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Differentiable bundles of subspaces and operators in Banach spaces

by

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Abstract. This paper follows the line of ideas of R. Janz [3], [4] who introduced a notion of continuous and holomorphic bundles of subspaces of a Banach space. His concepts constitute an elegant framework for the study of unbounded operators which depend on a parameter. Motivated by the work of Janz we investigate bundles of closed linear subspaces of a Banach space which are differentiable in a suitable sense. Our construction is based on lifting results for differentiable functions which have been established by the author in [6], [7].

0. Preliminaries. Throughout this note the letters E, F, G denote Banach spaces (real or complex). By $\mathrm{C}(E)$ we denote the set of all closed linear subspaces of E. For any two subspaces $M, N \in \mathrm{C}(E)$ let

$$\begin{aligned}
\delta(M, N) &:= \sup\{\operatorname{dist}(e, N) \mid e \in M,\ \|e\| \leqslant 1\},\\
\Delta(M, N) &:= \max\{\delta(M, N),\ \delta(N, M)\},\\
\gamma(M, N) &:= \inf\{\operatorname{dist}(e, N)/\operatorname{dist}(e, M \cap N) \mid e \in M \setminus N\},\\
\Gamma(M, N) &:= \min\{\gamma(M, N),\ \gamma(N, M)\}.
\end{aligned}$$

The reader may consult T. Kato [5], Chap. IV, for the properties of Δ and Γ. Especially we shall make use of the fact that there is a metric topology (the gap topology) on $\mathrm{C}(E)$ such that for each $M \in \mathrm{C}(E)$ the sets

$$U_\varepsilon(M) := \{N \in \mathrm{C}(E) \mid \Delta(M, N) < \varepsilon\}, \qquad \varepsilon > 0,$$

form a neighborhood basis of M. In terms of this topology we shall speak of continuous mappings $M\colon X \to \mathrm{C}(E)$ when X is a topological space. The significance of Γ stems from the fact that for any $M, N \in \mathrm{C}(E)$ we have $\Gamma(M, N) > 0$ iff $M + N \in \mathrm{C}(E)$ ([5], § IV.4.1).

0.1. THEOREM (Janz [4]). *Let X be a topological space and let $M, N\colon X \to \mathrm{C}(E)$ be continuous. Assume that $\Gamma(M(x_0), N(x_0)) > 0$ for some $x_0 \in X$. Then the following conditions are equivalent*:

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(i) $x \mapsto \Gamma(M(x), N(x))$ *is bounded away from* 0 *in a neighborhood of* x_0.
(ii) $x \mapsto \overline{M(x)+N(x)}$ *is continuous in a neighborhood of* x_0.
(iii) $M(x)+N(x) \in C(E)$ *in a neighborhood* U *of* x_0 *and* $x \mapsto M(x)+N(x)$ *is continuous on* U.
(iv) $x \mapsto M(x) \cap N(x)$ *is continuous in a neighborhood of* x_0.

We denote by $C(E, F)$ the set of all closed linear operators $A\colon \operatorname{dom}(A) \to F$ where $\operatorname{dom}(A)$ is a linear subspace (not necessarily dense) of E, and by $B(E, F)$ the subset of $C(E, F)$ which consists of all bounded operators $A\colon E \to F$. For any $A \in C(E, F)$ let

$$\ker(A) := \{e \in \operatorname{dom}(A) \mid Ae = 0\} \subset E,$$

$$\operatorname{ran}(A) := \{Ae \mid e \in \operatorname{dom}(A)\} \subset F,$$

$$\operatorname{gra}(A) := \{(e, Ae) \mid e \in \operatorname{dom}(A)\} \subset E \times F,$$

$$\Gamma(A) := \inf\{\|Ae\|/\operatorname{dist}(e, \ker(A)) \mid e \in \operatorname{dom}(A) \setminus \ker(A)\}.$$

We have (cf. [5], § IV.5)

$$\Gamma(A) > 0 \quad \text{iff} \quad \operatorname{ran}(A) \in C(F),$$

$$\Gamma(\operatorname{gra}(A), E \times \{0\}) = \Gamma(A)\left(1+\Gamma(A)^2\right)^{-1/2}$$

if the norm on $E \times F$ is defined to be $\|(e, f)\| := (\|e\|^2+\|f\|^2)^{1/2}$.

By identifying each operator $A \in C(E, F)$ with its graph the set $C(E, F)$ can (and will) be viewed as a topological subspace of $C(E \times F)$. It is not hard to see that $C(E, F)$ induces on $B(E, F)$ just the usual operator norm topology (cf. [5], § IV.2.4).

0.2. Theorem. *Let* X *be a topological space and let* $A\colon X \to C(E, F)$ *be continuous. Assume that* $\Gamma(A(x_0)) > 0$ *for some* $x_0 \in X$. *Then the following conditions are equivalent*:

(i) $x \mapsto \Gamma(A(x))$ *is bounded away from* 0 *in a neighborhood of* x_0.
(ii) $x \mapsto \overline{\operatorname{ran}(A(x))}$ *is continuous in a neighborhood of* x_0.
(iii) $\operatorname{ran}(A(x)) \in C(F)$ *in a neighborhood* U *of* x_0 *and* $x \mapsto \operatorname{ran}(A(x))$ *is continuous on* U.
(iv) $x \mapsto \ker(A(x))$ *is continuous in a neighborhood of* x_0.

