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# Isomorphic classification and lifting theorems for spaces of differentiable functions with Lipschitz conditions

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**Abstract.** If $F$ is a sequentially complete locally convex space, $\Omega$ a compact subset of $\mathbf{R}^d$ and $\varkappa$ a Lipschitz function we consider the sets $\Lambda^{m,\varkappa}(\Omega, F)$, $\lambda^{m,\varkappa}(\Omega, F)$ of $m$ times differentiable functions $\Omega \to F$ whose derivatives of order $m$ satisfy an $O(\varkappa)$- resp. $o(\varkappa)$-Lipschitz condition. For a broad class of functions $\varkappa$ we produce explicit isomorphisms $\Lambda^{m,\varkappa}(\Omega, F) \cong l_\infty(F)$ and $\lambda^{m,\varkappa}(\Omega, F) \cong c_0(F)$. The special construction used enables us to improve lifting theorems which have been established by W. Kaballo [13]. Furthermore, in the case of Banach spaces we solve vector function equations $T(x)e(x) \equiv f(x)$ where all data are $\Lambda^{m,\varkappa}$- resp. $\lambda^{m,\varkappa}$-dependent on the parameter $x$.

**0. Introduction.** The general lifting problem may be described as follows. Let a set $\Omega$ and a linear surjection $T\colon E \to F$ of two locally convex spaces (l.c.s.) $E$, $F$ be given. Then we look for function spaces $\mathscr{F}_1(\Omega, E)$ and $\mathscr{F}_2(\Omega, F)$ such that for any $f \in \mathscr{F}_2(\Omega, F)$ the equation $Te(x) = f(x)$, $x \in \Omega$, has a solution $e \in \mathscr{F}_1(\Omega, E)$. The function $e$ is called a *lifting* of $f$. For example we may consider a topological space $\Omega$ and ask whether any continuous $f\colon \Omega \to F$ admits a continuous lifting $e\colon \Omega \to E$.

A systematic approach to such questions has been given by A. Grothendieck [7], [9], W. Kaballo [12], [13]. W. Kaballo–D. Vogt [14] and others (cf. the literature cited in [12]–[14]) through the concept of topological tensor products. Unfortunately this method seems to be suitable only for those function spaces $\mathscr{F}$ which are a priori scalarly defined, i.e. which satisfy $\mathscr{F}(\Omega, E) \cong \mathscr{F}(\Omega)\,\varepsilon\, E$ (Schwartz' $\varepsilon$-product). In many applications the lifting problem is additionally complicated by the fact that the operator $T$ may also depend on the variable $x$, and that the range of $T(x)$ may vary, too. In these cases the tensor product technique is of limited use only since it requires the lifting of $T(x)$ to an operator function $\tilde{T}(x)$ which is pointwise right-invertible (cf. Kaballo [12]). In order to overcome this difficulty a lifting procedure is needed which takes more heed of the infinitesimal behaviour of $T(x)$.

In [16] the author provided a constructive method for solving vector function equations of the type

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$$(*)\qquad T(x)\,e(x) = f(x), \quad x \in \Omega,$$

in Banach spaces where all the data depend differentiably (finitely or infinitely often) on a parameter $x \in \Omega \subset \mathbf{R}^d$ and $T$ enjoys a "regularity property". For given $f$ we were able to prove the existence of a differentiable solution $e$ of $(*)$. However, in the case $d \geqslant 2$ we had to consider a special function space $C^{m_1}(I)\check{\otimes}_\varepsilon \ldots \check{\otimes}_\varepsilon C^{m_d}(I)$ instead of $C^m(I^d)$, $I = [0, 1]$. It is known (cf. W. Kaballo [13]) that $C^m(I^d, F)$-functions ($m < \infty$, $d \geqslant 2$) in general do not admit $C^m(I^d, E)$-liftings, but the loss of smoothness can be made arbitrarily small. In order to measure the defect, Kaballo [13] employed the spaces $\Lambda^{m,\varkappa}$ and $\lambda^{m,\varkappa}$ of all $m$ times differentiable functions which satisfy an $O(\varkappa)$- resp. $o(\varkappa)$-Lipschitz condition (see § 3 below). His lifting theorems are based on the isomorphisms $\lambda^{m,\varkappa}(I^d) \cong c_0$ (cf. [1], [4]) and $\lambda^{m,\varkappa}(I^d, E) \cong \lambda^{m,\varkappa}(I^d)\,\varepsilon\, E$. However, the result for $\Lambda^{m,\varkappa}$-functions was not optimal because it had to be derived from the $\lambda^{m,\varkappa}$-case.

The aim of this paper is the following: For any sequentially complete l.c.s. $F$ we are going to construct an explicit imbedding $C^m(I^d, F) \to F^{\mathbf{N}}$ from the space of all $m$ times continuously differentiable $F$-valued functions into the space of all $F$-valued sequences (§ 2). This imbedding will turn out to induce simultaneous isomorphisms

$$(**)\qquad \Lambda^{m,\varkappa}(I^d, F) \cong l_\infty(F) \quad\text{and}\quad \lambda^{m,\varkappa}(I^d, F) \cong c_0(F)$$

for quite general Lipschitz functions $\varkappa$, including the standard Hölder functions $\varkappa(\varrho) = \varrho^\vartheta$, $0 < \vartheta < 1$ (§ 3). As an application we shall improve lifting theorems of [13] for the above function classes (§ 4). Results of the type $(**)$ have been established by many authors (cf. [1]–[5], [19]). The main justification for our construction (and the ultimate goal of this paper) is contained in §5: In the case of Banach spaces we are able to solve the vector function equation $(*)$, producing a solution $e$ which is as smooth as $f$ and $T$.

This result may be viewed as a supplement to the author's paper [16]. It is used in the article that follows to study bundles of subspaces and closed operators in Banach spaces.

**1. Preliminaries.** All locally convex spaces (l.c.s.) $E$, $F$ under consideration are assumed to be Hausdorff and sequentially complete. By $\sigma(E)$ we denote the set of continuous seminorms on $E$, and by $\mathrm{L}(E, F)$ the set of continuous linear mappings $E \to F$. The letters $\mathbf{R}$, $\mathbf{N}$ are reserved for the reals and positive integers respectively, and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. If $m \in \mathbf{N}_0$, $d \in \mathbf{N}$ and $\Omega \subset \mathbf{R}^d$ is a compact set satisfying $\Omega = \overline{\operatorname{int}\Omega}$ then we define $C^m(\Omega, F)$ to be the set of functions $f\colon \Omega \to F$ whose partial derivatives up to order $m$ exist in $\operatorname{int}\Omega$ and can be extended continuously to the whole of $\Omega$. We shall first restrict ourselves to the case $\Omega = I^d$, where $I = [0, 1]$ is the unit interval. The constants $m$, $d$ are assumed to be fixed throughout this paper, and $c$ denotes a positive constant which depends on $m$ and $d$ only. Let



$$\mathbf{M} := \{\alpha \in \mathbf{N}_0^d \mid |\alpha| \leqslant m\}, \qquad S(r) := \{s \in \mathbf{R}^d \mid r \cdot s \in \{0, \ldots, r\}^d\}, \quad r \in \mathbf{N}.$$

The set $S(r)$ forms a mesh of $(r+1)^d$ uniformly distributed points in $I^d$. By $\|\cdot\|$ we denote the maximum norm and by $|\cdot|$ the $l_1$-norm on $\mathbf{R}^d$. For multiindices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}_0^d$ we adopt the standard notations

$$\alpha! = \alpha_1! \ldots \alpha_d!, \quad \binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_d}{\beta_d}, \quad D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}.$$

Partition of unity. There is a $C^\infty$-function $\psi\colon \mathbf{R} \to [0, 1]$ with the following properties:

(a) $\psi(t) = 1$ for $|t| \leqslant \frac{1}{3}$, $\psi(t) = 0$ for $|t| \geqslant \frac{2}{3}$.
(b) $\psi(t) = 1 - \psi(t-1)$ for $0 \leqslant t \leqslant 1$.

(The existence of such a function $\psi$ is well known). If $r \in \mathbf{N}$ and $s \in S(r)$ we put

$$\varphi(r, s; x) := \prod_{i=1}^{d} \psi(r(x_i - s_i)).$$

Then $\varphi(r, s; \cdot) \in C^\infty(\mathbf{R}^d, [0, 1])$ and

(c) $\varphi(r, s; x) = 1$ for $\|x - s\| \leqslant 1/(3r)$, $\varphi(r, s; x) = 0$ for $\|x - s\| \geqslant 2/(3r)$.
(d) $\sum_{s \in S(r)} \varphi(r, s; x) = 1$, $x \in I^d$, where at most $2^d$ summands are $\neq 0$.
(e) $|D^\alpha \varphi(r, s; x)| \leqslant c r^{|\alpha|}$ for $x \in \mathbf{R}^d$, $|\alpha| \leqslant m+1$.

