Boundary estimates for derivatives of harmonic functions

by

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Abstract. Boundary $L^p$ estimates are obtained for maximal functions of gradients of harmonic functions in terms of $L^p$ norms of derivatives with respect to a fixed transverse vector field. As an application, it is shown that, for functions of several complex variables, pluriharmonic conjugation is a continuous operation on $H^p$ for $0 < p < \infty$.

1. Introduction. It is well known that if $u$ is a harmonic function in the unit disk such that a nontangential maximal function of $u$ is an $L^p$ function on the unit circle, then $u$ is the real part of a holomorphic function in the Hardy class $H^p$ on the unit disk. For $1 < p < \infty$, this is equivalent to a classical result of M. Riesz [6], and for $0 < p \leq 1$ it is due to Burkholder, Gundy, and Silverstein [2]. Higher dimensional analogues of this result in the setting of a half-space may be found in [7] and [5]. In this paper, we obtain an analogue for domains in (real) Euclidean space of arbitrary dimension.

We begin with a reformulation of the conjugation result in the disk which will be meaningful in the higher dimensional setting. Let $X$ denote the radial vector field in the disk defined by

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$ 

Then the above result in the disk is easily seen to be equivalent to the following: If $u$ is a harmonic function in the disk such that a nontangential maximal function of $Xu$ is an $L^p$ function on the unit circle, then any nontangential maximal functions of $\partial u/\partial x$ and $\partial u/\partial y$ are also $L^p$ functions on the unit circle. This formulation has a natural analogue for domains in $\mathbb{R}^n$ in which the radial vector field $X$ is replaced by a smooth vector field which is everywhere transverse to the boundary. Indeed, this higher dimensional analogue is the principal result of this paper (Theorem (3.43)).

The half-space analogue of our main result is well known, and amounts to the boundedness of the Riesz transforms on $L^p$ when $1 < p < \infty$, and on $H^p$


Key words and phrases: Hardy space, conjugation, maximal function, area integral.
when \(0 < p \leq 1\). (See \([5]\) and \([7]\).) We shall draw heavily on ideas from \([5]\) and \([7]\), but in our case the arguments are complicated by the fact that differentiation with respect to a transverse field does not preserve harmonicity.

Results analogous to ours with volume \(L^p\) norms have been obtained previously by Detraz \([4]\) and by Boas and Straube \([1]\). The case considered here is more delicate than the volume case.

The paper is organized as follows. In Section 2, we collect some definitions and recall some well-known results concerning maximal functions and area integrals. In Section 3 we prove a series of lemmas, culminating in our main result (Theorem (3.43)). Finally, in Section 4 we give an application to pluriharmonic conjugation, extending a result of Stout \([10]\).

Throughout the paper, we shall use the symbol \(C\) to denote various positive constants which are independent of the relevant parameters in the expression in which they occur.

2. Preliminaries. Throughout this paper, \(D\) will denote a fixed bounded domain in \(\mathbb{R}^N\) with \(C^2\) boundary, and \(X\) will denote a \(C^1\) vector field in a neighborhood of \(\partial D\) which is everywhere transverse to \(\partial D\). More precisely, \(D\) is assumed to be a bounded set in \(\mathbb{R}^N\) of the form \(D = \{x \in \mathbb{R}^N; \quad q(x) < 0\}\) where \(q\) is a function on \(\mathbb{R}^N\) with \(q(0) = 0\). The vector field \(X\) is assumed to be of the form \(X = \sum_{j=1}^{N} a_j \partial_j\), where each coefficient \(a_j\) is a \(C^1\) function in a neighborhood of \(\partial D\), and \(X(x) \neq 0\) for every \(x \in \partial D\). We shall denote by \(\sigma\) the Euclidean surface measure on \(\partial D\), and we denote the norm in \(L^p(\partial D; \partial\sigma)\) by \(\|u\|_p\). Also, for any function \(u\) on a set \(K\), we define \(\|u\|_p^K = \sup\{\|u(x)\| : x \in K\}\).

For \(1 \leq j \leq N\), we shall denote partial differentiation with respect to the \(j\)th variable in \(\mathbb{R}^N\) by \(\partial_j\). Similarly, for any multi-index \(\alpha = (\alpha_1, \ldots, \alpha_N)\), let \(\partial_{\alpha} = \partial_{\alpha_1} \cdots \partial_{\alpha_N}\). For any nonnegative integer \(k\) and any \(C^k\) function \(u\), we shall denote by \(\partial_{\alpha}^k u\) the vector \(\partial_{\alpha}^k u\), where \(\alpha\) varies over all multi-indices of weight at most \(k\).

We choose and fix a neighborhood \(U\) of \(\partial D\) and a \(C^1\) diffeomorphism \(\Phi: \overline{U} \to \partial D \times [-2, 2]\) with the following properties:

(i) \(\Phi^{-1}(\partial D \times (0, 2]) = \overline{U} \cap \overline{\partial D}\) and \(\Phi^{-1}(\partial D \times [-2, 0)) = \overline{U} \setminus \partial D\);

(ii) \(\Phi(x) = (x, 0)\) for any \(x \in \partial D\);

(iii) in each connected component of \(\overline{U}\) the vector field \(\Phi_* X\) is a constant multiple of \(\partial_t\), where \(t\) denotes the second coordinate of a point in \(\partial D \times [-2, 2]\).

For any \(x \in \partial D\) and any \(\alpha > 0\), we denote by \(I_{\alpha}^r(x)\) the truncated cone in \(\partial D \times (0, 1]\) defined by \(I_{\alpha}^r(x) = \{(y, t) \in \partial D \times (0, 1]; \quad |y| < \alpha t\}\), and we denote the preimage of \(I_{\alpha}^r(x)\) under \(\Phi\) by \(I_{\alpha}^r(x)\). For any function \(u\) on \(\partial D \cap \overline{U}\), we shall denote the function \(u \circ \Phi^{-1}\) on \(\partial D \times (0, 2]\) by \(\tilde{u}\), and we shall denote the product measure \(\partial D \times dt\) on \(\partial D \times [0, 2]\) by \(d\tilde{V}\). Finally, we let \(U_0 = \Phi^{-1}(\partial D \times (-1, 1))\), and \(K = D \setminus U_0\).

It will be convenient to have a defining function for \(D\) which is harmonic near \(\partial D\). We choose a point \(x_0 \in \text{int} K\), and denote by \(\delta\) the Green function for \(D\) with singularity \(x_0\). Thus, \(\delta\) is harmonic in \(D \setminus \{x_0\}\) and \(\delta(x, t) = \alpha(t)\) is comparable with \(t\) on \(\partial D \times (0, 1]\).