Proof. With $M(x) = \operatorname{gra}(A(x))$, $N(x) = E \times \{0\}$ the assertion follows from Th. 0.1 since $M(x)+N(x) = E \times \operatorname{ran}(A(x))$ and $M(x) \cap N(x) = \ker(A(x)) \times \{0\}$. ∎

0.3. Definition. Let X be a topological space and let $M, N\colon X \to C(E)$ resp. $A\colon X \to C(E, F)$ be continuous. The pair (M, N) resp. the mapping A are said to be *regular* [at $x_0 \in X$] if one (hence all) of the conditions of Th. 0.1 resp. of Th. 0.2 are satisfied for each $x_0 \in X$ [at $x_0 \in X$].

We pause to give some useful criteria for regularity.

0.4. Proposition. *Let* X *be a metric space and* $A\colon X \to C(E, F)$ *a continuous mapping such that* $\operatorname{ran}(A(x))$ *is closed*, $x \in X$. *Let* $x_0 \in X$.

(a) *Assume that* $\ker(A(x_0))$ *is complemented in* E [$\operatorname{ran}(A(x_0))$ *is complemented in* F]. *Then* A *is regular at* x_0 *iff there exists a closed subspace* E_0 *in* E [F_0 *in* F] *and a neighborhood* Y *of* x_0 *such that* $E_0 \oplus \ker(A(x)) = E$ [$F_0 \oplus \operatorname{ran}(A(x)) = F$] *for* $x \in Y$.
(b) *Assume that* $\dim\ker(A(x_0)) < \infty$ [$\operatorname{codim}\operatorname{ran}(A(x_0)) < \infty$]. *Then* A *is regular at* x_0 *iff there exists a neighborhood* Y *of* x_0 *such that* $\dim\ker(A(x))$ [$\operatorname{codim}\operatorname{ran}(A(x))$] *is constant for* $x \in Y$.

Proof. According to R. Janz [4] there exist Banach spaces D, G and regular continuous operator functions $U\colon X \to B(D, E \times F)$, $V\colon X \to B(E \times F, G)$ with $\operatorname{ran}(U(x)) \equiv \operatorname{gra}(A(x)) \equiv \ker(V(x))$. Put

$$\tilde U\colon X \to B(E \times D, E \times F), \qquad \tilde U(x)(a, b) := (a, 0)+U(x)b,$$

$$\tilde V\colon X \to B(E \times F, F \times G), \qquad \tilde V(x)(a, b) := (b, V(x)(a, b)).$$

Then $\operatorname{ran}(\tilde U(x)) \equiv E \times \operatorname{ran}(A(x))$ is closed and $\ker(\tilde V(x)) \equiv \ker(A(x)) \times \{0\}$. Furthermore, from [3], (1.5), it follows that $\Gamma(\tilde V(x)) > 0$ (since $\Gamma(E \times \{0\}, \operatorname{gra}(A(x))) > 0$), i.e. $\operatorname{ran}(\tilde V(x))$ is closed, too. So from Th. 0.2 it follows that A is regular iff $\tilde U$ is regular iff $\tilde V$ is regular. The assertion is now obtained from the corresponding result for bounded-operator functions (cf. [9], § I.2) applied to $\tilde U$, $\tilde V$. ∎

In a similar way the proof of the following result can be reduced to the case of bounded-operator functions ([8], Lemma 1.9).

0.5. Proposition. *Let* X *be a metric space and* $A\colon X \to C(E, F)$, $B\colon X \to C(F, G)$ *continuous such that* $\operatorname{ran}(A(x)) = \ker(B(x))$, $\operatorname{ran}(B(x)) \in C(G)$ *for* $x \in X$. *Then both* A *and* B *are regular.*

There is a well developed theory which makes it possible to check the conditions of Prop. 0.4 in the case of an operator pencil $A(x) = T-xS$ (cf. [1] and the literature cited there).

Next we wish to define a notion of differentiability for mappings $M\colon X \to C(E)$. For technical reasons we restrict ourselves to the case of a compact cube $X = I^d$ where $I = [0, 1]$ and $d \in \mathbf{N}$. However, by using local coordinates and suitable partitions of unity all our results can analogously be proved for functions defined on a real C^∞-manifold X with (or without) boundary. For $m \in \mathbf{N}_0$ let

$$C^m(I^d, E) := \{f\colon I^d \to E \mid f \text{ is } m \text{ times continuously differentiable}\},$$

$$C^\infty(I^d, E) := \bigcap_{m=0}^{\infty} C^m(I^d, E).$$

Let $\varkappa\colon [0, \infty) \to [0, \infty)$ be a continuous function with the properties

$$\varkappa(\varrho)=0 \quad \text{iff} \quad \varrho=0, \qquad \varkappa \text{ and } \varrho/\varkappa(\varrho) \text{ are nondecreasing.}$$

Such a function will be called a *Lipschitz function* (for short: Lf). For any $f \in C^m(I^d, E)$ and $\varrho > 0$ let

$$\omega_m(f;\varrho) := \sum_{|\alpha|=m} \sup\{\|D^\alpha f(x)-D^\alpha f(y)\| \mid x, y \in I^d,\ \|x-y\| \leqslant \varrho\},$$

where $\alpha=(\alpha_1, \ldots, \alpha_d)$ is a multiindex of nonnegative integers and $|\alpha| := \sum_{i=1}^d |\alpha_i|$, $\|x\| := \max_{i=1}^d |x_i|$. The function spaces

$$\Lambda^{m,\varkappa}(I^d, E) := \{f \in C^m(I^d, E) \mid \|f\|_{m,\varkappa} := \|f\|_m + \sup_{\varrho>0} \omega_m(f;\varrho)/\varkappa(\varrho) < \infty\}$$

