|\mathscr{D}^{k+1} u\|_K^2$$

$$\leqslant C\,[\int_{\partial D} (N(X^k u)^2 + \|\mathscr{D}^{2k} u\|_K^2)(N_0 v)\, d\sigma + \|\mathscr{D}^{k+1} u\|_K^2]$$

$$\leqslant C\,[\|N(X^k u)\|_p^2 + \|\mathscr{D}^{2k} u\|_K^2].$$

Inserting (3.39)–(3.41) into (3.37) and taking the supremum over φ gives

$$\|S(X^{k+1} u)\|_p^2 \leqslant C\,[\|N(X^k u)\|_p^2 + \|N(X^k u)\|_p\, \|S(X^{k+1} u)\|_p + \|\mathscr{D}^{2k} u\|_K^2],$$

from which we conclude

$$\|S(X^{k+1} u)\|_p \leqslant C\,[\|N(X^k u)\|_p + \|\mathscr{D}^{2k} u\|_K] \tag{3.42}$$

for any harmonic function u such that $\|S(X^{k+1} u)\|_p < \infty$.

To complete the proof, we must argue that $S(X^{k+1} u) \in L^p(\partial D)$ whenever $N(X^k u) \in L^p(\partial D)$. Since ∂D is compact, it suffices to verify that $S(X^{k+1} u)$ is locally in $L^p(\partial D)$, which can be deduced from (3.42) by the following simple device. Fix $x_0 \in \partial D$, and choose a small neighborhood W of x_0 such that the outward normal ν to ∂D at x_0 is transverse at each point of $\partial D \cap W$. Next, choose a domain D_0 with C^2 boundary with $D_0 \subset D$ and with $\partial D_0 \cap \partial D \subset W$, and choose a vector field X_0 in a neighborhood of ∂D_0 which is everywhere transverse to ∂D_0 and which agrees with X near x_0. Let N^0 and S^0 be the nontangential maximal and area integral operators respectively on D_0. For any function U on D, we let $U_\varepsilon(x) = U(x - \varepsilon\nu)$. Assume that u is a harmonic function on D with $\|N(X^k u)\|_p < \infty$. Then for $\varepsilon > 0$ sufficiently small, u_ε is harmonic in a neighborhood of $\bar{D}_0$, and, by (3.42),

$$\|S^0(X_0^{k+1} u_\varepsilon)\|_{L^p(\partial D_0)} \leqslant C_a\,[\|N_a^0(X_0^k u_\varepsilon)\|_{L^p(\partial D_0)} + \|\mathscr{D}^{2k} u_\varepsilon\|_{K_0}],$$

where K_0 is some compact subset of D_0. If the aperture a is sufficiently large, then it follows easily from the Monotone Convergence Theorem that the right hand side of the above inequality converges as $\varepsilon \to 0$ to the analogous expression with u_ε replaced by u, which is necessarily finite in view of the finiteness of $\|N(X^k u)\|_p$. It thus follows from Fatou's Lemma that

$$\|S^0(X_0^{k+1} u)\|_{L^p(\partial D_0)} < \infty.$$

Since X_0 agrees with X in a neighborhood of x_0, it follows that $(S(X^{k+1} u))^p$ is integrable in a neighborhood of x_0. Thus, we have shown that $(S(X^{k+1} u))^p$ is locally integrable on ∂D. Since ∂D is compact, it follows that $S(X^{k+1} u) \in L^p(\partial D)$, and the proof is complete.

We are now prepared to formulate and prove our main result.

(3.43) THEOREM. *Let D be a bounded domain in $\mathbf{R}^N$ with C^2 boundary, and let X be a C^{k+1} vector field in a neighborhood of ∂D. Let u be a harmonic function on D such that $N(X^k u) \in L^p(\partial D)$ for some $0 < p < \infty$. Then $N(\mathscr{D}^k u) \in L^p(\partial D)$. Moreover, there is a positive constant C and a compact subset K_1 of D such that*

$$\|N(\mathscr{D}^k u)\|_p \leqslant C\,[\|N(X^k u)\|_p + \|u\|_{K_1}]$$

for every harmonic function u on D.

Proof. By Lemmas (3.18), (3.23), and (3.36), $S(\mathscr{D}^{k+1} u) \in L^p(\partial D)$ and

$$\|S(\mathscr{D}^{k+1} u)\|_p \leqslant C\,[\|N(X^k u)\|_p + \|\mathscr{D}^{2k} u\|_K]. \tag{3.44}$$

Thus, since all the components of $\mathscr{D}^{k+1} u$ are harmonic, it follows from Lemma (2.2) that $\partial^\alpha u \in H^p(D)$ for every multi-index α with $|\alpha| \leqslant k$, and that

$$\|N(\partial^\alpha u)\|_p \leqslant C\,\|S(\mathscr{D}\,\partial^\alpha u)\|_p.$$

Combining this estimate with (3.44) gives

$$\|N(\mathscr{D}^k u)\|_p \leqslant C\,[\|N(X^k u)\|_p + \|\mathscr{D}^{2k} u\|_K].$$

But if K_1 is any compact subset of D containing K in its interior, then

$$\|\mathscr{D}^{2k} u\|_K \leqslant C\,\|u\|_{K_1}$$

for any harmonic function u on D, so the result follows.

There is also a local version of Theorem (3.43), which we will need in the next section.