For any continuous function $f\colon \Omega \to F$ and any $q \in \sigma(F)$ the function

$$\omega^q(f; \varrho) := \sup\{q(f(x) - f(y)) \mid x, y \in \Omega,\ \|x - y\| \leqslant \varrho\}, \qquad \varrho \geqslant 0,$$

is called the *q-modulus of continuity* of $f$. If $f \in C^m(\Omega, F)$ and $q \in \sigma(F)$ we put

$$\|f\|_m^q := \sum_{\alpha \in \mathbf{M}} \sup\{q(D^\alpha f(x)) \mid x \in \Omega\}, \qquad \omega_m^q(f; \varrho) := \max_{|\alpha| = m} \omega^q(D^\alpha f; \varrho), \quad \varrho \geqslant 0.$$

The next lemma constitutes the principal tool in our subsequent investigations.

1.1. Lemma. *Let $f \in C^m(I^d, F)$ and for fixed $r \in \mathbf{N}$ let there be given a set of functions $B(s; \cdot) \in C^m(I^d, F)$, $s \in S(r)$, such that $D^\alpha B(s; s) = D^\alpha f(s)$, $\alpha \in \mathbf{M}$. Then the function*

$$\Delta(x) := \sum_{s \in S(r)} \varphi(r, s; x)\, B(s; x)$$

*satisfies*

(a) $\Delta \in C^m(I^d, F)$.
(b) $D^\alpha \Delta(s) = D^\alpha f(s)$ *for* $s \in S(r)$, $\alpha \in \mathbf{M}$.
(c) *There is a constant* $c_1 = c_1(m, d) > 0$ *such that, if* $q \in \sigma(F)$ *and* $\tau^q\colon [0, \infty) \to [0, \infty)$ *is a nondecreasing function with* $\omega_m^q(B(s; \cdot); \varrho) \leqslant \tau^q(\varrho)$ *for each* $s \in S(r)$, *then*

$$(1.1)\quad \omega_m^q(\Delta; \varrho) \leqslant c_1 \varrho r [\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})] + c_1 \tau^q(\varrho), \qquad 0 \leqslant \varrho \leqslant r^{-1}.$$

Proof. The statements (a) and (b) are evident. For the proof of (c) we partition the variables $\xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$ into $\xi_1$ and $\tilde{\xi} = (\xi_2, \ldots, \xi_d) \in \mathbf{R}^{d-1}$. (Check that our considerations remain valid also in the case $d = 1$!) We write

$$\tilde{S}(r) := \{\tilde{s} \mid s \in S(r)\}, \qquad \tilde{\varphi}(r, \tilde{s}; \tilde{x}) := \prod_{i=2}^{d} \psi\big(r(x_i - s_i)\big) \quad \text{for } \tilde{s} \in \tilde{S}(r),$$

$$A_t(x) := \sum_{\tilde{s} \in \tilde{S}(r)} \tilde{\varphi}(r, \tilde{s}; \tilde{x})\, B\big((t/r, \tilde{s}); x\big) \qquad \text{for } t \in \{0, \ldots, r\}.$$

With the definition $\psi_t(x_1) := \psi\big(r(x_1 - t/r)\big)$, $t = 0, \ldots, r$, we see that

$$\varphi(r, s; x) = \psi_{r \cdot s_1}(x_1)\, \tilde{\varphi}(r, \tilde{s}; \tilde{x}) \quad \text{and} \quad \Delta(x) = \sum_{t=0}^{r} \psi_t(x_1) A_t(x).$$

If $x \in I^d$ then $(t-1)/r \leqslant x_1 \leqslant t/r$ for some $t \in \{1, \ldots, r\}$, and

$$\Delta(x) = A_{t-1}(x) + \psi_t(x_1)\big(A_t(x) - A_{t-1}(x)\big),$$

since $\psi_{t-1}(x_1) = 1 - \psi_t(x_1)$. If now $\tilde{x} \in I^{d-1}$ is fixed and $(t-1)/r \leqslant u, v, x_1 \leqslant t/r$ we write $x = (x_1, \tilde{x})$, $y = (u, \tilde{x})$, $z = (v, \tilde{x})$. A short computation shows

$$\text{(1.2)} \quad D^\alpha \Delta(y) - D^\alpha \Delta(z)$$
$$= \big(1 - \psi_t(v)\big)\big(D^\alpha A_{t-1}(y) - D^\alpha A_{t-1}(z)\big) + \psi_t(v)\big(D^\alpha A_t(y) - D^\alpha A_t(z)\big)$$
$$+ \sum_{\beta_1=0}^{\alpha_1} \binom{\alpha_1}{\beta_1} \big(D^{\alpha_1-\beta_1}\psi_t(u) - D^{\alpha_1-\beta_1}\psi_t(v)\big) D^{(\beta_1, \tilde{\alpha})} V_t(y)$$
$$+ \sum_{\beta_1=0}^{\alpha_1-1} \binom{\alpha_1}{\beta_1} D^{\alpha_1-\beta_1}\psi_t(v)\big(D^{(\beta_1, \tilde{\alpha})} V_t(y) - D^{(\beta_1, \tilde{\alpha})} V_t(z)\big), \qquad \alpha \in \mathbf{M},$$

where $V_t(x) := A_t(x) - A_{t-1}(x)$. Our task will be to estimate the right hand side of (1.2) for $|\alpha| = m$. To this end consider the functions $B_f(s; x) := B(s; x) - f(x)$, $s \in S(r)$. Since $D^\beta B_f(s; s) = 0$, $\beta \in \mathbf{M}$, Taylor's formula yields

$$q\big(D^\beta B_f(s; x)\big) \leqslant c r^{|\beta| - m} \omega_m^q(B_f; r^{-1})$$
$$\leqslant c r^{|\beta| - m}[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})] \qquad \text{if } \|x - s\| \leqslant r^{-1}, \ \beta \in \mathbf{M}.$$

Observing that $\sum_{\tilde{s} \in \tilde{S}(r)} D^{\tilde{\beta}} \tilde{\varphi}(r, \tilde{s}; \tilde{x}) \equiv 0$ for $\tilde{\beta} \neq 0$, and

$$q\big(h(u) - h(v)\big) \leqslant |u - v| \sup_{(t-1)/r \leqslant x_1 \leqslant t/r} q\big(h'(x_1)\big) \qquad \text{for any } h \in C^1(I, F)$$

we get ($\sum'_{\tilde{\beta} \leqslant \tilde{\alpha}}$ means that $\tilde{\beta} \neq \tilde{\alpha}$ and $\beta_1 = \alpha_1$ under the sum)

$$q\big(D^\alpha A_{r \cdot s_1}(y) - D^\alpha A_{r \cdot s_1}(z)\big)$$
$$\leqslant 2^d \max_{\substack{\tilde{s} \in \tilde{S}(r) \\ \|\tilde{x} - \tilde{s}\| \leqslant 1/r}} q\Big(\sum_{\tilde{\beta} \leqslant \tilde{\alpha}}' \binom{\tilde{\alpha}}{\tilde{\beta}} D^{\tilde{\alpha} - \tilde{\beta}} \tilde{\varphi}(r, \tilde{s}; \tilde{x})\big(D^\beta B_f(s; y) - D^\beta B_f(s; z)\big)$$



$$+ \tilde{\varphi}(r, \tilde{s}; \tilde{x})\big(D^\alpha B(s; y) - D^\alpha B(s; z)\big)\Big)$$
$$\leqslant c \sum_{\tilde{\beta} \leqslant \tilde{\alpha}}' r^{|\tilde{\alpha} - \tilde{\beta}|} |u - v| r^{|\beta| + 1 - m}[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})] + \tau^q(|u - v|).$$

From this it follows that the $q$-norm of the first line of the right hand side in (1.2) is bounded by

$$c|u - v| r[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})] + \tau^q(|u - v|), \qquad |\alpha| = m.$$