with $\|f\|_m := \sum_{|\alpha|\leqslant m} \sup\{\|D^\alpha f(x)\| \mid x \in I^d\}$, and

$$\lambda^{m,\varkappa}(I^d, E) := \{f \in \Lambda^{m,\varkappa}(I^d, E) \mid \lim_{\varrho\to 0} \omega_m(f;\varrho)/\varkappa(\varrho) = 0\}$$

are complete under the norm $\|\cdot\|_{m,\varkappa}$ [2]. Note that for any Lf $\varkappa$ we have $\lambda^{m+1,\varkappa} \subset \Lambda^{m+1,\varkappa} \subset C^{m+1} \subset \Lambda^{m,\varkappa}$ and $C^{m+1} \subset \lambda^{m,\varkappa}$ if $\lim_{\varrho\to 0} \varrho/\varkappa(\varrho) = 0$.

0.6. Definition. Let the symbol $\mathscr{F}$ stand for one of the above function spaces and let $M\colon I^d \to \mathrm{C}(E)$ be given. Then we denote by

$$\mathscr{F}(I^d, M) := \{e \in \mathscr{F}(I^d, E) \mid e(x) \in M(x),\ x \in I^d\}$$

the set of all (global) $\mathscr{F}$-sections in M. We say that M is an *$\mathscr{F}$-bundle* and write $M \in \mathscr{F}(I^d, \mathrm{C}(E))$ if the following holds:

(i) $M\colon I^d \to \mathrm{C}(E)$ is continuous.

(ii) For any $x_0 \in I^d$ and $\xi_0 \in M(x_0)$ there is $e \in \mathscr{F}(I^d, M)$ with $e(x_0) = \xi_0$.

In particular, Def. 0.6 yields a notion of differentiability for mappings $T\colon I^d \to \mathrm{C}(E, F)$ by requirinq that $x \mapsto \mathrm{gra}(T(x))$ be of class $\mathscr{F}(I^d, \mathrm{C}(E\times F))$. We write $T \in \mathscr{F}(I^d, \mathrm{C}(E, F))$ then. In order to avoid ambiguities we shall use the notation $T \in \mathscr{F}(I^d, \mathrm{B}(E, F))$ for operator functions $T\colon I^d \to \mathrm{B}(E, F)$ which are differentiable with respect to the uniform operator norm (see, however, Cor. 2.2 below).

0.7. Lemma. *Let D be a linear subspace of $\Lambda^{m,\varkappa}(I^d, G)$ endowed with a norm $\|\cdot\|$ stronger than $\|\cdot\|_{m,\varkappa}$. For each $x \in I^d$ let the operator $S(x)\colon D \to G$ be defined by $S(x)g := g(x)$. Then the mapping $x \mapsto S(x)$ is of class $\Lambda^{m,\varkappa}(I^d, \mathrm{B}(D, G))$.*

Proof. Obviously $S(x) \in \mathrm{B}(D, G)$, $x \in I^d$. In the case $m = 0$ a direct calculation shows that $\|S\|_{0,\varkappa} \leqslant 2K$ if $\|\cdot\|_{0,\varkappa} \leqslant K\|\cdot\|$. If $m \geqslant 1$ an iterated application of Taylor's formula yields $S \in C^m(I^d, \mathrm{B}(D, G))$ and $(D^\alpha S(x))g = D^\alpha g(x)$ for each $g \in D$, $x \in I^d$ and $|\alpha| \leqslant m$. ∎

A Lf $\varkappa$ will be called *admissible* (1) if the following two conditions are satisfied:

(A) There is $a > 0$ with $\int_0^s (\varkappa(t)/t)\,dt \leqslant a\varkappa(s)$, $0 < s \leqslant 1$;

(1) The class of admissible Lf's has been denoted by (A_0B_1)Lf in the preceding article.

(B) There is $b > 0$ with $\int_s^1 (\varkappa(t)/t^2)\,dt \leqslant b\varkappa(s)/s$, $0 < s \leqslant 1$.

The most important examples of admissible Lf's are the Hölder functions $\varkappa(s) = s^p$ where $0 < p < 1$. The following theorem is quoted from [6] and [7].

0.8. Theorem. *Let the symbol $\mathscr{F}$ stand for one of the function spaces C^∞, $\lambda^{m,\varkappa}$, $\Lambda^{m,\varkappa}$ where $\varkappa$ is an admissible Lf and $m \in \mathbf{N}_0$. Let $T \in \mathscr{F}(I^d, \mathrm{B}(E, F))$ be regular. Let $f \in \mathscr{F}(I^d, F)$ with $f(x) \in \mathrm{ran}(T(x))$, $x \in I^d$, and $x_0 \in I^d$, $\xi_0 \in E$ such that $T(x_0)\xi_0 = f(x_0)$. Then there is a function $e \in \mathscr{F}(I^d, E)$ which satisfies $e(x_0) = \xi_0$ and $T(x)e(x) = f(x)$, $x \in I^d$.*

It is one of our aims to generalize part of Theorem 0.8 to the case where the operator function T takes its values in the set of unbounded operators, $T \in \mathscr{F}(I^d, \mathrm{C}(E, F))$.

1. Representation theorems for differentiable bundles. Let F' denote the dual of a Banach space F and $\iota_F\colon F \to F''$ the canonical imbedding into its second dual. For each subset V of F let $V^\circ := \{w \in F' \mid |w(v)| \leqslant 1,\ v \in V\}$.