For any function \(u\) on \(D \cap U_0\) and any \(\alpha > 0\), we define the nontangential maximal function of \(u\) by

\[N_u(x) = \sup \{ |u(y)| : y \in I_{\alpha}^r(x) \}\]

whenever \(x \in \partial D\). Thus, \(N_u\) is a lower semicontinuous function on \(\partial D\). We define the radial maximal function by

\[N_{\alpha} u(x) = \sup \{|u(x, t)| : 0 < t \leq 1\}\]

for \(x \in \partial D\). If \(u\) is continuously differentiable on \(U \cap D\), then we define the Littlewood–Paley \(g\)-function by

\[g_u(x) = \left( \int_0^1 |\tilde{u}(x, t)|^2 \, dt \right)^{1/2}\]

For \(q > 0\), we denote the \(q\)-area integral by

\[S_{\alpha}^q u(x) = \int_{I_{\alpha}^r} |\tilde{u}(y, t)|^q \, dt^{-N} \, d\tilde{V} \, dt^{1/2}\]

where, once again, \(x \in \partial D\). In all of the above notations, we shall often drop the subscript \(\alpha\) when the aperture is 1. We shall also denote the classical area integral \(S_{\alpha}^1 u\) by the simpler notation \(S u\). We will use the above operators for both scalar and vector-valued functions. When \(u\) is vector-valued, \(|u|\) is to be interpreted as the Euclidean norm of \(u\).

We will need to compare \(L^p(\partial D)\) norms of each of the above quantities. These comparisons are facilitated by the fact that finiteness of \(\|N_u\|_p\) and \(\|S u\|_p\) is independent of the aperture \(\alpha\).

(2.1) Lemma. Let \(0 < p < \infty\), and let \(a, b > 0\). There are constants \(C = C(a, b, p)\) and \(C' = C(a, b, p)\) such that

(i) \(\|N u\|_p \leq C \|N_a u\|_p\) for every function \(u\) on \(D\);

(ii) \(\|S^1 u\|_p \leq C \|S_{\alpha}^1 u\|_p\) for every measurable function \(u\) on \(D\).

For functions in half-spaces, parts (i) and (ii) of the above lemma appear in \([5, \text{p. 166}]\) and \([3, \text{p. 309}]\) respectively. The proof in the present context is identical to those appearing in \([3]\) and \([5]\) and will be omitted.

We are now in a position to define the harmonic Hardy spaces on the domain \(D\). For \(0 < p \leq \infty\), we define \(H^p = H^p(D)\) to consist of all harmonic functions on \(D\) such that \(N u \in L^p(\partial D)\) for some (and hence every) \(\alpha > 0\). With the “norm” defined by

\[\|u\|_{H^p} = \|N u\|_p,

\]
for any nonnegative integer $i < k$ and any $0 < a < b$. The lemma will follow from (3.2) by a simple inductive argument.

For $z = \Phi^{-1}(x, \tau)$ and $w = \Phi^{-1}(y, \tau)$ in $U$ with $|z-w| < \epsilon \tau$, we have, since $\Phi$ is a diffeomorphism,

$$|x-y| + |y-z| < c_1 \epsilon \tau,$$

where $c_1$ is a constant depending only on $\Phi$ and thus

$$(1-c_1 \epsilon) \tau < \tau < (1+c_1 \epsilon) \tau.$$  

It follows that if $(x, \tau) \in \Gamma_\alpha(x_0)$ for some $x_0 \in \partial D$, and if $(y, \tau)$ is as above with $0 < \epsilon < c_1^{-1}$, then

$$|y-x|_0 \leq |x-x_0| + |x-y| < a \tau + c_1 \epsilon \tau < \frac{a+c_1 \epsilon}{1-c_1 \epsilon}.$$  

We choose and fix $0 < \epsilon < c_1^{-1}$ sufficiently small that $(a+c_1 \epsilon)/(1-c_1 \epsilon) < b$ so that, with $(x, \tau)$ and $(y, \tau)$ as above,

$$|y-x|_0 < b.$$  

For any $(x, \tau) = \Phi(z) \in \partial D \times (0, 1)$, let $B(x, \tau) = \{w \in \mathbb{R}^3 : |w-z| < \epsilon \tau\}$, and let $\tilde{B}(x, \tau) = \Phi[B(x, \tau)]$. It thus follows that $B(x, \tau) \subseteq \Gamma_\alpha(x_0) \cup K$ whenever $(x, \tau) \in \tilde{\Gamma}_\alpha(x_0)$.

We next fix a boundary point, which we assume for simplicity to be the origin, and let $(x, \tau) \in \tilde{\Gamma}_\alpha(0)$. Then for $1 \leq j \leq N,$

$$|X^{i_j} \partial_j u|^{-} (x, \tau) = \frac{1}{t} \int_i^t \partial_i [X^{i_j} \partial_j u]^{-} (x, \tau) d\tau + [X^{i_j} \partial_j u]^{-} (x, 1)$$

$$= C \int \frac{[X^{i_j} \partial_j u]^{-} (x, \tau) d\tau + [X^{i_j} \partial_j u]^{-} (x, 1)}{t}.$$  

For any fixed $(x, \tau) \in \tilde{\Gamma}_\alpha(0)$, let $L$ denote the constant coefficient differential operator obtained by freezing the coefficients of $X^{i_j} \partial_j$ at $\Phi^{-1}(x, \tau)$. Then $L$ commutes with each $\partial_j$, and $Lu$ is harmonic in $D$, so it follows that

$$|[X^{i_j} \partial_j u]^{-} (x, \tau)|^2 \leq |L \partial_j u (\Phi^{-1}(x, \tau))^2| = |\partial_j (L \partial_j (\Phi^{-1}(x, \tau)))|^2$$

$$\leq C \frac{1}{t^2} \int_{B(x, \tau)} |Lu|^2 dV$$

$$\leq C \frac{1}{t^2} \int_{B(x, \tau)} |X^{i_j} \partial_j u|^2 dV + \frac{C}{t^{2+\tau}} \int_{B(x, \tau)} |(L-X^{i_j} \partial_j) u|^2 dV.$$  

But since $i+1$ is at most $k$, the coefficients of $X^{i+1}$ satisfy a Lipschitz condition of order $\eta_i$, so the integrand in the last integral on the right is bounded by a constant multiple of $\tau^{2+\eta_i} |\partial^{i+1} u|^2$, which yields the estimate.
\[
\|X^{i+1} \partial_j u\|_{\mathcal{D}}^2 \leq C \int_S \|X^\perp u\|_2^2 d\mathcal{V} + C \int_S \|\partial Y u\|_2^2 d\mathcal{V}.
\]

Since the Jacobian of \(F\) is bounded, the integrals on the right hand side may be replaced by the corresponding integrals over the regions \(\mathcal{B}(x, \tau)\) in \(\partial D \times (0, 2)\). But in view of (3.4) and (3.5), \(\mathcal{B}(x, \tau)\) is contained in the strip
\[
S = \{(x, \tau) \in \partial D \times (0, 2) : |x| < bt \text{ and } (1 - c_1 \tau) \tau < \tau < (1 + c_1 \tau)\},
\]
where \(c_1\) is as in (3.3), and therefore
\[
\|X^{i+1} \partial_j u\|_{\mathcal{D}}^2 \leq C \int_{S_{\tau}} \|X^\perp u\|_2^2 d\mathcal{V} + C \int_{S_{\tau}} \|\partial Y u\|_2^2 d\mathcal{V} = I_1(\tau) + I_2(\tau).
\]

Combining this estimate with (3.6) yields
\[
\|X \partial_j u\|_{\mathcal{D}} \leq C \left[ \frac{1}{r} I_1(\tau)^{1/2} + I_2(\tau)^{1/2} \right] dt + \|\partial Y u\|_2 k,
\]
Squaring both sides, multiplying by \(r^{2-N}\), and integrating over \(\Gamma_\infty(0)\) gives
\[
(S^g(X \partial_j u)(0))^2 \leq C \sum_{j=1}^2 \left[ \frac{1}{r} I_1(\tau)^{1/2} dt \right] + \|\partial Y u\|_2 k.
\]