Similarly we get for $(t-1)/r \leqslant x_1 \leqslant t/r$ the estimate

$$\text{(1.3)} \quad q\big(D^{(\beta_1, \tilde{\alpha})} V_t(x)\big)$$
$$\leqslant \sum_{\tilde{s} \in \tilde{S}(r)} \sum_{\tilde{\beta} \leqslant \tilde{\alpha}} \binom{\tilde{\alpha}}{\tilde{\beta}} |D^{\tilde{\alpha} - \tilde{\beta}} \tilde{\varphi}(r, \tilde{s}; \tilde{x})|\, q\Big(D^\beta B_f\Big(\Big(\frac{t}{r}, \tilde{s}\Big); x\Big) - D^\beta B_f\Big(\Big(\frac{t-1}{r}, \tilde{s}\Big); x\Big)\Big)$$
$$\leqslant c r^{\beta_1 + |\tilde{\alpha}| - m}[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})], \qquad 0 \leqslant \beta_1 \leqslant \alpha_1.$$

This yields for the terms in the second line of the right hand side of (1.2) a bound

$$|D^{\alpha_1 - \beta_1}\psi_t(u) - D^{\alpha_1 - \beta_1}\psi_t(v)|\, q\big(D^{(\beta_1, \tilde{\alpha})} V_t(y)\big)$$
$$\leqslant c|u - v| r[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})], \qquad 0 \leqslant \beta_1 \leqslant \alpha_1, \ |\alpha| = m.$$

For the last line of (1.2) we observe that $\beta_1 + |\tilde{\alpha}| < |\alpha| = m$ there. Thus we may use (1.3) to obtain

$$|D^{\alpha_1 - \beta_1}\psi_t(v)|\, q\big(D^{(\beta_1, \tilde{\alpha})} V_t(y) - D^{(\beta_1, \tilde{\alpha})} V_t(z)\big)$$
$$\leqslant c|u - v| r[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})], \qquad 0 \leqslant \beta_1 < \alpha_1, \ |\alpha| = m.$$

Collecting the above inequalities, we subsume that

$$q\big(D^\alpha \Delta(y) - D^\alpha \Delta(z)\big) \leqslant c|u - v| r[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})] + \tau^q(|u - v|)$$

for $y = (u, \tilde{x})$, $z = (v, \tilde{x})$, $(t-1)/r \leqslant u, v \leqslant t/r$ (for some $t \in \{1, \ldots, r\}$) and $|\alpha| = m$. In the case $(t-1)/r \leqslant u \leqslant t/r < v \leqslant (t+1)/r$ we note that

$$\text{(1.4)} \quad q\big(D^\alpha \Delta(y) - D^\alpha \Delta(z)\big)$$
$$\leqslant q\big(D^\alpha \Delta(y) - D^\alpha \Delta(t/r, \tilde{x})\big) + q\big(D^\alpha \Delta(t/r, \tilde{x}) - D^\alpha \Delta(z)\big)$$
$$\leqslant c|u - v| r[\tau^q(r^{-1}) + \omega_m^q(f; r^{-1})] + 2\tau^q(|u - v|).$$

If now $d = 1$ the proof is finished. In the case $d \geqslant 2$ we repeat the above arguments with respect to each variable $x_2, \ldots, x_d$. For any $y, z \in I^d$ with $\|y - z\| \leqslant r^{-1}$ it then follows from (1.4) that

$$q\big(D^\alpha \Delta(y) - D^\alpha \Delta(z)\big) \leqslant \sum_{i=1}^{d} q\big(D^\alpha \Delta(y_1, \ldots, y_i, z_{i+1}, \ldots z_d) - D^\alpha \Delta(y_1, \ldots, y_{i-1}, z_i, \ldots, z_d)\big)$$

$$\leqslant c_1 \|y-z\| r [\tau^q (r^{-1}) + \omega_m^q (f; r^{-1})] + c_1 \tau^q (\|y-z\|). \blacksquare$$

**2. Series expansions for differentiable functions.** With the aid of Lemma 1.1 we are now going to expand each function $f \in C^m(I^d, F)$ in a series $f = \sum_{k=1}^{\infty} \Delta_k^r$, where the $\Delta_k^r$ are $C^\infty$-functions with values in finite-dimensional subspaces $F_k$ of $F$. Furthermore, the partial sums $\sum_{k=1}^{n} \Delta_k^r$ interpolate $f$ and its derivatives up to order $m$ on a set $S(r^n) \subset I^d$.

2.1. CONSTRUCTION. Let $f \in C^m(I^d, F)$ and $r \in \mathbf{N}$. We define

$$B(f, r, s; x) := \sum_{\alpha \in \mathbf{M}} \frac{(x-s)^\alpha}{\alpha!} D^\alpha f(s), \quad s \in S(r),$$

$$\Delta(f, r; x) := \sum_{s \in S(r)} \varphi(r, s; x) B(f, r, s; x), \quad x \in I^d.$$

Then $B(f, r, s; \cdot) \in C^\infty(I^d, F)$ and $D^\alpha B(f, r, s; s) = D^\alpha f(s)$, $\alpha \in \mathbf{M}$. For $|\alpha| = m$, $D^\alpha B(f, r, s; \cdot)$ is a constant, i.e. $\omega_m^q(B(f, r, s; \cdot); \varrho) \equiv 0$, $q \in \sigma(F)$. From Lemma 1.1 it therefore follows that

(2.1) $$\Delta(f, r; \cdot) \in C^\infty(I^d, F),$$

(2.2) $$D^\alpha \Delta(f, r; s) = D^\alpha f(s) \quad \text{for } s \in S(r),\ \alpha \in \mathbf{M},$$

(2.3) $$\omega_m^q(\Delta(f, r; \cdot); \varrho) \leqslant c_1 \varrho r \omega_m^q(f; r^{-1}), \quad q \in \sigma(F),\ 0 \leqslant \varrho \leqslant r^{-1}.$$

We now choose $r \in \mathbf{N}_{\geqslant 2}$ and define a sequence of functions $f_n^r \in C^m(I^d, F)$ by

(2.4) $$f_0^r := f, \quad f_n^r(x) := f_{n-1}^r(x) - \Delta(f_{n-1}^r, r^n; x), \quad n \geqslant 1. \blacksquare$$

Note that $f_n^r(x) = f(x) - \sum_{k=1}^{n} \Delta(f_{k-1}^r, r^k; x)$. From (2.2) we see that

(2.5) $$D^\alpha f_n^r|_{S(r^n)} \equiv 0, \quad \alpha \in \mathbf{M},\ n \geqslant 1,$$

and from Taylor's formula

(2.6) $$\|D^\alpha f_n^r\|_0^q \leqslant c r^{-n(m-|\alpha|)} \omega_m^q(f_n^r; r^{-n}), \quad \alpha \in \mathbf{M},\ n \geqslant 1.$$

By (2.3), (2.4) we have

(2.7) $$\omega_m^q(f_n^r; \varrho) \leqslant \omega_m^q(f_{n-1}^r; \varrho) + c_1 \varrho r^n \omega_m^q(f_{n-1}^r; r^{-n}), \quad \varrho \leqslant r^{-n},\ n \geqslant 1.$$

This yields by induction

(2.8) $$\omega_m^q(f_n^r; \varrho) \leqslant \omega_m^q(f_0^r; \varrho) + c_1 \varrho \sum_{k=1}^{n} (c_1+1)^{n-k} \omega_m^q(f_0^r; r^{-k}) r^k, \quad \varrho \leqslant r^{-n}.$$

Now fix $0 < \zeta < 1$ and choose $r \in \mathbf{N}_{\geqslant 2}$ such that $r^{1-\zeta} \geqslant c_1 + 1$. Then from (2.8) it follows that

$$\omega_m^q(f_n^r; r^{-n}) \leqslant c \sum_{k=1}^{n} r^{(k-n)\zeta} \omega_m^q(f_0^r; r^{-k}), \quad n \geqslant 1.$$

Together with (2.6) this implies



(2.9) $$\sum_{\alpha \in \mathbf{M}} r^{n(m-|\alpha|)} \|D^\alpha f_n^r\|_0^q \leqslant c \sum_{k=1}^{n} r^{(k-n)\zeta} \omega_m^q(f_0^r; r^{-k}), \quad n \geqslant 1.$$

Since $f_0^r = f \in C^m(I^d, F)$ we have $\lim_{k \to \infty} \omega_m^q(f_0^r; r^{-k}) = 0$, and the right hand side in (2.9) tends to 0 as $n \to \infty$. From this it is clear that

(2.10) $$f(x) = \sum_{k=1}^{\infty} \Delta(f_{k-1}^r, r^k; x) =: \sum_{k=1}^{\infty} \Delta_k^r(f; x),$$

the series converging with respect to each seminorm $\|\cdot\|_m^q$, $q \in \sigma(F)$.