1.1. Lemma. *Let $M \in \mathrm{C}(F)$ and $A \in \mathrm{B}(F'', G)$ with $\ker(A) = M^{\circ\circ}$. Then the restriction $A|_F = A \circ \iota_F$ satisfies $\ker(A|_F) = M$ and $\Gamma(A|_F) \geqslant \Gamma(A)$. In particular, if $\mathrm{ran}(A)$ is closed then also $\mathrm{ran}(A|_F)$ is closed.*

Proof. Apply Lemmas 1.1 and 1.2 of [4]. ∎

1.2. Theorem. *Let $M\colon I^d \to \mathrm{C}(F)$ be continuous, let $m \in \mathbf{N}_0$ and $\varkappa$ an admissible Lf. Then the following conditions are equivalent:*

(i) *$M \in \Lambda^{m,\varkappa}(I^d, \mathrm{C}(F))$.*

(ii) *There is a Banach space E and a (regular) operator function $S \in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(E, F))$ such that $\mathrm{ran}(S(x)) = M(x)$, $x \in I^d$.*

(iii) *There is a Banach space G and a (regular) operator function $T \in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(F, G))$ such that $\ker(T(x)) = M(x)$ and $\mathrm{ran}(T(x)) \in \mathrm{C}(G)$, $x \in I^d$.*

(iv) *$M^\circ\colon x \mapsto (M(x))^\circ$ is of class $\Lambda^{m,\varkappa}(I^d, \mathrm{C}(F'))$.*

Note that the regularity of the operator functions S, T is a consequence of Theorem 0.2.

Proof. (i)⇒(ii). Let $E := \Lambda^{m,\varkappa}(I^d, M)$ be endowed with the norm $\|\cdot\|_{m,\varkappa}$. Then E is a Banach space and by Lemma 0.7 the operator function $x \mapsto S(x)$, $S(x)e := e(x)$ belongs to $\Lambda^{m,\varkappa}(I^d, \mathrm{B}(E, F))$. From the assumption it follows that $\mathrm{ran}(S(x)) = M(x)$ for each $x \in I^d$.

(ii)⇒(iv). Let $S \in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(E, F))$ satisfy (ii) and consider the transpose $S'(x)\colon F' \to E'$ of $S(x)$. Then $S' \in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(F', E'))$ and S' is regular because $\gamma(S'(x)) = \gamma(S(x))$ [5]. Furthermore, $\ker(S'(x)) = M^\circ(x)$ for $x \in I^d$, and thus $M^\circ\colon I^d \to \mathrm{C}(F')$ is continuous by Th. 0.2. Fix $x_0 \in I^d$ and $\xi_0 \in M^\circ(x_0)$. Then from Th. 0.8 follows the existence of a function $f' \in \Lambda^{m,\varkappa}(I^d, F')$ such that $f'(x_0) = \xi_0$ and $S'(x)f'(x) \equiv 0$. Thus $f' \in \Lambda^{m,\varkappa}(I^d, M^\circ)$ and it follows that $M^\circ \in \Lambda^{m,\varkappa}(I^d, \mathrm{C}(F'))$. The same argument shows that (iii)⇒(i) holds.

(iv)⇒(iii). By what is already proved above there is a Banach space $\tilde{G}$ and a regular operator function $\tilde{T}\in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(\tilde{G}, F'))$ such that $\operatorname{ran}(\tilde{T}(x)) = M^\circ(x)$, $x\in I^d$. Put $T(x) := \tilde{T}'(x)_{|F}$. Then $T\in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(F, \tilde{G}'))$ and by Lemma 1.1 we have $\ker(T(x)) = M(x)$, $\operatorname{ran}(T(x))\in \mathrm{C}(\tilde{G}')$ for $x\in I^d$. ∎

The analogue of Theorem 1.2 for C^∞-functions reads:

1.3. Theorem. *Let $M: I^d \to \mathrm{C}(F)$ be continuous. Then the following conditions are equivalent:*

(i) $M\in C^\infty(I^d, \mathrm{C}(F))$.

(ii) $M\in C^m(I^d, \mathrm{C}(F))$ *for each* $m\in \mathbf{N}_0$.

(iii) *There is a Banach space E and a (regular) operator function* $S\in C^\infty(I^d, \mathrm{B}(E, F))$ *such that* $\operatorname{ran}(S(x)) = M(x)$, $x\in I^d$.

(iv) *There is a Banach space G and a (regular) operator function* $T\in C^\infty(I^d, \mathrm{B}(F, G))$ *such that* $\ker(T(x)) = M(x)$ *and* $\operatorname{ran}(T(x))\in \mathrm{C}(G)$, $x\in I^d$.

(v) $M^\circ\in C^\infty(I^d, \mathrm{C}(F'))$.

If 1.3(iii) holds then from Theorem 0.8 it follows that the Fréchet space $C^\infty(I^d, M)$ (endowed with the subspace topology of $C^\infty(I^d, F)$) is a quotient of the space $C^\infty(I^d, E)\cong (s)\hat{\otimes}_\pi E$, where (s) denotes the Fréchet space of all rapidly decreasing sequences. D. Vogt [10] characterized the quotient spaces of $(s)\hat{\otimes}_\pi E$ (E a Banach space) by a topological invariant (Ω). The following lemma essentially shows that for any bundle $M\in \bigcap_{m=0}^\infty C^m(I^d, \mathrm{C}(F))$ the space $C^\infty(I^d, M)$ has in fact property (Ω). We use the notation $\Lambda^{m,1/2} := \Lambda^{m,\varkappa}$ with $\varkappa(\varrho) := \varrho^{1/2}$.