The two integrals on the right of (3.7) can be estimated by Hardy's inequality. Letting \(c_2\) denote the characteristic function of the strip \(S_{\tau} \cap \Gamma_\infty(0)\), we have
\[
\int_0^{r^{1/2}} \left( \frac{1}{r} I_1(\tau)^{1/2} dt \right) dt \leq C \left[ \frac{1}{r} I_1(\tau)^{1/2} \right] d\mathcal{V} dt + \|\partial Y u\|_2 k,
\]

\[
= C \left[ \frac{1}{r} I_1(\tau)^{1/2} \right] \int_{S_{r^{-1/2}}} \|X^\perp u\|_2^2 d\mathcal{V} dt + \|\partial Y u\|_2 k,
\]

\[
\leq C \left[ \int_{\mathcal{B}(x, \tau)} \|X^\perp u\|_2^2 d\mathcal{V} dt + \|\partial Y u\|_2 k \right],
\]

\[
\leq C \left[ \int_{\mathcal{B}(x, \tau)} \|X^\perp u\|_2^2 d\mathcal{V} dt + \|\partial Y u\|_2 k \right],
\]

\[
= C \left[ \|X^\perp u\|_2^2 + \|\partial Y u\|_2 k \right],
\]

Here \(c_2 = (1 + c_1 \delta)^{-1}\) and \(c_3 = (1 - c_1 \delta)^{-1}\) with \(c_1\) as in (3.3). The same technique may be used to estimate the second integral on the right in (3.7). The result is
\[
\int_0^{r^{1/2}} \left( \frac{1}{r} I_2(\tau)^{1/2} dt \right) dt \leq C \left[ \|S^g Y u\|_2^2 + \|\partial Y u\|_2 k \right].
\]

Inserting these last two estimates into (3.7) gives (3.2).

Applying (3.2) inductively, with \(a\) and \(b\) replaced by suitable intermediate values, gives
\[
S^g(X \partial_j u)(0) \leq C_{a,b,r} \left[ \sum_{j=0}^{k-1} \|X^\perp u\|_2^2 \right] + \|\partial Y u\|_2 k.
\]

Let \(0 < a < b\), and let \(r\) be a positive integer. Choose \(a < a_0 < a_1 < \ldots < a_t = b\). By applying (3.8) with \(a\) and \(b\) replaced by \(a_i\) and \(a_{i+1}\) for \(0 < i \leq r - 1\), we obtain
\[
S^g(\partial Y u)(0) \leq C_{a,b,r} \left[ \sum_{j=0}^{k-1} \|X^\perp u\|_2^2 \right] + \|\partial Y u\|_2 k.
\]

But each term in the sum on the right is dominated by \(S^g(X \partial_j u)(0)\), so, since \(r\) is at our disposal, it follows that for any \(M > 0\),
\[
S^g(\partial Y u)(0) \leq C_{a,b,r,M} \left[ \sum_{j=0}^{k-1} \|X^\perp u\|_2^2 \right] + \|\partial Y u\|_2 k.
\]

For any \((x, \tau) \in \Gamma_\infty(0)\), the radius of the ball \(B(x, \tau)\) is comparable with \(\tau\), so we have
\[
|\partial Y u(\Phi^{-1}(x, \tau))|^2 \leq C \tau^{-2k-N} \int_{B(x, \tau)} |u|^2 d\mathcal{V} \leq C \tau^{-2k-N} \int |u(\bar{y}, \tau')|^2 d\bar{\mathcal{V}}(y, \tau).
\]

Since, by (3.4), \(\tau\) is comparable with \(r\) in \(\mathcal{B}(x, \tau)\), it follows that
\[
|\partial Y u(\Phi^{-1}(x, \tau))|^2 \leq C \tau^{-2k-N} \int |u(\bar{y}, \tau')|^2 d\bar{\mathcal{V}}(y, \tau) \leq C_{a,b} \left[ \tau^{-2k-N} \right] \leq C_{a,b} \left[ \tau^{-2k-N} \right],
\]

Multiplying by \(\tau^{-N+2k+1}\) and integrating over \(\Gamma_\infty(0)\) gives
\[
S^g(\partial Y u)(0) \leq C_{a,b,r} \left[ \sum_{j=0}^{k-1} \|X^\perp u\|_2^2 \right] + \|\partial Y u\|_2 k.
\]

whenever \(0 < a < b\). For any \(0 < a < b\), choose \(b'\) between \(a\) and \(b\). Applying (3.9) with \(b\) replaced by \(b'\) and \(M = 2k+1\), and (3.10) with \(a\) replaced by \(b'\) gives
\[
S^g(\partial Y u)(0) \leq C_{a,b,r} \left[ \sum_{j=0}^{k-1} \|X^\perp u\|_2^2 \right] + \|\partial Y u\|_2 k.
\]

But since \(u\) can be recovered from \(X^\perp u\) by \(k\)-fold integration, the second term on the right of (3.11) can be absorbed by the first and the last, and the lemma is proved.

(3.12) COROLLARY. With \(X\) and \(k\) as above, there is a constant \(C\) such that for any harmonic function \(u\) on \(D\)
\[
g(\partial Y u) \leq C \left[ \|X^\perp u\|_2^2 + \|\partial Y u\|_2 k \right].
\]
Proof. The case $k = 0$ is proved exactly as in the case of a half-space. (See [8, p. 90].) But applying the case $k = 0$ to the components of $\mathcal{D} u$ gives

$$g(\mathcal{D} u) \leq C(S(\mathcal{D} u) + ||\mathcal{D} u||_K)$$

and the result follows from Lemma (3.1).

(3.13) Lemma. Let $X$ and $k$ be as in Lemma (3.1). For any $0 < a < b$ there is a constant $C$ such that for any harmonic function $u$ on $D$ and any $x \in \partial D$

$$N_a(\delta \mathcal{D}^{k+1} u)(x) \leq C [N_b(X^k u)(x) + ||\mathcal{D}^{2k} u||_K].$$

Proof. We fix a point $\partial D$, which we assume is the origin. For $(x, t) \in \bar{B}_a(0)$ we have

$$[X^k \mathcal{D}^{k+1} u]^{-}(x, t) = C \int_{t_1}^{t} \int_{u_{t_1}}^{u_t} [X^k \mathcal{D}^{k+1} u]^{-}(x, t_1) \, dt_1 \, dx_1 + \delta$$

where $\delta \in \mathcal{E}(x, t)$ satisfies $|\delta| \leq C||\mathcal{D}^{2k} u||_K$. For any fixed $x$ and $t$, let $L(u, x)$ denote the constant coefficient operator obtained by freezing the coefficients of $X^k$ at $\Phi^{-1}(x, t)$. Then, letting $B(x, t)$ be as in the proof of Lemma (3.1),

$$[[X^k \mathcal{D}^{k+1} u]^{-}(x, t)] = [[\mathcal{D}^{k+1} L(u, x) u]^{-}(x, t)] \leq C \frac{1}{x-2} ||L(u, x)||_{B(x, t)}$$

$$\leq C \frac{1}{x-2} ||X^k u||_{B(x, t)} + ||[L(u, x) - X^k u)||_{B(x, t)}.$$

But the coefficients of $X^k$ are Lipschitz of order $\eta$, so we obtain

$$[[X^k \mathcal{D}^{k+1} u]^{-}(x, t)] \leq C [\tau^{-k+1} ||X^k u||_{B(x, t)} + \tau^{-k+1} ||\mathcal{D}^{2k} u||_{B(x, t)}].$$