Let $\mathbf{L}(r) := \bigcup_{n=1}^{\infty} S(r^n)$ and $\mathbf{K}(r) := \mathbf{L}(r) \times \mathbf{M}$. We can now construct a linear imbedding $\Phi_F^r$: $C^m(I^d, F) \to F^{\mathbf{K}(r)}$. To this end note that $\mathbf{L}(r)$ is the disjoint union of the sets $\mathbf{L}(r, n) := S(r^n) \setminus S(r^{n-1})$, $n \geqslant 2$, and $\mathbf{L}(r, 1) := S(r^1)$. For fixed $r \in \mathbf{N}$, $r > c_1 + 1$, and any $f \in C^m(I^d, F)$ we define the sequence $(f_k^r)_{k=0}^{\infty} \subset C^m(I^d, F)$ through (2.4). Then the mapping

(2.11) $$\Phi_F^r\colon C^m(I^d, F) \to F^{\mathbf{K}(r)}, \quad (\Phi_F^r(f))_{(s,\alpha)} := r^{n(m-|\alpha|)} D^\alpha f_n^r(s)$$
$$\text{for } n \in \mathbf{N}_0,\ s \in \mathbf{L}(r, n+1),\ \alpha \in \mathbf{M},$$

is clearly linear. Recall that by (2.5) and the definition of $\Delta$ the function $\Delta(f_{k-1}^r, r^k; \cdot)$ vanishes identically if $D^\alpha f_{k-1}^r(s) = 0$ for $s \in \mathbf{L}(r, k)$, $\alpha \in \mathbf{M}$. Therefore $\Phi_F^r(f) = 0$ implies $\Delta(f_{k-1}^r, r^k; \cdot) \equiv 0$, $k \in \mathbf{N}$, hence $f \equiv 0$ by (2.10). So $\Phi_F^r$ is injective.

**3. Classification of differentiable functions with Lipschitz conditions.** A continuous function $\varkappa$: $[0, \infty) \to [0, \infty)$ will be called a *Lipschitz function* (for short: Lf) if the following holds:

$$\varkappa(\varrho) = 0 \text{ iff } \varrho = 0; \quad \varkappa \text{ and } \varrho \mapsto \varrho/\varkappa(\varrho) \text{ are nondecreasing.}$$

Such a function $\varkappa$ is always subadditive, i.e. $\varkappa(\varrho_1 + \varrho_2) \leqslant \varkappa(\varrho_1) + \varkappa(\varrho_2)$. For a compact set $\Omega \subset \mathbf{R}^d$ with $\Omega = \overline{\operatorname{int} \Omega}$ we introduce the function space

$$\Lambda^{m,\varkappa}(\Omega, F) := \{f \in C^m(\Omega, F) \mid \|f\|_{m,\varkappa}^q := \|f\|_m^q + \sup_{\varrho > 0} \omega_m^q(f; \varrho)/\varkappa(\varrho) < \infty,\ q \in \sigma(F)\}$$

and its subspace

$$\lambda^{m,\varkappa}(\Omega, F) := \{f \in \Lambda^{m,\varkappa}(\Omega, F) \mid \lim_{\varrho \to 0} \omega_m^q(f; \varrho)/\varkappa(\varrho) = 0,\ q \in \sigma(F)\}.$$

Note that, if $\Omega$ is convex, $C^{m+1}(\Omega, F) \subset \Lambda^{m,\varkappa}(\Omega, F)$ for any Lf $\varkappa$, and even $C^{m+1}(\Omega, F) \subset \lambda^{m,\varkappa}(\Omega, F)$ if $\lim_{\varrho \to 0} \varrho/\varkappa(\varrho) = 0$. Note further that in the definition of the spaces $\Lambda^{m,\varkappa}$, $\lambda^{m,\varkappa}$ only the behaviour of $\varkappa$ near 0 is important. If $\varkappa$ satisfies the conditions of a Lf only on an interval $[0, s_0]$, we may therefore replace $\varkappa(s)$ by $\tilde{\varkappa}(s) := \varkappa(s_0)$ for $s \geqslant s_0$, thus obtaining an "equivalent" Lf $\tilde{\varkappa}$.

3.1. DEFINITION. A Lf $\varkappa$ belongs to the class $(\mathrm{A}_\varepsilon \mathrm{B}_\zeta)$Lf $(0 \leqslant \varepsilon < 1,\ 0 < \zeta \leqslant 1)$ if

($\mathrm{A}_\varepsilon$) There is $a > 0$ with $\int_0^s \frac{\varkappa(t)}{t^{1+\varepsilon}} dt \leqslant a \frac{\varkappa(s)}{s^\varepsilon}$, $0 < s \leqslant 1$,

($B_\zeta$) There is $b > 0$ with $\int\limits_s^1 \frac{\varkappa(t)}{t^{1+\zeta}}\,dt \leqslant b\,\frac{\varkappa(s)}{s^\zeta}$, $0 < s \leqslant 1$.

Approximating the integrands by step functions it is easily seen that $(A_\varepsilon)\Leftrightarrow(A'_\varepsilon)$ and $(B_\zeta)\Leftrightarrow(B'_\zeta)$, where

($A'_\varepsilon$) For any (some) $r > 1$ there is $a > 0$ with

$$\sum_{k=n}^{\infty} r^{(k-n)\varepsilon}\varkappa(r^{-k}) \leqslant a\,\varkappa(r^{-n}), \qquad n\in\mathbf{N},$$

($B'_\zeta$) For any (some) $r > 1$ there is $b > 0$ with

$$\sum_{k=1}^{n} r^{(k-n)\zeta}\varkappa(r^{-k}) \leqslant b\,\varkappa(r^{-n}), \qquad n\in\mathbf{N}.$$

Remarks. (i) The conditions $(A_0)$ and $(B_1)$ have been introduced by Z. Ciesielski [2]. The most important example of a function $\varkappa\in(A_0\,B_1)$Lf is $\varkappa(s) = s^\alpha$ where $0 < \alpha < 1$. Further examples are obtained by

$$\varkappa_1(s) := s^\alpha|\ln s|^\beta, \quad \varkappa_2(s) := s^\alpha\exp(\beta|\ln s|^\gamma), \quad \varkappa_i(0) := 0,$$

where $0 < \alpha < 1$, $\gamma < 1$, $\beta\in\mathbf{R}$.

(ii) Note that $\lim_{s\to 0}\varkappa(s)/s^\varepsilon = 0$, $\lim_{s\to 0}\varkappa(s)/s^\zeta = \infty$ for any $\varkappa\in(A_\varepsilon\,B_\zeta)$Lf.

3.2. Lemma. *For any Lf $\varkappa$ the following holds:*

(i) *$\varkappa$ satisfies $(A_0)$ iff $\varkappa$ satisfies $(A_\varepsilon)$ for some $0 < \varepsilon < 1$.*

(ii) *$\varkappa$ satisfies $(B_1)$ iff $\varkappa$ satisfies $(B_\zeta)$ for some $0 < \zeta < 1$.*

Proof. The "if" part is easy to see. Let $\varkappa$ satisfy $(A_0)$ with constant $a > 1$. We then put $A := e^{2a}$ and obtain for $0 < s \leqslant 1/A$

$$2\varkappa(s) = \frac{\varkappa(s)}{a}\int\limits_s^{As}\frac{1}{t}\,dt \leqslant \frac{1}{a}\int\limits_s^{As}\frac{\varkappa(t)}{t}\,dt \leqslant \varkappa(As).$$

Now let $0 < u \leqslant v \leqslant 1/A$. Choose $n\in\mathbf{N}_0$ such that $A^n u \leqslant v < A^{n+1}u$. With $\delta := \ln 2/(2a) < 1$ it follows that

$$\frac{\varkappa(u)}{u^\delta} \leqslant \frac{\varkappa(Au)}{2u^\delta} \leqslant \ldots \leqslant \frac{\varkappa(A^n u)}{2^n u^\delta} \leqslant 2^{-n}A^{(n+1)\delta}\frac{\varkappa(v)}{v^\delta} = 2\,\frac{\varkappa(v)}{v^\delta}.$$