1.4. Lemma. *Let $M\in \bigcap_{m=0}^\infty C^m(I^d, \mathrm{C}(F))$. Then there are constants $K_m, N_m > 0$ $(m\in \mathbf{N}_0)$ such that for each $x_0\in I^d$ and $r\geq 1$ any $f\in \Lambda^{m,1/2}(I^d, M)$ admits a decomposition $f = f_1+f_2$, where*

$$f_1\in \Lambda^{m,1/2}(I^d, M),\quad f_1(x_0) = 0,\quad \|f_1\|_m \leq r^{-1}\|f\|_{m,1/2},$$
$$f_2\in \Lambda^{m+1,1/2}(I^d, M),\quad \|f_2\|_{m+1,1/2} \leq K_m r^{N_m}\|f\|_{m,1/2}.$$

Proof. Fix $m\in \mathbf{N}_0$. Since $M\in \Lambda^{m+1,1/2}(I^d, \mathrm{C}(F))$ there is a Banach space E_{m+1} and a regular operator function $S_{m+1}\in \Lambda^{m+1,1/2}(I^d, \mathrm{B}(E_{m+1}, F))$ such that $\operatorname{ran}(S_{m+1}(x)) \equiv M(x)$ (apply Th. 1.2). Now let $f\in \Lambda^{m,1/2}(I^d, M)$. By Th. 0.8 and the open mapping theorem (applied to the operator π: $\Lambda^{m,1/2}(I^d, E_{m+1}) \to \Lambda^{m,1/2}(I^d, F)$, $(\pi e)(x) := S_{m+1}(x)e(x)$) there is a constant $\mu_m > 0$ which depends on S_{m+1} only and a function $e\in \Lambda^{m,1/2}(I^d, E_{m+1})$ such that

$$S_{m+1}(x)e(x) \equiv f(x),\quad \|e\|_{m,1/2} \leq \mu_m\|f\|_{m,1/2}.$$

We construct a decomposition $e = e_1+e_2$ in the following way: First choose a function $\psi\in C^\infty(\mathbf{R})$ with the properties

$$\psi(t) = 1 \quad \text{for } |t|\leq \tfrac{1}{2},\qquad \psi(t) = 0 \quad \text{for } |t|\geq \tfrac{2}{3},$$
$$\psi(t) = 1-\psi(t-1) \quad \text{for } 0\leq t\leq 1.$$

For $r = 1, 2, \ldots$ let $\sigma(r) := \{s\in \mathbf{R}^d \mid r\cdot s\in \{0, \ldots, r\}^d\} \subset I^d$ and

$$\Phi(e, r; x) := \sum_{s\in\sigma(r)} \prod_{i=1}^d \psi(r(x_i-s_i)) \sum_{|\alpha|\leq m} \frac{(x-s)^\alpha}{\alpha!} D^\alpha e(s).$$

Put

$$e_1(x) := e(x)-\Phi(e, r; x),\qquad e_2(x) := \Phi(e, r; x).$$

Then clearly $e_2\in C^\infty(I^d, E_{m+1})$ and

$$\|e_2\|_{m+1,1/2} \leq c_0\|e_2\|_{m+2} \leq c_0 r^{m+2}\|e\|_m \leq c_0\mu_m r^{m+2}\|f\|_{m,1/2}$$

with a constant $c_0 > 0$ which depends on m and the choice of ψ only. From Lemma 1.1 in [7] it follows that

$$D^\alpha e_1(s) = 0 \quad \text{for } s\in\sigma(r),\ |\alpha|\leq m,$$
$$\omega_m(e_1; \varrho) \leq \omega_m(e; \varrho)+c_1\varrho r\omega_m(e; r^{-1}),\qquad 0\leq \varrho\leq r^{-1},$$

with a different constant c_1. This yields by Taylor's formula

$$\|e_1\|_m \leq c_2\omega_m(e_1; r^{-1}) \leq c_3\omega_m(e; r^{-1})$$
$$\leq c_3 r^{-1/2}\|e\|_{m,1/2} \leq c_3\mu_m r^{-1/2}\|f\|_{m,1/2}.$$

Letting $\tilde{e}_1(x) := e_1(x)-e_1(x_0)$, $\tilde{e}_2(x) := e_2(x)+e_1(x_0)$ and $f_i(x) := S_{m+1}(x)\tilde{e}_i(x)$ $(i = 1, 2)$ we obtain $f = f_1+f_2$ with

$$f_1\in \Lambda^{m,1/2}(I^d, M),\quad f_1(x_0) = 0,\quad \|f_1\|_m \leq K'_m r^{-1/2}\|f\|_{m,1/2},$$
$$f_2\in \Lambda^{m+1,1/2}(I^d, M),\quad \|f_2\|_{m+1,1/2} \leq K''_m r^{m+2}\|f\|_{m,1/2}.$$

From this the assertion follows with $N_m = 2m+4$. ∎

1.5. Lemma. *Let $M\in \bigcap_{m=0}^\infty C^m(I^d, \mathrm{C}(F))$. Then there are constants $L_m > 0$ $(m\in \mathbf{N}_0)$ such that for any $x_0\in I^d$ and $\xi_0\in M(x_0)$ there exists a section $f\in C^\infty(I^d, M)$ with*

$$f(x_0) = \xi_0,\quad \|f\|_m \leq L_m\|\xi_0\| \quad \textit{for } m\in \mathbf{N}_0.$$