For $0 < \alpha < 1$ and $\alpha > 0$, let

$$\bar{F}_a^{(0)}(x) = \{ (x, t) \in \bar{F}_a(x) : t > a \}, \quad \bar{F}_a^{(0)}(x) = \Phi^{-1}[\bar{F}_a^{(0)}(x)].$$

Then for $(x, t) \in \bar{F}_a^{(0)}(0)$ and $t \leq \tau \leq 1$, we have $B(x, t) = \tau^{\alpha} \mathcal{E}(0) \cup K$, where $0 < \alpha < 1$ and $x$ depends only on $a$ and $b$. Thus, letting

$$N^{\alpha}_a(x) = \sup \{ |u| : x \in \bar{F}_a^{(0)}(x) \},$$

it follows from (3.15) that

$$[[X^k \mathcal{D}^{k+1} u]^{-}(x, t)] \leq C_{a, b} [\tau^{-k+1} N_b(X^k u)(0) + \tau^{-k+1} \alpha N^{\alpha}_a(\mathcal{D} u)(0)].$$

Inserting this estimate in (3.14) and evaluating the resulting integral gives

$$[[\mathcal{D}^{k+1} u]^{-}(x, t)] \leq C \frac{1}{x} [N_b(X^k u)(0) + ||\mathcal{D}^{2k} u||_K]$$

and hence, since $t$ is comparable with $\delta$,

$$[[\mathcal{D}^{k+1} u]^{-}(x, t)] \leq C \frac{1}{x} [N_b(X^k u)(0) + ||\mathcal{D}^{2k} u||_K].$$

(3.17) $N_a(\delta \mathcal{D}^{k+1} u)(0) \leq C_{a, b} [N_b(X^k u)(0) + N_b(\delta \mathcal{D}^{k} u)(0) + ||\mathcal{D}^{2k} u||_K].$

To complete the proof, we must argue that the second term on the right of (3.17) can be absorbed by the first and the last. The argument which led to (3.16) gives, for any $(x, t) \in \bar{F}_a(0)$ and any $b > a$, the estimate

$$[[\mathcal{D}^{k} u]^{-}(x, t)] \leq C_{a, b} [\log(1/t) N_b(X^k u)(0) + t^a N^{\alpha}_a(\mathcal{D} u)(0) + ||\mathcal{D}^{2k+1} u||_K],$$

from which it follows that, for any $\beta > 0$,

$$N_a(\delta \mathcal{D}^{k} u)(0) \leq C_{a, b, \beta} [N_b(X^k u)(0) + N_a(\delta \mathcal{D}^{k+1} u)(0) + ||\mathcal{D}^{2k+1} u||_K].$$

Iteration of this last estimate gives, for any $M > 0$,

$$N_a(\delta \mathcal{D}^{M} u)(0) \leq C_{a, b, \beta, M} [N_b(X^k u)(0) + N_a(\delta \mathcal{D}^{M+1} u)(0) + ||\mathcal{D}^{2k+1} u||_K].$$

On the other hand, we also have the elementary inequalities

$$N_a(\delta \mathcal{D}^{k} u)(0) \leq C_{a, b} (N_b(X^k u)(0) + ||\mathcal{D}^{2k} u||_K)$$

for any $0 < a < b$, and

$$N_a u(0) \leq N_b(X^k u)(0) + C ||\mathcal{D}^{2k-1} u||_K.$$

Combining these last three inequalities gives

$$N_a(\delta \mathcal{D}^{k} u)(0) \leq C_{a, b, \beta} [N_b(X^k u)(0) + ||\mathcal{D}^{2k} u||_K],$$

which, in view of (3.17), completes the proof.

The remainder of this section will be devoted to demonstrating that the $L^p$ norms of area integrals of gradients are dominated by $L^p$ norms of maximal functions of transverse derivatives. We will consider separately the cases $0 < p < 2$, $p = 2$, and $2 < p < \infty$. We begin with the simplest case, $p = 2$.

(3.18) Lemma. Let $k$ be a nonnegative integer, and assume that the vector field $X$ is of class $C^{k+1}$. There is a constant $C$ such that $||S(\mathcal{D}^{k+1} u)||_2 \leq C ||S(X^k u)||_2 + ||\mathcal{D}^{2k} u||_K$ for any harmonic function $u$ on $D$.

Proof. There is no loss of generality in assuming that $u$ is real-valued and it will be convenient to do so. By Lemma (3.1), it suffices to estimate $||S(\mathcal{D} u)||_2$. We have

$$\int_{\partial D} \{ S(\mathcal{D} u)^2 \} \, d\sigma = \int_{\partial D} \{ [\mathcal{D} u]^2 \}^{-1} \, t^{-2} \, d\sigma \quad \text{and} \quad \int_{\partial D} \{ \phi \}^{-1} \, d\sigma \leq C \int_{\partial D} \{ \mathcal{D} u \}^2 \, d\sigma \quad \text{for any} \, \phi \in \mathcal{D}^{k+1}.$$
It will be convenient to extend $X$ to be a $C^{k+1}$ vector field in a neighborhood of $\bar{D}$ which vanishes in a neighborhood of the singularity of $\delta$. (Recall that $\delta$ was chosen to be a Green function for $D$.) Thus
\[
\int_{\partial D} (S(PX^k u))^2 \, d\sigma \leq C \int_{\bar{D}} |PX^k u|^2 \, dV = C \lim_{\varepsilon \to 0^+} \int_{\partial D} |PX^k u|^2 \, dV
\]
where $D_\varepsilon = K \cup \Phi^{-1}[\partial D \times (\varepsilon, 1)]$. Using the identity
\[
\Delta (U^2) = 2|U|^2 + 2U \Delta U,
\]
we obtain
\[
\int_{\partial D} (S(PX^k u))^2 \, d\sigma \leq C \left[ \lim_{\varepsilon \to 0^-} \int_{\partial D} \Delta \left((X^k u)^2\right) \, dV \right] + \int_{\bar{D}} |X^k u| |\Delta X^k u| \, dV.
\]
But by Green's Formula,
\[
\int_{\partial D} \Delta \left((X^k u)^2\right) \, dV = \int_{\bar{D}} \partial_{D_{\varepsilon}} \left((X^k u)^2\right) \, d\sigma + \int_{\partial D_\varepsilon} (X^k u)^2 \, d\sigma \delta_{\varepsilon},
\]
where $d\sigma_{\varepsilon}$ is the Euclidean volume element on $\partial D_{\varepsilon}$, and $\delta_{\varepsilon}$ denotes outward normal differentiation along $\partial D_{\varepsilon}$. Thus
\[
\int_{\partial D} \Delta \left((X^k u)^2\right) \, dV \leq C \left[ \int_{\bar{D}} N_0(X^k u) N_0 \left(\Theta^{k+1} u\right) \, d\sigma + \int_{\partial D} \left((N(X^k u))^2\right) \, d\sigma \right].
\]
It follows from Lemma (3.13) that
\[
\int_{\partial D} \Delta \left((X^k u)^2\right) \, dV \leq C \left[ \int_{\partial D} |N(X^k u)|^2 \, d\sigma + |\Theta^{2k} u|^2 \right].
\]
As for the second term on the right of (3.20), note that, since $u$ is harmonic, $\Delta X^k u = [A, X^k] u$, where $[A, X^k]$ denotes the commutator of the operators $A$ and $X^k$, which is of order $k+1$. It thus follows that
\[
\int_{\partial D} |X^k u| |\Delta X^k u| \, dV \leq C \left[ \int_{\partial D} \left|\Delta (X^k u)^2\right| + \left|\Theta^{2k} u\right|^2 \right].
\]
Inserting (3.21) and (3.22) into (3.20) gives
\[
|S(\Theta^{k+1} u)|_p \leq C \left[ \int_{\partial D} |N(X^k u)|^2 \right]^\frac{1}{2} + |\Theta^j u|_{L^2},
\]
with $J = \max\{1, 2k\}$. This is the required estimate if $k > 0$. On the other hand, if $k = 0$, and if $|N(u)| < \infty$, then $u$ is the Poisson integral of a function in $L^p(\partial D)$, and therefore the last term on the right of the above inequality can be absorbed into the first, and again the required estimate obtains.