This yields for $0 < s \leqslant 1/A$ and $0 < \varepsilon < \delta$

$$\int\limits_0^s \frac{\varkappa(t)}{t^{1+\varepsilon}}\,dt \leqslant 2\,\frac{\varkappa(s)}{s^\delta}\int\limits_0^s t^{\delta-\varepsilon-1}\,dt = \frac{2}{\delta-\varepsilon}\,\frac{\varkappa(s)}{s^\varepsilon}.$$

Hence $\varkappa$ satisfies $(A_\varepsilon)$ for $\varepsilon < \delta$. If $\varkappa$ satisfies $(B_1)$ with constant $b > 1$ we put $B := e^{2b}$ and obtain for $0 < s \leqslant 1/B$

$$\varkappa(Bs) \leqslant \frac{Bs}{2b}\,\frac{\varkappa(Bs)}{Bs}\int\limits_s^{Bs}\frac{1}{t}\,dt \leqslant \frac{Bs}{2b}\int\limits_s^{Bs}\frac{\varkappa(t)}{t^2}\,dt \leqslant \frac{B}{2}\,\varkappa(s).$$



A similar calculation as above yields for $\delta := 1-\ln 2/(2b) < 1$

$$\frac{\varkappa(u)}{u^\delta} \geqslant \frac{2}{B}\,\frac{\varkappa(v)}{v^\delta}, \qquad 0 < u \leqslant v \leqslant \frac{1}{B}.$$

From this it follows that $\varkappa$ satisfies $(B_\zeta)$ for $\zeta > \delta$. ∎

This result is due to Hilsmann ([11], Lemma 1.5). We have reproduced the proof here for the convenience of the reader. Now let $c_1 = c_1(m, d)$ be the constant in (1.1). Choose $r\in\mathbf{N}$, $r > c_1+1$ and define the mapping $\Phi_F^r$: $C^m(I^d, F)\to F^{\mathbf{K}(r)}$ through (2.11). For $y\in F^{\mathbf{K}(r)}$ we define the expressions

$$\delta_n^q(y) := \max_{s\in\mathbf{L}(r,n+1)}\sum_{\alpha\in\mathbf{M}} q(y_{(s,\alpha)}), \qquad q\in\sigma(F),\ n\in\mathbf{N}_0.$$

3.3. Lemma. *Let $0 < \zeta < 1$ and $r^{1-\zeta} \geqslant c_1+1$. Then for any $\varkappa\in(A_0\,B_\zeta)$Lf there is a constant $K = K(r, m, d, \zeta, \varkappa) > 0$ such that for each $f\in C^m(I^d, F)$, $q\in\sigma(F)$:*

(a) $\omega_m^q(f;\varrho) \leqslant \eta\varkappa(\varrho)$ *implies* $\delta_n^q(\Phi_F^r(f)) \leqslant K\begin{cases}\|f\|_m^q & \text{if } n = 0,\\ \eta\varkappa(r^{-n}) & \text{if } n\geqslant 1.\end{cases}$

(b) $\omega_m^q(f;\varrho) = o(\varkappa(\varrho))$, $\varrho\to 0$, *implies* $\delta_n^q(\Phi_F^r(f)) = o(\varkappa(r^{-n}))$, $n\to\infty$.

Proof. Fix $q\in\sigma(F)$. By assumption we have $\omega_m^q(f;r^{-k}) = \varkappa(r^{-k})\,t_k$ with $t_k\geqslant 0$, $t_k\leqslant\eta$ in case (a), and $t_k\to 0$ in case (b). From (2.9), (2.11) it follows that

$$\delta_n^q := \delta_n^q(\Phi_F^r(f)) = \max_{s\in\mathbf{L}(r,n+1)}\sum_{\alpha\in\mathbf{M}} r^{n(m-|\alpha|)}\,q(D^\alpha f_n^r(s))$$

$$\leqslant c\sum_{k=1}^{n} r^{(k-n)\zeta}\varkappa(r^{-k})\,t_k, \qquad n\geqslant 1,$$

$$\delta_0^q = \max_{s\in\mathbf{L}(r,1)}\sum_{\alpha\in\mathbf{M}} q(D^\alpha f(s)) \leqslant c\,\|f\|_m^q.$$

In case (a) we obtain from $(B'_\zeta)$ the estimate

$$\delta_n^q \leqslant cb\eta\varkappa(r^{-n}), \qquad n\geqslant 1.$$

In case (b) we write $t_k^* := \sup_{i=k}^{\infty} t_i$ and find for $1\leqslant l\leqslant n$

$$\delta_n^q \leqslant c\sum_{k=1}^{l} r^{(k-n)\zeta}\varkappa(r^{-k})\,t_k + c\sum_{k=l+1}^{n} r^{(k-n)\zeta}\varkappa(r^{-k})\,t_k$$

$$\leqslant ct_1^*\,r^{(l-n)\zeta}\sum_{k=1}^{l} r^{(k-l)\zeta}\varkappa(r^{-k}) + ct_{l+1}^*\sum_{k=1}^{n} r^{(k-n)\zeta}\varkappa(r^{-k})$$

$$\leqslant cb\varkappa(r^{-n})\left\{t_1^*\,\frac{r^{l\zeta}\varkappa(r^{-l})}{r^{n\zeta}\varkappa(r^{-n})} + t_{l+1}^*\right\}.$$

Since $\lim_{l\to\infty} t_l^* = 0$ and $\lim_{n\to\infty} r^{n\zeta}\varkappa(r^{-n}) = \infty$, assertion (b) is proved. ∎

3.4. LEMMA. *For any* $\varkappa \in (\mathrm{A}_0\mathrm{B}_1)$ Lf *there is a constant* $K = K(r, m, d, \varkappa) > 0$ *such that for each* $f \in C^m(I^d, F)$, $q \in \sigma(F)$:

(a) $\delta_n^q(\Phi_F^r(f)) \leqslant \eta\varkappa(r^{-n})$ *implies* $\omega_m^q(f; \varrho) \leqslant K\eta\varkappa(\varrho)$ *and* $\|f\|_m^q \leqslant K\eta$.

(b) $\delta_n^q(\Phi_F^r(f)) = o(\varkappa(r^{-n}))$, $n \to \infty$, *implies* $\omega_m^q(f; \varrho) = o(\varkappa(\varrho))$, $\varrho \to 0$.

Proof. Recall that by 2.1 and (2.5)

$$\Delta(f_{n-1}^r, r^n; x) = \sum_{s \in \mathbf{L}(r,n)} \varphi(r^n, s; x) \sum_{\alpha \in \mathbf{M}} \frac{(x-s)^\alpha}{\alpha!} D^\alpha f_{n-1}^r(s), \qquad n \in \mathbf{N}.$$

For any multiindex $\beta \in \mathbf{N}_0^d$, $|\beta| \leqslant m+1$, and $x \in I^d$ we compute

$$\begin{aligned}(3.1)\qquad q(D^\beta \Delta(f_{n-1}^r, r^n; x)) &\leqslant c \max_{s \in \mathbf{L}(r,n)} \sum_{\gamma \leqslant \beta} r^{n|\beta-\gamma|} \sum_{\substack{\alpha \in \mathbf{M}\\ \alpha \geqslant \gamma}} r^{-n|\alpha-\gamma|} q(D^\alpha f_{n-1}^r(s)) \\ &\leqslant c r^{m+1+(n-1)(|\beta|-m)} \max_{s \in \mathbf{L}(r,n)} \sum_{\alpha \in \mathbf{M}} r^{(n-1)(m-|\alpha|)} q(D^\alpha f_{n-1}^r(s)) \\ &= c r^{m+1+(n-1)(|\beta|-m)} \delta_{n-1}^q,\end{aligned}$$

where $\delta_{n-1}^q := \delta_{n-1}^q(\Phi_F^r(f))$. For $|\beta| = m$ it follows that

$$\omega^q(D^\beta \Delta(f_{n-1}^r, r^n; \cdot); \varrho) \leqslant c\varrho \sum_{|\gamma| = m+1} \|D^\gamma \Delta(f_{n-1}^r, r^n; \cdot)\|_0^q \leqslant c\varrho r^{m+n} \delta_{n-1}^q.$$