Proof. Fix $x_0\in I^d$ and $\xi_0\in M(x_0)$ with $\|\xi_0\| = 1$. According to Th. 1.2 we may choose a Banach space E_0 and a regular operator function $S_0\in \Lambda^{0,1/2}(I^d, \mathrm{B}(E_0, F))$ with $\operatorname{ran}(S_0(x)) \equiv M(x)$. Because of the regularity of S_0 there is a constant $0 < \tilde{l}_0 := 2\inf\{\Gamma(S_0(x)) \mid x\in I^d\}^{-1} < \infty$ and $e_0\in E_0$ with

$$S_0(x_0)e_0 = \xi_0,\quad \|e_0\| \leq \tilde{l}_0.$$

Put $f_0(x) := S_0(x)e_0$, $x\in I^d$. We inductively assume to have already constructed a function $f_m\in \Lambda^{m,1/2}(I^d, M)$ such that

$$f_m(x_0) = \xi_0,\quad \|f_m\|_{m,1/2} \leq l_m$$

with a constant $l_m\geq 1$ which depends on the given bundle M only. Then by Lemma 1.4 there is a decomposition $f_m = f_m^1+f_m^2$ with

$$f_m^1 \in \Lambda^{m,1/2}(I^d, M), \quad f_m^1(x_0) = 0, \quad \|f_m^1\|_m \leqslant \frac{1}{l_m 2^m}\|f_m\|_{m,1/2} \leqslant 2^{-m},$$

$$f_m^2 \in \Lambda^{m+1,1/2}(I^d, M), \quad \|f_m^2\|_{m+1,1/2} \leqslant K_m (l_m 2^m)^{N_m} \|f_m\|_{m,1/2} \leqslant l_{m+1},$$

where K_m, N_m and thus $l_{m+1} := \max\{1, K_m (l_m 2^m)^{N_m} l_m\}$ also depend on M and m only. We then put $f_{m+1} := f_m^2$ and see that

$$f_{m+1} \in \Lambda^{m+1,1/2}(I^d, M), \quad f_{m+1}(x_0) = \xi_0,$$

$$\|f_{m+1} - f_m\|_m \leqslant 2^{-m}, \quad \|f_{m+1}\|_{m+1,1/2} \leqslant l_{m+1}.$$

With the definition $f(x) := \lim_{m\to\infty} f_m(x)$ we now obtain $f \in C^\infty(I^d, M)$, $f(x_0) = \xi_0$ and

$$\|f\|_m \leqslant \|f_m\|_m + \sum_{k=m}^{\infty} \|f_{k+1} - f_k\|_k \leqslant l_m + \sum_{k=m}^{\infty} 2^{-k} =: L_m. \ \blacksquare$$

Proof of Theorem 1.3. (i)⇒(ii): trivial. (ii)⇒(iii). Let the constants L_m be as in Lemma 1.5 and put

$$E := \{f \in C^\infty(I^d, M) \mid \|f\|_E := \sup_{m=0}^{\infty} (1/L_m)\|f\|_m < \infty\}.$$

Then E is a Banach space. For $x \in I^d$ let $S(x)\colon E \to F$ be defined through $S(x)e := e(x)$. From Lemma 0.7 follows that $S \in \bigcap_{m=0}^{\infty} \Lambda^{m,1/2}(I^d, \mathrm{B}(E, F)) = C^\infty(I^d, \mathrm{B}(E, F))$. Furthermore, by the definition of E and Lemma 1.5 we have $\operatorname{ran}(S(x)) \equiv M(x)$.

The rest of the proof is analogous to that of Th. 1.2. ■

The methods employed above yield simple proofs for approximation results such as the following one:

1.6. Proposition. *Let $M \in C^\infty(I^d, \mathrm{C}(F))$. Let $m \in \mathbf{N}_0$ and $\varkappa$ an admissible Lf. Then for each $f \in \lambda^{m,\varkappa}(I^d, M)$, $x_0 \in I^d$ and $\varepsilon > 0$ there is $f_\varepsilon \in C^\infty(I^d, M)$ such that $D^\alpha f_\varepsilon(x_0) = D^\alpha f(x_0)$, $|\alpha| \leqslant m$, and $\|f_\varepsilon - f\|_{m,\varkappa} \leqslant \varepsilon$.*

Proof. Choose $S \in C^\infty(I^d, \mathrm{B}(E, F))$ as in 1.3(iii). From condition (B) it follows that $\lim_{\varrho\to 0} \varrho/\varkappa(\varrho) = 0$, which implies that $S \in \lambda^{m,\varkappa}(I^d, \mathrm{B}(E, F))$. Using our lifting result 0.8 for $\lambda^{m,\varkappa}$-functions and the fact that $C^\infty(I^d, E)$ is dense in $\lambda^{m,\varkappa}(I^d, E)$ (cf. [7], proof of Th. 4.2) the assertion follows easily. ■

2. The sum-intersection property and vector function equations. In this section let $m \in \mathbf{N}_0$ be fixed and $\varkappa$ an admissible Lf. Let the symbol $\mathscr{F}$ stand for either $\Lambda^{m,\varkappa}$ or C^∞.

2.1. Theorem. *Let $M_1, M_2 \in \mathscr{F}(I^d, \mathrm{C}(F))$. Then the following conditions are equivalent:*

(i) *The pair (M_1, M_2) is regular* (cf. Def. 0.3).