(3.23) Lemma. Assume that $X$ is of class $C^{k+1}$. If $u$ is harmonic in $D$, and if $N(X^k u) \in L^p(\partial D)$ for some $0 < p < 2$, then $S(\Theta^{k+1} u) \in L^p(\partial D)$ and
\[
\|S(\Theta^{k+1} u)\|_p \leq C_p \left[ \int_{\partial D} |N(X^k u)|^p \right]^\frac{1}{p} + |\Theta^{2k} u|_{L^2}.
\]

Proof. As in the proof of Lemma (3.18), it will be convenient to assume that $u$ is real-valued. By Lemma (3.1), it suffices to estimate $|S(PX^k u)|_p$. We will estimate the distribution function of $S(PX^k u)$ using a technique of Fefferman and Stein [5, pp. 162–163]. Let $a > 0$. Then according to Lemma (2.2), $\|N(X^k u)\|_p \leq C_p \|N(X^k u)\|_p$. Fix $a > 0$, and let $E = E_a = \{x \in \partial D: N(X^k u) < a\}$ and $F = F_a = \partial D \backslash E_a = \{x \in \partial D: N(X^k u) > a\}$. Then $E$ is closed by the lower semicontinuity of $N(X^k u)$. Let $R = R_a = \{x \in \partial D: x \in E\}$ and $\bar{R} = \Phi[R]$. It follows from Fubini's Theorem that
\[
\int_{\partial D} |S(PX^k u)|_p \, d\sigma \leq C \|\|PX^k u\|_p^2 \, dV \leq C \|PX^k u\|_p^2 \, dV.
\]
We will estimate the integral on the right by applying Green's Formula in a sequence of piecewise smooth domains which approximate $R$ from within. Choose a sequence $\phi_j$ of $C^2$ functions on $\partial D$ such that the tangential gradients of the $\phi_j$'s are bounded uniformly in $j$ and $\alpha$, and such that $\phi_j$ decreases to $\phi = \text{dist}(x, E)$ as $j \to \infty$. (The existence of such a sequence follows from the fact that $\phi$ is Lipschitz of order 1.) Let $\bar{R}_j = \{x, t \in \partial D: t > \phi_j(x)\}$. By (3.24) and the identity (3.19),
\[
\int_{\partial D} |S(PX^k u)|_p \, d\sigma \leq C \left[ \int_{\partial D} |N(X^k u)|^2 \, d\sigma \right]^\frac{1}{2} + C \left[ \int_{\partial D} |\Theta^{2k} u|^2 \, d\sigma \right]^\frac{1}{2}.
\]
But Green's Formula gives
\[
\int_{\partial D} \Delta \left((X^k u)^2\right) \, dV = \int_{\partial D} \partial_{D_{\varepsilon}} \left((X^k u)^2\right) \, d\sigma + \int_{\partial D} (X^k u)^2 \, d\sigma \delta_{\varepsilon},
\]
where $\delta_{\varepsilon}$ denotes the surface measure on $\partial D_{\varepsilon}$, and $\partial_{D_{\varepsilon}}$ denotes the outer normal derivative on $\partial D_{\varepsilon}$. To estimate these integrals, we split $\partial D_{\varepsilon}$ into three pieces $\partial D_{\varepsilon} = \partial D_{\varepsilon}^1 \cup \partial D_{\varepsilon}^2 \cup \partial D_{\varepsilon}^3$, where $\Phi[\partial D_{\varepsilon}] = \partial D_{\varepsilon}^1 \cup \partial D_{\varepsilon}^2 \cup \partial D_{\varepsilon}^3 \subset E(0, 1)$, and $\Phi[\partial D_{\varepsilon}] = F(0, 1)$. We will denote the image of $\partial D_{\varepsilon}$ under $\Phi$ by $\partial B_{\varepsilon}$.

We will now estimate the first term on the right of (3.26). For the integrand, we have the pointwise estimate
\[
|\partial_{\varepsilon} \left((X^k u)^2\right)| \leq C \|X^k u\|_{L^2} |\Theta^{k+1} u|_p,
\]
from which it follows trivially that
\[
\left| \int_{\partial D_{\varepsilon}^1} \partial_{\varepsilon} \left((X^k u)^2\right) \, d\sigma \right| \leq C \left[ \int_{\partial D_{\varepsilon}^1} |X^k u|^2 \left|\Theta^{k+1} u\right|_p \right]^\frac{1}{2}.
\]
For the integral over $B_j^+$, note that, in view of the fact that the functions $\varphi_j$ which define the regions $R_j$ have uniformly bounded derivatives, we have $d\sigma_j(x,t) \leq C\sigma(x)$ along $\overline{B_j^+} \cup B_j^+$, where $d\sigma$ is obtained by pulling back the measure $d\sigma_j$ by $\varphi^{-1}$. Moreover, by Lemma (13.3) we have

\begin{equation}
N(\delta B_{j+1}^+) \leq C_n [N_n(\delta B_{j+1}^+) + \|\delta B_{j+1}^+ u\|_2],
\end{equation}

so (3.27) implies

\begin{equation}
\left| \delta_+ (X_{ij} u)^2 \right| \leq C \left[ N(\delta B_{j+1}^+) \right] \leq C \left[ N_n(\delta B_{j+1}^+) \right] \leq C \left[ N_n(X_{ij} u)^2 + \|\delta B_{j+1}^+ u\|_2 \right].
\end{equation}

Thus we have the estimate

\begin{equation}
\left| \int_{B_j^+} \delta_+ (X_{ij} u)^2 \right| \leq C \left[ \int_{B_j^+} \left( N_n(X_{ij} u)^2 + \|\delta B_{j+1}^+ u\|_2 \right) \right].
\end{equation}