If $x, y \in I^d$ and $\|x-y\| \leqslant r^{-n}$ for some $n$, then by (2.10)

$$\begin{aligned} q(D^\beta f(x) - D^\beta f(y)) &= q\Big(D^\beta \sum_{k=1}^\infty \Delta(f_{k-1}^r, r^k; x) - D^\beta \sum_{k=1}^\infty \Delta(f_{k-1}^r, r^k; y)\Big) \\ &\leqslant \sum_{k=1}^n q(D^\beta \Delta(f_{k-1}^r, r^k; x) - D^\beta \Delta(f_{k-1}^r, r^k; y)) + 2 \sum_{k=n+1}^\infty \|D^\beta \Delta(f_{k-1}^r, r^k; \cdot)\|_0^q \\ &\leqslant c r^{m+1} \Big\{\sum_{k=0}^{n-1} r^{k-n} \delta_k^q + \sum_{k=n}^\infty \delta_k^q\Big\}, \qquad |\beta| = m. \end{aligned}$$

With $\delta_k^q = \varkappa(r^{-k}) t_k$, $t_k \geqslant 0$, and $t_k^* := \sup_{i=k}^\infty t_i$ this becomes by virtue of $(\mathrm{A}_0')$

$$q(D^\beta f(x) - D^\beta f(y)) \leqslant c r^{m+1} \varkappa(r^{-n}) \Big\{\sum_{k=0}^{n-1} t_k \frac{r^k \varkappa(r^{-k})}{r^n \varkappa(r^{-n})} + a t_n^*\Big\}, \qquad |\beta| = m.$$

And, using $(\mathrm{B}_1')$,

$$\begin{aligned}(3.2)\qquad \omega_m^q(f; r^{-n}) &\leqslant c r^{m+1} \varkappa(r^{-n}) \Big\{t_0^* \sum_{k=0}^{l-1} \frac{r^k \varkappa(r^{-k})}{r^n \varkappa(r^{-n})} + t_l^* \sum_{k=l}^{n-1} \frac{r^k \varkappa(r^{-k})}{r^n \varkappa(r^{-n})} + a t_n^*\Big\} \\ &\leqslant c r^{m+1} \varkappa(r^{-n}) \Big\{t_0^* b \frac{r^l \varkappa(r^{-l})}{r^n \varkappa(r^{-n})} + t_l^*(b+a)\Big\}, \qquad 0 \leqslant l \leqslant n.\end{aligned}$$

Observing that $\lim_{n\to\infty} r^n \varkappa(r^{-n}) = \infty$ we conclude that

$$(3.3)\qquad \omega_m^q(f; r^{-n}) \leqslant K_1 \varkappa(r^{-n}) \tilde{t}_n,$$



where $(\tilde{t}_n)_{n=1}^\infty$ depends on $(t_n)_{n=1}^\infty$, $r$, and $\varkappa$ only; $\sup_n \tilde{t}_n \leqslant \sup_n t_n$, and $\tilde{t}_n \to 0$ if $t_n \to 0$. Finally, for $r^{-(n+1)} < \varrho < r^{-n}$ we get

$$\omega_m^q(f; \varrho) \leqslant \omega_m^q(f; r^{-n}) \leqslant K_1 \varkappa(r^{-n}) \tilde{t}_n \leqslant K_1 \varkappa(r\varrho) \tilde{t}_n \leqslant K_1 r \varkappa(\varrho) \tilde{t}_n,$$

from which follow the statements concerning $\omega_m^q(f; \cdot)$. Now recall that for $k \geqslant 1$, $\alpha \in \mathbf{M}$ we have $D^\alpha f_k^r(0) = 0$ by (2.5), since $0 \in S(r^k)$. Thus $D^\alpha \Delta(f_k^r, r^{k+1}; 0) = 0$ by (2.2), and

$$\sum_{\alpha \in \mathbf{M}} q(D^\alpha f(0)) = \sum_{\alpha \in \mathbf{M}} q(D^\alpha \Delta(f_0^r, r^1; 0)) \leqslant c r^{m+1} \delta_0^q \leqslant c r^{m+1} \eta$$

in view of (3.1). For $|\alpha| = m$

$$\|D^\alpha f\|_0^q \leqslant q(D^\alpha f(0)) + \omega_m^q(f; 1) \leqslant c r^{m+1} \eta + K_1 r \varkappa(1) \eta, \quad \text{i.e.}$$

$$\sum_{|\alpha| = m} \|D^\alpha f\|_0^q \leqslant K_2 \eta.$$

For $0 \leqslant k \leqslant m-1$ observe that $\omega_k^q(f; \varrho) \leqslant c\varrho \max_{|\alpha| = k+1} \|D^\alpha f\|_0^q$. This recursively leads to

$$\sum_{\alpha \in \mathbf{M}} \|D^\alpha f\|_0^q \leqslant K\eta. \ \blacksquare$$

We now introduce the weighted sequence spaces

$$l_\infty^\varkappa(\mathbf{K}(r), F) := \{y \in F^{\mathbf{K}(r)} \mid |y|_\varkappa^q := \sup_{n \in \mathbf{N}_0} \delta_n^q(y)/\varkappa(r^{-n}) < \infty, \ q \in \sigma(F)\},$$

$$c_0^\varkappa(\mathbf{K}(r), F) := \{y \in l_\infty^\varkappa(\mathbf{K}(r), F) \mid \lim_{n \to \infty} \delta_n^q(y)/\varkappa(r^{-n}) = 0, \ q \in \sigma(F)\}.$$

The subspace $c_0^\varkappa$ will be endowed with the topology induced by $l_\infty^\varkappa$. In terms of these spaces we may rephrase Lemmata 3.3 and 3.4 as follows.

3.5. THEOREM. *Let* $0 < \zeta < 1$, *choose* $r \in \mathbf{N}$ *such that* $r^{1-\zeta} \geqslant c_1 + 1$ *and let the imbedding* $\Phi_F^r$: $C^m(I^d, F) \to F^{\mathbf{K}(r)}$ *be defined through* (2.11). *Then for any* $\varkappa \in (\mathrm{A}_0\mathrm{B}_\zeta)$Lf *the mapping* $\Phi_F^r$ *induces simultaneous topological isomorphisms*

$$(3.4)\qquad \Lambda^{m,\varkappa}(I^d, F) \to l_\infty^\varkappa(\mathbf{K}(r), F) \quad \text{and} \quad \lambda^{m,\varkappa}(I^d, F) \to c_0^\varkappa(\mathbf{K}(r), F).$$

*Furthermore,* $(\Phi_F^r)^{-1}$ *identifies the subset of finite sequences in* $F^{\mathbf{K}(r)}$ *with a linear subspace of* $C^\infty(I^d, F)$.

Proof. From Lemma 3.3 it follows that $\Phi_F^r$ induces continuous linear imbeddings $\Lambda^{m,\varkappa} \to l_\infty^\varkappa$ and $\lambda^{m,\varkappa} \to c_0^\varkappa$. If $\hat{f} \in l_\infty^\varkappa(\mathbf{K}(r), F)$ we define

$$(3.5)\qquad f(x) := \sum_{n=0}^\infty \Big\{\sum_{s \in \mathbf{L}(r,n+1)} \varphi(r^{n+1}, s; x) \sum_{\alpha \in \mathbf{M}} \frac{(x-s)^\alpha}{\alpha!} r^{n(|\alpha|-m)} \hat{f}_{(s,\alpha)}\Big\}, \qquad x \in I^d.$$

As is seen from (3.1) for any $q \in \sigma(F)$ and $n \in \mathbf{N}_0$ the $\|\cdot\|_m^q$-norm of the function in braces is bounded by

$$c r^{m+1} \delta_n^q(\hat{f}) \leqslant c r^{m+1} |\hat{f}|_\varkappa^q \varkappa(r^{-n}).$$

Since $\sum_{n=0}^{\infty} \varkappa(r^{-n}) < \infty$ the sum (3.5) converges to $f(x)$ in the topology of $C^m(I^d, F)$. By induction it is easily seen that $\sum_{n=l}^{\infty}\{\ldots\} = f_l^r$, where $f_l^r \in C^m(I^d, F)$ is given through (2.4). Thus $\Phi_F^r(f) = \hat{f}$, and from Lemma 3.4(a) we obtain $f \in \Lambda^{m,\varkappa}(I^d, F)$, $\|f\|_{m,\varkappa}^q \leqslant K|\hat{f}|_\varkappa^q$. If $\hat{f} \in c_0^\varkappa$ then from part (b) of Lemma 3.4 it follows that even $f \in \lambda^{m,\varkappa}$. Thus the arrows in (3.4) are bijections, and the inverse mappings are continuous. The last assertion is clear from the definition of $\Phi_F^r$. ∎