(ii) *$M_1 \cap M_2\colon x \mapsto M_1(x) \cap M_2(x)$ is of class $\mathscr{F}(I^d, \mathrm{C}(F))$.*

(iii) *$M_1 + M_2\colon x \mapsto M_1(x) + M_2(x)$ is of class $\mathscr{F}(I^d, \mathrm{C}(F))$.*

Let one (hence all) of the above conditions be satisfied. Then the mapping

$$+\colon \mathscr{F}(I^d, M_1) \times \mathscr{F}(I^d, M_2) \to \mathscr{F}(I^d, M_1 + M_2), \quad (v_1 + v_2)(x) := v_1(x) + v_2(x),$$

is surjective. More precisely: Let $w \in \mathscr{F}(I^d, M_1 + M_2)$, $x_0 \in I^d$, $\xi_1 \in M_1(x_0)$ and $\xi_2 \in M_2(x_0)$ such that $\xi_1 + \xi_2 = w(x_0)$. Then there exist $v_1 \in \mathscr{F}(I^d, M_1)$, $v_2 \in \mathscr{F}(I^d, M_2)$ such that $v_1(x_0) = \xi_1$, $v_2(x_0) = \xi_2$ and $v_1 + v_2 = w$.

Proof. By Th. 1.2 resp. Th. 1.3 there exist Banach spaces E_i, G_i and regular operator functions $S_i \in \mathscr{F}(I^d, \mathrm{B}(E_i, F))$, $T_i \in \mathscr{F}(I^d, \mathrm{B}(F, G_i))$ with $\operatorname{ran}(S_i(x)) \equiv M_i(x) \equiv \ker(T_i(x))$ $(i = 1, 2)$. Define

$$S \in \mathscr{F}(I^d, \mathrm{B}(E_1 \times E_2, F)), \quad S(x)(e_1, e_2) := S_1(x)e_1 + S_2(x)e_2,$$

$$T \in \mathscr{F}(I^d, B(F, G_1 \times G_2)), \quad T(x)f := (T_1(x)f, T_2(x)f).$$

Then $\operatorname{ran}(S(x)) \equiv M_1(x) + M_2(x)$ and $\ker(T(x)) \equiv M_1(x) \cap M_2(x)$. Furthermore, $\operatorname{ran}(T(x))$ is closed iff $M_1(x) + M_2(x)$ is closed (apply (1.5) of [3]). Thus by Ths. 0.1 and 0.2 condition (i) is equivalent to the regularity of S or to the regularity of T. The equivalence of (i)–(iii) again follows from Th. 0.1 and Th. 1.2 resp. Th. 1.3.

Now assume that (M_1, M_2) is regular and let w, x_0, ξ_1, ξ_2 be as above. Choose $\hat{\xi}_1 \in E_1$, $\hat{\xi}_2 \in E_2$ with $S_1(x_0)\hat{\xi}_1 = \xi_1$, $S_2(x_0)\hat{\xi}_2 = \xi_2$. By Th. 0.8 there is a solution $(\hat{v}_1, \hat{v}_2) \in \mathscr{F}(I^d, E_1 \times E_2)$ of

$$(\hat{v}_1(x_0), \hat{v}_2(x_0)) = (\hat{\xi}_1, \hat{\xi}_2), \quad S(x)(\hat{v}_1(x), \hat{v}_2(x)) \equiv w(x).$$

Put $v_1(x) := S_1(x)\hat{v}_1(x)$ and $v_2(x) := S_2(x)\hat{v}_2(x)$. ■

2.2. Corollary. *For any mapping $T\colon I^d \to \mathrm{B}(E, F)$ the following conditions are equivalent:*

(i) *$x \mapsto T(x)e$ belongs to $\mathscr{F}(I^d, F)$ for each $e \in E$.*

(ii) *$T \in \mathscr{F}(I^d, \mathrm{B}(E, F))$.*

(iii) *$T \in \mathscr{F}(I^d, \mathrm{C}(E, F))$.*

Proof. Consider the case $\mathscr{F} = \Lambda^{m,\varkappa}$ first.

(i)⇒(ii). Let the space

$$D := \Lambda^{m,\varkappa}(I^d, \operatorname{gra}(T(\cdot))) = \{(e, f) \in \Lambda^{m,\varkappa}(I^d, E \times F) \mid T(x)e(x) \equiv f(x)\}$$

be endowed with the norm $\|e\|_{m,\varkappa} + \|f\|_{m,\varkappa}$ and define

$$S(x)\colon D \to E \times F, \quad S(x)(e, f) := (e(x), f(x)) \quad \text{for } x \in I^d.$$

By Lemma 0.7 the operator function $x \mapsto S(x)$ is of class $\Lambda^{m,\varkappa}(I^d, \mathrm{B}(D, E \times F))$. By assumption for each $\xi \in E$ the function $g_\xi(x) := (\xi, T(x)\xi)$ belongs to D. Thus there is a linear imbedding $j\colon E \to D$, $\xi \mapsto g_\xi$, which is bounded by the closed graph theorem. With the canonical projection $\pi_F\colon E \times F \to F$ we have $T(x) = \pi_F \circ S(x) \circ j$, which shows that in fact $T \in \Lambda^{m,\varkappa}(I^d, \mathrm{B}(E, F))$.

(ii)⇒(iii): easy to see. (iii)⇒(i): analogous to the proof of Cor. 10 in [3], using our Th. 2.1.