For the integral over $B_j^+$, note that for any $x \in \mathbb{R}$ we have, by (3.27) and (3.29),

\begin{equation}
\left| \delta_+ (X_{ij} u)^2 \right| \leq C \left[ \sup_{B_j^+} (N_n(X_{ij} u)^2 + \|\delta B_{j+1}^+ u\|_2) \right] \leq C \left[ x^2 + \|\delta B_{j+1}^+ u\|_2 \right].
\end{equation}

so

\begin{equation}
\left| \int_{B_j^+} \delta_+ (X_{ij} u)^2 \right| \leq C \left[ x^2 \sigma(F) + \|\delta B_{j+1}^+ u\|_2 \right].
\end{equation}

Combining the three estimates (3.28), (3.31), and (3.32) gives an estimate for the first integral on the right of (3.26):

\begin{equation}
\left| \int_{B_j^+} \delta_+ (X_{ij} u)^2 \right| \leq C \left[ \left( \int_{B_j^+} \delta_+ (X_{ij} u)^2 \right) \right].
\end{equation}

where $J = \max \{2k, 1\}$. A similar argument shows that the right hand side of (3.33) also dominates the remaining integral on the right of (3.26), and we thus have

\begin{equation}
\lim_{j \to \infty} \sup_{B_j^+} \left| \int_{B_j^+} \delta_+ (X_{ij} u)^2 \right| \leq C \left[ \left( \int_{B_j^+} \delta_+ (X_{ij} u)^2 \right) \right].
\end{equation}

We now turn our attention to the last term on the right of (3.25). Since $u$ is harmonic, $|A X_{ij} u| \leq C|\delta B_{j+1}^+ u|$. (Here $[A, X_{ij}]$ denotes the commutator of $A$ and $X_{ij}$.) Thus, letting $\chi$ denote the characteristic function of $R$, we have

\begin{equation}
\left| \int_{R} \right| X_{ij} u \right| A (X_{ij} u) \left| \delta dV \leq C \left| \int_{R} \right| X_{ij} u \right| \delta B_{j+1}^+ u \left| \delta dV \right|
\end{equation}

Inserting (3.34) and (3.35) into (3.25) gives

\begin{equation}
\left| \int_{R} \right| S(\nabla X_{ij} u)^2 \right| \delta dV \leq C \left| \int_{R} \right| X_{ij} u \right| \delta B_{j+1}^+ u \left| \delta dV \right|
\end{equation}

By Chebychev's Inequality, it follows that

\begin{align*}
\sigma \{ x \in \mathbb{R} : S(\nabla X_{ij} u)^2 \right| \delta dV \leq C \left[ \int_{R} \right| S(\nabla X_{ij} u)^2 \right| \delta dV \right]
\end{align*}

But since the right hand side trivially dominates $\sigma(F)$, it in fact bounds $\sigma \{ x \in \mathbb{R} : S(\nabla X_{ij} u)^2 \right| \delta dV \} \leq \sigma(F)$. On the other hand, we also have the trivial estimate

\begin{align*}
\sigma \{ x \in \mathbb{R} : S(\nabla X_{ij} u)^2 \right| \delta dV \} \leq \sigma(\delta D) < \infty,
\end{align*}

and so letting $M = ||\delta D||_\infty$, we have

\begin{align*}
\left| \int_{R} \right| S(\nabla X_{ij} u)^2 \right| \delta dV \right|
\end{align*}

This proves the lemma in the case $k > 0$. But when $k = 0$ the first term on the right of the above inequality can be absorbed by the second, since $u$ is harmonic, and so the proof is complete.
(3.36) Lemma. Let \( k \) be a nonnegative integer, and assume that the vector field \( X \) is of class \( C^{k+1} \). For any \( 2 < p < \infty \) there is a constant \( C_p \) such that for any harmonic function \( u \) on \( D \)

\[
\| S \partial^{k+1} u \|_p \leq C_p \left( \| N X^k u \|_p + \| \partial^{2k} u \|_{L^p} \right).
\]

Proof. When \( k = 0 \), the result is contained in Lemma (2.2). We shall show that the case \( k \geq 1 \) also follows from Lemma (2.2) via a duality argument (cf. [7, pp. 455–458]). As in the proof of Lemma (3.18), it will be convenient to assume that \( u \) is real-valued and that the vector field \( X \) is defined in a full neighborhood of \( D \) and vanishes in a neighborhood of the singularity of \( \delta \). By Lemma (3.1), it suffices to prove

\[
\| S (X^{k+1} u) \|_{L^p} \leq C_p \| N (X^k u) \|_{L^p} + \| \partial^{2k} u \|_{L^p}.
\]

We have

\[
\| S (X^{k+1} u) \|_{L^p}^p = \| (SX^{k+1} u)^2 \|_{L^p} = \sup \int_{\partial D} \langle S (X^{k+1} u)^2 \rangle \varphi \, d\sigma
\]

where the supremum is taken over all nonnegative functions \( \varphi \) in \( L^1(\partial D) \) satisfying \( \| \varphi \|_1 \leq 1 \). We begin with the elementary estimate

\[
\int_{\partial D} (S X^{k+1} u)^2 \varphi \, d\sigma = \int_{\partial D} \left( \int \langle (X^{k+1} u)^2 \rangle \varphi (x) \, dx \right) \, d\sigma (x)
\]

where \( \chi_t \) denotes the characteristic function of the interval \([0, t] \). But, letting \( \chi_t \) denote the Poisson kernel of \( D \) on \( D \times \partial D \), we have \( \chi_t = C \varphi (x, t, \xi) \), \( \varphi \) denote the Poisson indicator of \( \varphi \), it thus follows that

\[
(3.37) \quad \int_{\partial D} (S X^{k+1} u)^2 \varphi \, d\sigma
\]

\[
\leq C \int \langle (X^{k+1} u)^2 \rangle \varphi \left( (X^{k+1} u)^2 \right) \, dt \, d\sigma (x)
\]

\[
\leq C \int \| X^{k+1} u \|_p^2 \delta \, dV \leq C \int \| X^{k+1} u \|_p^2 \delta \, dV
\]

\[
= C \int \delta \, d \left( V \right) - V \left( X^{k+1} u \right) \cdot V \delta \, dV
\]

\[
\leq C \int \delta \, d \left( V \right) - V \left( X^{k+1} u \right) \cdot V \delta \, dV
\]

\[
+ C \int \| X^{k+1} u \|_p \delta \, dV
\]

Here we have used the identity

\[
|\mathcal{F} U| \leq \frac{1}{2} (U^2 V) - \frac{1}{2} U^2 \mathcal{A} V - V (U^2) \cdot V \mathcal{A} U \mathcal{V}.
\]

The first term on the right of (3.37) may be estimated by applying Green’s Formula in the approximating domains \( D_t = \{ \Phi^{-1} (x, t) \mid t \in \partial K \} \), and letting \( t \to 0^+ \) to obtain

\[
(3.38) \quad \int_D \delta \left( (X^{k+1} u)^2 \right) \, d\sigma \leq C \int_D \left( N_0 \left( X^{k+1} u \right)^2 \right) \left( N_0 \varphi + N_0 \left( \delta V \mathcal{A} u \right) \right) \, d\sigma
\]

\[
+ C \int_D N_0 \left( X^{k+1} u \right) \left( \delta V \mathcal{A} u \right) \, d\sigma.
\]