**4. Lifting theorems.** In this section $E$, $F$ are two l.c.s. and $T \in \mathrm{L}(E, F)$. Let $e \in C^m(I^d, E)$ and $f := T \circ e \in C^m(I^d, F)$. If $(e_k^r)_{k=0}^{\infty}$, $(f_k^r)_{k=0}^{\infty}$ are the corresponding functions defined in (2.4) for fixed $r \in \mathbf{N}_{\geqslant 2}$ then it is clear that $f_k^r = T \circ e_k^r$. Hence, by definition of $\Phi^r$ we have

$$\Phi_F^r(T \circ e) = \big(T(\Phi_E^r(e))_{(s,\alpha)}\big)_{(s,\alpha)\in \mathbf{K}(r)}, \tag{4.1}$$

i.e. the functor $\Phi^r$ commutes with continuous linear mappings. Let $\eta$: $\mathbf{N} \to \mathbf{K}(r)$ be a fixed bijection, and for any $n \in \mathbf{N}$ let $l(n) \in \mathbf{N}_0$ be the unique number which satisfies $\eta(n) \in \mathbf{L}(r, l(n)+1) \times \mathbf{M}$. Put

$$\Psi_F^{r,\varkappa}\colon F^{\mathbf{K}(r)} \to F^{\mathbf{N}}, \quad (\Psi_F^{r,\varkappa}(y))_n := \frac{1}{\varkappa(r^{-l(n)})} y_{\eta(n)} \quad \text{for } y \in F^{\mathbf{K}(r)},\ n \in \mathbf{N}.$$

Then the mapping $\Psi_F^{r,\varkappa}$ induces isomorphisms

$$l_\infty^\varkappa(\mathbf{K}(r), F) \cong l_\infty(\mathbf{N}, F) := \{z \in F^{\mathbf{N}} \mid |z|^q := \sup_{n=1}^{\infty} q(z_n) < \infty,\ q \in \sigma(F)\} \quad \text{and}$$

$$c_0^\varkappa(\mathbf{K}(r), F) \cong c_0(\mathbf{N}, F) := \{z \in l_\infty(\mathbf{N}, F) \mid \lim_{n\to\infty} q(z_n) = 0,\ q \in \sigma(F)\}.$$

Note that the functor $\Psi^{r,\varkappa}$ also commutes with continuous linear mappings. These observations enable us to establish lifting theorems for differentiable functions: Let $T \in \mathrm{L}(E, F)$ be surjective. $T$ is said to have the *$(l_\infty, c_0)$-lifting property* if for any $y \in c_0(\mathbf{N}, F)$ there is $x \in l_\infty(\mathbf{N}, E)$ such that $Tx_n = y_n$ for all $n$. In an analogous way the $(l_\infty, l_\infty)$- or $(c_0, c_0)$-lifting property is defined.

4.1. THEOREM *Let $\varkappa \in (\mathrm{A}_0\mathrm{B}_1)\mathrm{Lf}$. Then $T$ has the $\left\{\begin{matrix}(l_\infty, l_\infty)\\ (l_\infty, c_0)\\ (c_0, c_0)\end{matrix}\right\}$-lifting property iff for any $f \in \left\{\begin{matrix}\Lambda^{m,\varkappa}(I^d, F)\\ \lambda^{m,\varkappa}(I^d, F)\\ \lambda^{m,\varkappa}(I^d, F)\end{matrix}\right\}$ there is $e \in \left\{\begin{matrix}\Lambda^{m,\varkappa}(I^d, E)\\ \Lambda^{m,\varkappa}(I^d, E)\\ \lambda^{m,\varkappa}(I^d, E)\end{matrix}\right\}$ with $T \circ e = f$.*

Proof. This follows immediately from Lemma 3.2 and Theorem 3.5 by application of $\Psi^{r,\varkappa} \circ \Phi^r$ resp. $(\Psi^{r,\varkappa} \circ \Phi^r)^{-1}$ in $E$ and $F$. ∎

Remarks. This theorem sharpens a result of W. Kaballo [13]. If $T$: $E \to F$ is a quotient map of two l.c.s. then $T$ has the $(l_\infty, l_\infty)$-lifting property e.g. if

(i) $E$, $F$ are $(DF)$-spaces ([8], Th. 9),

(ii) the kernel of $T$ is a quasinormable Fréchet space [17],



(iii) $E$ is a Fréchet space and $F$ is a Fréchet–Montel space.

As is well known the operator $T$ enjoys the $(c_0, c_0)$-lifting property if $E$ and $F$ are Fréchet spaces. Additional information can be found in [13], 1.22, and the literature cited there.

For any $0 < \vartheta \leqslant 1$ we introduce the function spaces $\Lambda^{m,\vartheta}$, $\lambda^{m,\vartheta}$ corresponding to $\varkappa_\vartheta(s) := s^\vartheta$.

4.2. THEOREM. *Suppose that $T$ has the $(l_\infty, c_0)$-lifting property and that the kernel $\mathrm{N}(T)$ of $T$ is a Fréchet space. Then any $f \in \Lambda^{m,1}(I^d, F)$ admits a lifting $e \in \bigcap_{0<\vartheta<1} \lambda^{m,\vartheta}(I^d, E)$.*

Proof. Let $\vartheta_n := 1-1/(n+1)$, $n \in \mathbf{N}$, and let $p_1 \leqslant p_2 \leqslant \ldots$ be a fundamental sequence of seminorms in $\mathrm{N}(T)$. Since $f \in \lambda^{m,\vartheta_{n+1}}(I^d, F)$, by Theorem 4.1 we obtain liftings $e_n \in \Lambda^{m,\vartheta_{n+1}}(I^d, E) \subset \lambda^{m,\vartheta_n}(I^d, E)$. Define a sequence of functions $e_n' \in \lambda^{m,\vartheta_n}(I^d, E)$ satisfying $T \circ e_n' = f$ in the following way: Let $e_1' := e_1$ and assume that the functions $e_1', \ldots, e_n'$ are already defined so that $d_n := e_{n+1} - e_n' \in \lambda^{m,\vartheta_n}(I^d, \mathrm{N}(T))$. Since $\lambda^{m,\vartheta_n}(I^d, \mathrm{N}(T)) \cong c_0(\mathbf{N}, \mathrm{N}(T))$ and since the set of finite sequences is dense in the latter space we get from Th. 3.5 a function $d_n' \in C^\infty(I^d, \mathrm{N}(T))$ such that $\|d_n' - d_n\|_{m,\vartheta_n}^{p_n} \leqslant 2^{-n}$. Then we define $e_{n+1}' := e_{n+1} - d_n'$. By construction we have

$$\|e_{n+1}' - e_n'\|_{m,\vartheta_n}^{p_n} \leqslant 2^{-n}, \quad \text{hence} \quad \|e_{n+1}' - e_n'\|_{m,\vartheta_k}^{p_k} \leqslant 2^{-k} \quad \text{for } n \geqslant k,$$

because of $\|\cdot\|_{m,\vartheta_k}^{p_k} \leqslant \|\cdot\|_{m,\vartheta_n}^{p_n}$. Invoking the sequential completeness of $\lambda^{m,\vartheta_n}(I^d, E)$ (consequence of 3.5) we conclude that $e(x) := \lim_{n\to\infty} e_n'(x)$ belongs to $\bigcap_{n=1}^{\infty} \lambda^{m,\vartheta_n}(I^d, E)$, and $T \circ e = f$. ∎

Note that if $\varkappa$ is a Lf satisfying $(\mathrm{B}_1)$ then by Lemma 3.2 there is $0 < \vartheta < 1$ and $K > 0$ with $\varkappa(s) \geqslant Ks^\vartheta$, $0 \leqslant s \leqslant 1$. It follows that the solution $e$ in Theorem 4.2 is of class $\lambda^{m,\varkappa}(I^d, E)$ also. G. M. Henkin [10] proved that in general there is no lifting $e \in \Lambda^{m,\varkappa}(I^d, E)$, $\varkappa(s) = s|\ln s|$, even in the case of Banach spaces $E$, $F$, when $d \geqslant 2$.

Now let $\Omega \subset \mathbf{R}^d$ be a compact set with $\Omega = \overline{\operatorname{int}\Omega}$. In order to carry over our results to functions defined on $\Omega$ we employ a theorem of G. Glaeser ([6], Ch. I, 12–14).