(3.45) COROLLARY. *Let D be a domain in $\mathbf{R}^N$ with C^2 boundary, and let V and V_0 be neighborhoods of $0 \in \partial D$ with $V_0 \Subset V$. Let k be a nonnegative integer, and let X_0 be a C^{k+1} vector field on V which is transverse to ∂D at each point of $\partial D \cap V$. Then for any harmonic function u on $D \cap V$ such that $N(X_0^k u) \in L^p(\partial D \cap V)$ for some $0 < p < \infty$, we have $N(\mathscr{D}^k u) \in L^p(\partial D \cap V_0)$. Moreover, there is a compact subset K_0 of $V \cap D$ such that*

$$\|\mathscr{D}^k u\|_{L^p(\partial D \cap V_0)} \leqslant C\,(\|X^k u\|_{L^p(\partial D \cap V)} + \|u\|_{K_0})$$

for every harmonic function u on $D \cap V$.

Proof. Choose an open set V_1 with $V_0 \Subset V_1 \Subset V$, and a bounded domain D_0 with C^2 boundary such that $D_0 \subset D \cap V_1$ and $0 \in \partial D_0 \cap V_0 \subset \partial D$. Choose also a C^{k+1} vector field X_1 in a neighborhood of ∂D_0 such that X_1 is everywhere transverse to ∂D_0 and such that X_1 agrees with X_0 in a neighboorhood of $\partial D \cap V_0$. It is clear that our hypothesis implies that $N^0(X_1^k u) \in L^p(\partial D_0)$, where N^0 is a nontangential maximal operator for the domain D_0, and that

$$\|N^0(X_1^k u)\|_{L^p(\partial D_0)} \leqslant C\,[\|N(X_0^k u)\|_{L^p(\partial D \cap V_1)} + \|u\|_L],$$

where L is some fixed compact subset of $V \cap D$. Thus applying Theorem (3.43) in the domain D_0 gives the desired result.

4. Pluriharmonic conjugation. In this section, we show that, for C^2 domains in $\mathbf{C}^n$, pluriharmonic conjugation is a continuous operation on H^p for $0 < p < \infty$. In the case $1 < p < \infty$, this result is due to Stout [10], with a different proof.

(4.1) THEOREM. *Let D be a bounded, connected open set in $\mathbf{C}^n$ with C^2 boundary which contains the origin, and let f be a holomorphic function in D such that* $\operatorname{Re} f \in H^p(D)$ *for some* $0 < p < \infty$. *Then* $f \in H^p(D)$, and

$$\|Nf\|_p \leqslant C_p [\|N(\operatorname{Re} f)\|_p + |\operatorname{Im} f(0)|].$$

Proof. Let $\zeta \in \partial D$, and by a unitary change of coordinates, assume that $\partial/\partial x_n$ is the outward unit normal at ζ. (Here the coordinates in $\mathbf{C}^n$ are $z_j = x_j + iy_j$ for $1 \leqslant j \leqslant n$.) By integrating f with respect to z_n, we can find a neighborhood V of ζ and a holomorphic function F on $V \cap D$ such that $\partial F/\partial z_n = f$ and such that for any compact set L in D there is a compact set L' in D such that

$$\|F\|_{L \cap V} \leqslant C \|f\|_{L'} \tag{4.2}$$

with a constant C independent of f. Thus, if $u = \operatorname{Re} F$, then, by the Cauchy–Riemann equations,

$$f = \frac{\partial u}{\partial x_n} - i \frac{\partial u}{\partial y_n}.$$

By shrinking V, we may assume that $\partial/\partial x_n$ is transverse to ∂D in V, so if V_0 is a neighborhood of ζ with $V_0 \Subset V$, it follows from Corollary (3.45) that

$$\begin{aligned}\|N(\operatorname{Im} f)\|_{L^p(V_0 \cap \partial D)} &\leqslant C [\|N(\operatorname{Re} f)\|_{L^p(V \cap \partial D)} + \|F\|_{K_0}] \\ &\leqslant C [\|N(\operatorname{Re} f)\|_{L^p(V \cap \partial D)} + \|f\|_{K_0'}]\end{aligned}$$

where K_0 is some compact subset of $D \cap V$, and K_0' is as in (4.2). Since ∂D is compact, it follows that

$$\|N(\operatorname{Im} f)\|_p \leqslant C [\|N(\operatorname{Re} f)\|_p + \|f\|_{K_1}]$$

for some compact subset K_1 of D. But since $\operatorname{Re} f$ is harmonic on D, it follows that

$$\|\operatorname{Re} f\|_{K_1} \leqslant C \|N(\operatorname{Re} f)\|_p$$

and thus

$$\begin{aligned}\|N(\operatorname{Im} f)\|_p &\leqslant C [\|N(\operatorname{Re} f)\|_p + \|\operatorname{Im} f\|_{K_1}] \\ &\leqslant C [\|N(\operatorname{Re} f)\|_p + |\operatorname{Im} f(0)| + \|\operatorname{Im} f - \operatorname{Im} f(0)\|_{K_1}].\end{aligned} \tag{4.3}$$

Let K_2 be a compact subset of D with connected interior containing $K_1 \cup \{0\}$ in its interior. Then

$$\|\operatorname{Im} f - \operatorname{Im} f(0)\|_{K_1} \leqslant C \|\nabla(\operatorname{Im} f)\|_{K_2} = C \|\nabla(\operatorname{Re} f)\|_{K_2}.$$

But since $\operatorname{Re} f$ is harmonic in D, the last term on the right is bounded by a constant multiple of $\|N(\operatorname{Re} f)\|_p$, so the theorem follows from (4.3).

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Added in proof (November 1990). It has been pointed out to me by J. Bruna that Theorem (4.1) on pluriharmonic conjugation can also be deduced directly from the area integral characterization of H^p. If $f = u + iv$ is holomorphic on D and $u \in H^p(D)$, then $S(\nabla u) \in L^p(\partial D)$. But it follows from the Cauchy–Riemann equations that $S(\nabla v) = S(\nabla u)$, and so $v \in H^p(D)$.