From what is already proved the case $\mathcal{F} = C^\infty$ follows by observing that $C^\infty = \bigcap_{m=0}^{\infty} \Lambda^{m,1/2}$. ■

2.3. Corollary. *Let $M \in \mathcal{F}(I^d, \mathrm{C}(F))$. Assume that $M(x_0)$ is complemented in F for some $x_0 \in I^d$. Then:*

(i) *There is a neighborhood Y of x_0 and a projection-valued function $P \in \mathcal{F}(Y, \mathrm{B}(F, F))$ such that $\operatorname{ran}(P(x)) = M(x)$ for $x \in Y$.*

(ii) *There is a neighborhood Y of x_0 and a function $T \in \mathcal{F}(Y, \mathrm{B}(F, F))$ such that $T(x)$ is invertible and $T(x)M(x_0) = M(x)$ for $x \in Y$.*

Proof. Analogous to the proof of Cor. 11 in [3]. ■

2.4. Corollary. *Let $T \in \mathcal{F}(I^d, \mathrm{C}(E, F))$, $\operatorname{ran}(T(x))$ closed for each x. Then the following conditions are equivalent:*

(i) *T is regular.*

(ii) $\operatorname{ran}(T(\cdot)) \in \mathcal{F}(I^d, \mathrm{C}(F))$.

(iii) $\ker(T(\cdot)) \in \mathcal{F}(I^d, \mathrm{C}(E))$.

Let one (hence all) of the above conditions be satisfied. Let $f \in \mathcal{F}(I^d, \operatorname{ran}(T(\cdot)))$, $x_0 \in I^d$ and $\xi_0 \in \operatorname{dom}(T(x_0))$ with $T(x_0)\xi_0 = f(x_0)$. Then there exists a function $e \in \mathcal{F}(I^d, E)$ such that $e(x_0) = \xi_0$ and $e(x) \in \operatorname{dom}(T(x))$, $T(x)e(x) = f(x)$ for $x \in I^d$.

Proof. Consider $M_1(x) = \operatorname{gra}(T(x))$, $M_2(x) = E \times \{0\}$ in $E \times F$ and $w = (0, f) \in \mathcal{F}(I^d, M_1 + M_2)$. Then $M_1(x) + M_2(x) = E \times \operatorname{ran}(T(x))$ and $M_1(x) \cap M_2(x) = \ker(T(x)) \times \{0\}$. The assertion follows by application of Th. 2.1. ■

The most frequent situation where this result can be applied is the following one:

2.5. Proposition. *Let D be a linear subspace of E and $T\colon I^d \to \mathrm{C}(E, F)$ such that $\operatorname{dom}(T(x)) = D$, $x \in I^d$. Assume that for each $e \in D$ the function $x \mapsto T(x)e$ belongs to $\mathcal{F}(I^d, F)$. Then $T \in \mathcal{F}(I^d, \mathrm{C}(E, F))$.*

Proof. We only have to show the continuity of the mapping $x \mapsto \operatorname{gra}(T(x))$. To this end we consider the norms

$$\|e\|_x := \|e\| + \|T(x)e\| \quad \text{for } x \in I^d,$$

$$\|e\|_* := \|e\| + \sup\{\|T(x)e\| \mid x \in I^d\}$$

on D. Fix $x \in I^d$ and let $D_x := (D, \|\cdot\|_x)$, $D_* := (D, \|\cdot\|_*)$. Then D_x is a Banach space and by Cor. 2.2 we have $T \in \mathcal{F}(I^d, \mathrm{B}(D_x, F))$. Thus there is $c_x > 0$ with $\|\cdot\|_x \leqslant \|\cdot\|_* \leqslant c_x \|\cdot\|_x$. Since $T\colon I^d \to \mathrm{B}(D_*, F)$ is continuous there exists a neighborhood Y of x such that

$$\|T(x) - T(y)\|_{\mathrm{B}(D_*, F)} \leqslant 1/(2c_x) \quad \text{for } y \in Y.$$

This implies

$$\|e\|_* \leqslant c_x\|e\|_x \leqslant c_x\big(\|e\|_y + \|(T(y) - T(x))e\|\big) \leqslant c_x\|e\|_y + \tfrac{1}{2}\|e\|_*,$$

i.e. $\|e\|_* \leqslant 2c_x\|e\|_y$ for $y \in Y$ and $e \in D$.

For $y_1, y_2 \in Y$ we obtain

$$\begin{aligned}\delta\big(\operatorname{gra}(T(y_1)), \operatorname{gra}(T(y_2))\big) &\leqslant \sup\{\|(T(y_1) - T(y_2))e\| \mid e \in D,\ \|e\| + \|T(y_1)e\| \leqslant 1\} \\ &\leqslant \sup\{\|(T(y_1) - T(y_2))e\| \mid e \in D,\ \|e\|_* \leqslant 2c_x\}\end{aligned}$$

if the norm on $E \times F$ is defined to be $\|e\| + \|f\|$. It follows that

$$\lim_{y \to x} \Delta\big(\operatorname{gra}(T(y)), \operatorname{gra}(T(x))\big) \leqslant 2c_x \lim_{y \to x} \|T(y) - T(x)\|_{\mathrm{B}(D_*, F)} = 0. \ ■$$

Remark. If $R \in \mathrm{C}(E \times F)$ is a closed relation then the definitions

$$\operatorname{dom}(R) := \{e \in E \mid \text{there is } f \in F \text{ with } (e, f) \in R\},$$

$$\operatorname{ran}(R) := \{f \in F \mid \text{there is } e \in E \text{ with } (e, f) \in R\},$$

$$\ker(R) := \{e \in E \mid (e, 0) \in R\},$$

$$\operatorname{gra}(R) := R$$

are natural. With a slight abuse of notation we may write "$Re = f$" iff $(e, f) \in R$. Then Cor. 2.4 is easily seen to hold also in the more general situation where the operator function T is replaced by a relation mapping $R \in \mathcal{F}(I^d, \mathrm{C}(E \times F))$.

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