Since \( N_0 \delta V \mathcal{A} u \leq C \left( N_0 + \| \varphi \|_p \right) \), it follows from Hölder’s Inequality that the first integral on the right is majorized by \( \| N_0 \left( X^{k+1} u \right) \|_p \| \varphi \|_p \). For the last term in (3.38), note that by Lemma (3.13), \( N_0 \delta V \mathcal{A} u \leq C \left( N \left( X^{k+1} u \right) + \| \partial^{2k} u \|_p \right) \), so again by Hölder’s Inequality, the last integral is majorized by \( \| N \left( X^{k+1} u \right) \|_p + \| \partial^{2k} u \|_p \| \varphi \|_p \). Since \( \mathcal{A} u \) is the Poisson integral of \( \varphi \), it follows that \( \| \mathcal{A} u \|_p \leq C \| \varphi \|_p \), so it follows from (3.38) that

\[
(3.39) \quad \int_D \delta \left( (X^{k+1} u)^2 \right) \, d\sigma \leq C \| N \left( X^{k+1} u \right) \|_p^2 + \| \partial^{2k} u \|_p^2
\]

The second term on the right of (3.37) is more delicate. We have

\[
\int_D (X^{k+1} u) \cdot V \mathcal{A} u \cdot V \delta \, dV
\]

\[
\leq C \int_D \left( \int \left( \delta V \mathcal{A} u \right) \cdot V \delta \, dV \right) \, d\sigma (x)
\]

\[
\leq C \int_D \left( \int \left( \delta V \mathcal{A} u \right) \cdot V \delta \, dV \right) \, d\sigma (x)
\]

\[
+ C \| \delta V \mathcal{A} u \|_p^2
\]

\[
= C \int_D \left( \int \left( \delta V \mathcal{A} u \right) \cdot V \delta \, dV \right) \, d\sigma (x)
\]

By Hölder’s Inequality, the integral on the right is dominated by \( \| N_0 \left( X^{k+1} u \right) \|_p \times \| \varphi \|_p \langle \delta V \mathcal{A} u \rangle \rangle \langle \delta V \rangle \), and by Lemma (2.2), \( \| \varphi \|_p \leq C \| \mathcal{N} \|_p \leq C \| \varphi \|_p \leq C \). In addition, by Corollary (3.12), \( \| \delta V \mathcal{A} u \|_p \rangle \rangle \langle \delta V \rangle \), and, trivially, the radial maximal function is dominated by any nonintegrable maximal function, so we obtain

\[
(3.40) \quad \int_D \left( \int \left( \delta V \mathcal{A} u \right) \cdot V \delta \, dV \right) \, d\sigma (x)
\]

\[
\leq C \| N \left( X^{k+1} u \right) \|_p \| S \left( X^{k+1} u \right) \|_p + \| N \left( X^{k+1} u \right) \|_p \| \partial^{2k} u \|_p + \| \partial^{2k} u \|_p
\]

Estimation of the last integral in (3.37) is straightforward: passing to the radial maximal function and integrating over \( \partial D \) gives

\[
(3.41) \quad \int_D \left( \int \left( \delta V \mathcal{A} u \right) \cdot V \delta \, dV \right) \, d\sigma (x)
\]
\[ \int_{\partial D} N_{0}(X u) \, d\sigma + C \| \mathcal{D}^{k+1} u \|_{\infty}^{2} \leq C \left[ \int_{\partial D} (N(X u)^{2} + \| \mathcal{D}^{2k} u \|_{\infty}^{2}) \, d\sigma + C \| \mathcal{D}^{k+1} u \|_{\infty}^{2} \right] \]

Inserting (3.39)–(3.41) into (3.37) and taking the supremum over \( \varphi \) gives

\[ \| S(X^{k+1} u) \|_{p}^{2} \leq C \left[ \| N(X u) \|_{p}^{2} + \| N(X u) \|_{p} \| S(X^{k+1} u) \|_{p} + \| \mathcal{D}^{2k} u \|_{\infty}^{2} \right], \]

from which we conclude

(3.42) \[ \| S(X^{k+1} u) \|_{p} \leq C \left( \| N(X u) \|_{p} + \| \mathcal{D}^{2k} u \|_{\infty} \right) \]

for any harmonic function \( u \) such that \( \| S(X^{k+1} u) \|_{p} < \infty \).

To complete the proof, we must argue that \( S(X^{k+1} u) \in B(\partial D) \) whenever \( N(X u) \in B(\partial D) \). Since \( \partial D \) is compact, it suffices to verify that \( S(X^{k+1} u) \) is locally in \( B(\partial D) \), which can be deduced from (3.42) by the following simple device. Fix \( x_{0} \in \partial D \) and choose a small neighborhood \( W \) of \( x_{0} \) such that the outward normal to \( \partial D \) at \( x_{0} \) is transverse at each point of \( \partial D \cap W \). Next, choose a domain \( D_{0} \) with \( C^{2} \) boundary with \( D_{0} \subset D \) and with \( \partial D_{0} \cap \partial D \subset W \), and choose a vector field \( X_{0} \) in a neighborhood of \( \partial D_{0} \) which is everywhere transverse to \( \partial D \) and which agrees with \( X \) near \( x_{0} \). Let \( N^{0} \) and \( \mathcal{S}^{0} \) be the nontangential maximal and area integral operators respectively on \( D_{0} \). For any function \( u \) on \( D \), we let \( U_{x}(x) = U(x - \varepsilon v) \). Assume that \( u \) is a harmonic function on \( D \) with \( \| N(X u) \|_{p} < \infty \). Then for \( \varepsilon > 0 \) sufficiently small, \( u_{\varepsilon} \) is harmonic in a neighborhood of \( D_{0} \), and by (3.42),

\[ \| \mathcal{S}^{0}(X^{k+1} u) \|_{L^{p}(D_{0})} \leq C_{0} \left[ \| N^{0}(X u) \|_{L^{p}(D_{0})} + \| \mathcal{D}^{2k} u \|_{\infty} \right], \]

where \( K_{0} \) is some compact subset of \( D \). If the aperture \( \varepsilon \) is sufficiently large, then it follows easily from the Monotone Convergence Theorem that the right-hand side of the above inequality converges as \( \varepsilon \to 0 \) to the analogous expression with \( u_{\varepsilon} \) replaced by \( u \), which is necessarily finite in view of the finiteness of \( \| N(X u) \|_{p} \). It thus follows from Fatou's Lemma that

\[ \| \mathcal{S}^{0}(X^{k+1} u) \|_{L^{p}(\partial D_{0})} < \infty. \]

Since \( X_{0} \) agrees with \( X \) in a neighborhood of \( x_{0} \), it follows that \( S(X^{k+1} u)^{p} \) is integrable in a neighborhood of \( x_{0} \). Thus, we have shown that \( S(X^{k+1} u)^{p} \) is locally integrable on \( \partial D \). Since \( \partial D \) is compact, it follows that \( S(X^{k+1} u) \in B(\partial D) \), and the proof is complete.

We are now prepared to formulate and prove our main result.

(3.43) Theorem. Let \( D \) be a bounded domain in \( \mathbb{R}^{N} \) with \( C^{2} \) boundary, and let \( X \) be a \( C^{k+1} \) vector field in a neighborhood of \( \partial D \). Let \( u \) be a harmonic function on \( D \) such that \( N(X u) \in B(\partial D) \) for some \( 0 < p < \infty \). Then \( N(X u) \in B(\partial D) \). Moreover, there is a positive constant \( C \) and a compact subset \( K_{1} \) of \( D \) such that

\[ \| N(X u) \|_{L^{p}(\partial D)} \leq C \left( \| N(X u) \|_{L^{p}(\partial D \cap V)} + \| u \|_{K_{1}} \right) \]

for every harmonic function \( u \) on \( D \).

Proof. By Lemmas (3.18), (3.23), and (3.36), \( S(X^{k+1} u) \in B(\partial D) \) and

\[ \| S(X^{k+1} u) \|_{p} \leq C \left( \| N(X u) \|_{p} + \| \mathcal{D}^{2k} u \|_{\infty} \right). \]

Thus, since all the components of \( S(X^{k+1} u) \) are harmonic, it follows from Lemma (2.2) that \( \mathcal{S}^{a} u \in H^{0}(D) \) for every multi-index \( a \) with \( |a| \leq k \), and that

\[ \| N(\mathcal{S}^{a} u) \|_{p} \leq C \| S(\mathcal{S}^{a} u) \|_{p}. \]

Combining this estimate with (3.44) gives

\[ \| S(X^{k+1} u) \|_{p} \leq C \left( \| N(X u) \|_{p} + \| \mathcal{D}^{2k} u \|_{\infty} \right). \]

But if \( K_{1} \) is any compact subset of \( D \) containing \( K \) in its interior, then

\[ \| \mathcal{D}^{2k} u \|_{\infty} \leq C \| u \|_{K_{1}} \]

for any harmonic function \( u \) on \( D \), so the result follows.

There is also a local version of Theorem (3.43), which we will need in the next section.

(3.45) Corollary. Let \( D \) be a domain in \( \mathbb{R}^{N} \) with \( C^{2} \) boundary, and let \( V \) and \( V_{0} \) be neighborhoods of \( 0 \in \partial D \) with \( V_{0} \subset V \). Let \( K \) be a nonnegative integer, and let \( X_{0} \) be a \( C^{k+1} \) vector field on \( V \) which is transverse to \( \partial D \) at each point of \( \partial D \cap V \). Then for any harmonic function \( u \) on \( \partial D \cap V \) such that \( N(X_{0} u) \in B(\partial D \cap V) \) for some \( 0 < p < \infty \), we have \( N(X_{0} u) \in B(\partial D \cap V_{0}) \). Moreover, there is a compact subset \( K_{0} \) of \( V \cap D \) such that

\[ \| \mathcal{D}^{k+1} u \|_{L^{p}(\partial D \cap V_{0})} \leq C \| [X_{0} u] \|_{L^{p}(\partial D \cap V)} \]

for every harmonic function \( u \) on \( \partial D \cap V \).

Proof. Choose an open set \( V_{1} \) with \( V_{0} \subset V_{1} \subset V \) and a bounded domain \( D_{0} \) with \( C^{2} \) boundary such that \( D_{0} \subset D \cap V_{1} \) and \( 0 \in \partial D_{0} \cap V_{0} < \partial D \). Choose also a \( C^{k+1} \) vector field \( X_{1} \) in a neighborhood of \( \partial D_{0} \) such that \( X_{1} \) is everywhere transverse to \( \partial D_{0} \) and such that \( X_{1} \) agrees with \( X_{0} \) in a neighborhood of \( \partial D \cap V_{0} \). It is clear that our hypothesis implies that \( N(X_{1} u) \in B(\partial D_{0}) \), where \( N^{0} \) is a nontangential maximal operator for the domain \( D_{0} \), and that

\[ \| N^{0}(X_{1} u) \|_{L^{p}(\partial D_{0})} \leq C \left[ \| N(X_{0} u) \|_{L^{p}(\partial D \cap V_{1})} + \| u \|_{K_{1}} \right], \]

where \( L \) is some fixed compact subset of \( V \cap D \). Thus applying Theorem (3.43) in the domain \( D_{0} \) gives the desired result.

4. Pluriharmonic conjugation. In this section, we show that, for \( C^{2} \) domains in \( \mathbb{C}^{n} \), pluriharmonic conjugation is a continuous operation on \( H^{0} \) for \( 0 < p < \infty \). In the case \( 1 < p < \infty \), this result is due to Stout [10], with a different proof.
Theorem. Let $D$ be a bounded, connected open set in $\mathbb{C}^n$ with $C^2$ boundary which contains the origin, and let $f$ be a holomorphic function in $D$ such that $\text{Re}f \in H^p(D)$ for some $0 < p < \infty$. Then $f \in H^p(D)$, and

$$
\|f\|_p \leq C_p [\|N(\text{Re}f)\|_p + \|\text{Im}f(0)\|_p].
$$

Proof. Let $\xi \in \partial D$, and by a unitary change of coordinates, assume that $\partial / \partial x_n$ is the outward unit normal at $\xi$. (Here the coordinates in $\mathbb{C}^n$ are $z_j = x_j + iy_j$ for $1 \leq j \leq n$.) By integrating $f$ with respect to $z_n$, we can find a neighborhood $V$ of $\xi$ and a holomorphic function $F$ on $\overline{V} \cap D$ such that $\partial F / \partial z_n = f$ and such that for any compact set $L$ in $D$ there is a compact set $L'$ in $V$ such that

$$
\|F\|_{L^p(V)} \leq C \|f\|_{L^p(D)}
$$

with a constant $C$ independent of $f$. Thus, if $u = \text{Re} F$, then, by the Cauchy–Riemann equations,

$$
f = \frac{\partial u}{\partial x_n} - \frac{1}{i} \frac{\partial u}{\partial y_n}.
$$

By shrinking $V$, we may assume that $\partial / \partial x_n$ is transverse to $\partial D$ in $V$, so if $V_0$ is a neighborhood of $\xi$ with $V_0 \subset V$, it follows from Corollary (3.45) that

$$
\|N(\text{Im}f)\|_{L^p(V \cap \partial D)} \leq C \|N(\text{Re}f)\|_{L^p(V \cap \partial D)} + \|F\|_{K_0}
$$

where $K_0$ is some compact subset of $D \cap V$. But since $\partial D$ is compact, it follows that

$$
\|N(\text{Im}f)\|_p \leq C \|N(\text{Re}f)\|_p + \|f\|_{K_1}
$$

for some compact subset $K_1$ of $D$. But since $\text{Re}f$ is harmonic on $D$, it follows that

$$
\|\text{Re}f\|_{K_1} \leq C \|N(\text{Re}f)\|_p
$$

and thus

$$
\|N(\text{Im}f)\|_p \leq C \|N(\text{Re}f)\|_p + \|\text{Im}f(0)\|_{K_1}.
$$

Let $K_2$ be a compact subset of $D$ with connected interior containing $K_1 \cup \{0\}$ in its interior. Then

$$
\|\text{Im}f - \text{Im}f(0)\|_{K_1} \leq C \|P(\text{Im}f)\|_{K_1} = C \|P(\text{Re}f)\|_{K_5}.
$$

But since $\text{Re}f$ is harmonic in $D$, the last term on the right is bounded by a constant multiple of $\|N(\text{Re}f)\|_p$, so the theorem follows from (4.3).