4.3. THEOREM. *Let $\varkappa \in \mathrm{Lf}$ and $Q \subset \mathbf{R}^d$ a compact cube containing $\Omega$. Then there exists a linear operator $\wedge$: $C^m(\Omega, F) \to C^m(Q, F)$ which takes $\Lambda^{m,\varkappa}(\Omega, F)$ continuously into $\Lambda^{m,\varkappa}(Q, F)$ such that $\hat{f}|_\Omega = f$ for each $f \in C^m(\Omega, F)$. If $\lim_{s\to 0} s/\varkappa(s) = 0$ then also $\wedge(\lambda^{m,\varkappa}(\Omega, F)) \subset \lambda^{m,\varkappa}(Q, F)$.*

(Glaeser essentially stated this result for Banach spaces $F$ and concave $\varkappa$, but there are no serious difficulties in proving our version of Theorem 4.3 with his methods.) Now the problem of lifting a function, say $f \in \Lambda^{m,\varkappa}(\Omega, F)$, reduces to finding a lifting of $\hat{f} \in \Lambda^{m,\varkappa}(Q, F)$. Another consequence of Theorem 4.3 is the following. Note that the operator $\wedge$ identifies $\Lambda^{m,\varkappa}(\Omega, F)$ with a complemented subspace of $\Lambda^{m,\varkappa}(Q, F)$. On the other hand, since $\operatorname{int}\Omega \neq \emptyset$ there exists a compact cube $Q' \subset \Omega$. Again by 4.3, $\Lambda^{m,\varkappa}(Q', F)$ may be identified with a complemented subspace of $\Lambda^{m,\varkappa}(\Omega, F)$. If $\varkappa \in (\mathrm{A}_0\mathrm{B}_1)\mathrm{Lf}$ then both $\Lambda^{m,\varkappa}(Q, F)$ and $\Lambda^{m,\varkappa}(Q', F)$ are isomorphic to $l_\infty(\mathbf{N}, F)$. Thus Pełczyński's decomposition method (cf. [18], Prop. 4) yields that $\Lambda^{m,\varkappa}(\Omega, F) \cong l_\infty(\mathbf{N}, F)$. The same considerations apply to show that $\lambda^{m,\varkappa}(\Omega, F) \cong c_0(\mathbf{N}, F)$. In fact, a closer inspection of the above arguments will show the following.

4.4. THEOREM. *Let $\varkappa \in (\mathrm{A}_0\mathrm{B}_1)\mathrm{Lf}$. Then there is a linear imbedding $C^m(\Omega, F) \to F^{\mathbf{N}}$ which induces simultaneous topological isomorphisms $\Lambda^{m,\varkappa}(\Omega, F) \to l_\infty(\mathbf{N}, F)$ and $\lambda^{m,\varkappa}(\Omega, F) \to c_0(\mathbf{N}, F)$.*

**5. Vector function equations in Banach spaces.** In this section $E$, $F$ are fixed Banach spaces. The space $\mathrm{L}(E, F)$ will be equipped with the uniform operator norm. If $A \in \mathrm{L}(E, F)$ then

$$\mathrm{N}(A) := \{e \in E \mid Ae = 0\}, \qquad \mathrm{R}(A) := \{Ae \mid e \in E\} \quad \text{and}$$
$$\gamma(A) := \inf\{\|Ae\|/\mathrm{dist}(e, \mathrm{N}(A)) \mid e \in E \setminus \mathrm{N}(A)\}.$$

$\gamma(A)$ is called the *minimum modulus* of $A$. If $\Omega$ is a topological space and $T\colon \Omega \to \mathrm{L}(E, F)$ an operator function then $T$ is said to be *regular* (see [16] for a discussion of this notion) if every point $x_0 \in \Omega$ has a neighbourhood $U$ in $\Omega$ such that

$$\gamma(T, U) := \inf\{\gamma(T(x)) \mid x \in U\} > 0.$$

In the sequel let $\varkappa \in (\mathrm{A}_0\mathrm{B}_1)\mathrm{Lf}$.

5.1. THEOREM. *Let the symbol $\mathscr{F}$ stand for one of the function spaces $\Lambda^{m,\varkappa}$, $\lambda^{m,\varkappa}$. Let $T \in \mathscr{F}(I^d, \mathrm{L}(E, F))$ be regular and $f \in \mathscr{F}(I^d, F)$ such that $f(x) \in \mathrm{R}(T(x))$, $x \in I^d$. Let $\xi \in I^d$ and $M$ a subset of $\mathbf{M}$ such that $\alpha \in M$, $\beta \leqslant \alpha$ imply $\beta \in M$. Then there is a solution $e \in \mathscr{F}(I^d, E)$ of the equation $T(x)e(x) \equiv f(x)$. If $D^\alpha f(\xi) = 0$ for $\alpha \in M$, then we may also achieve that $D^\alpha e(\xi) = 0$, $\alpha \in M$.*

5.2. COROLLARY. *Let the symbol $\mathscr{F}$ stand for one of the function spaces $\Lambda^{m,\varkappa}$, $\lambda^{m,\varkappa}$. Suppose there are given Banach spaces $E_p, E_{p-1}, \ldots, E_0$ $(p \geqslant 2)$ and operator functions $T_j \in \mathscr{F}(I^d, \mathrm{L}(E_{j+1}, E_j))$, $j = 0, \ldots, p-1$, such that the sequence*

$$E_p \xrightarrow{T_{p-1}(x)} E_{p-1} \to \ldots \xrightarrow{T_1(x)} E_1 \xrightarrow{T_0(x)} E_0$$

*is exact for each $x \in I^d$. In the case $p = 2$ we additionally assume that $\mathrm{R}(T_0(x))$ is closed, $x \in I^d$. Then the induced sequence*

$$\mathscr{F}(I^d, E_p) \xrightarrow{[T_{p-1}]} \mathscr{F}(I^d, E_{p-1}) \to \ldots \xrightarrow{[T_1]} \mathscr{F}(I^d, E_1) \xrightarrow{[T_0]} \mathscr{F}(I^d, E_0)$$

*given by $([T_j]\, e_{j+1})(x) := T_j(x)\, e_{j+1}(x)$ is also exact.*

The proof of 5.2 is analogous to that of Theorem 6.1 in [16]. Our constructive proof of Theorem 5.1 is based on Lemma 1.1, Theorem 3.5 and the following lemma which is a consequence of Cor. 2.7 in [16]. Notations are adopted from the preceding sections with obvious simplifications such as $\omega(f; \varrho) := \omega^q(f; \varrho)$, $\|f\|_k := \|f\|_k^q$, where $q$ is the norm of $E$ resp. $F$ resp. $\mathrm{L}(E, F)$.



5.3. LEMMA. *Let $T \in C^m(I^d, \mathrm{L}(E, F))$ be regular and $f \in C^m(I^d, F)$ such that $f(x) \in \mathrm{R}(T(x))$, $x \in I^d$. Let*

$$\Gamma_k(T) := (1 + \|T\|_k)^k (1 + 1/\gamma(T, I^d))^{k+1}, \qquad k = 0, \ldots, m.$$

*Then for any $\xi \in I^d$ there are vectors $(e_\xi^\alpha)_{\alpha \in \mathbf{M}} \subset E$ such that*

(a) $$\|e_\xi^\alpha\| \leqslant c\Gamma_{|\alpha|}(T) \sum_{\beta \leqslant \alpha} \|D^\beta f(\xi)\|, \qquad \alpha \in \mathbf{M},$$

(b) $$D^\alpha f(\xi) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta T(\xi)\, e_\xi^{\alpha-\beta}, \qquad \alpha \in \mathbf{M}.$$

5.4. CONSTRUCTION. Let the assumptions be as in Theorem 5.1 and choose $r \in \mathbf{N}_{\geqslant 2}$. Below we shall impose conditions (I), (II), (III) on $r$. For the sake of simplicity let $\xi \in S(r^1)$. We inductively define sequences of functions $A_n \in \Lambda^{m,\varkappa}(I^d, E)$ and $C_n \in \Lambda^{m,\varkappa}(I^d, F)$ such that $C_n(x) \in \mathrm{R}(T(x))$, $x \in I^d$, as follows. Set $C_0 := f$ and assume that $C_0, \ldots, C_{n-1}$ are already defined, $n \geqslant 1$. According to Lemma 5.3, for each $s \in \mathbf{L}(r, n)$ choose $(e_s^\alpha)_{\alpha \in \mathbf{M}} \subset E$ with

(a) $$\|e_s^\alpha\| \leqslant c\Gamma_{|\alpha|}(T) \sum_{\beta \leqslant \alpha} \|D^\beta C_{n-1}(s)\|, \qquad \alpha \in \mathbf{M},$$

(b) $$D^\alpha C_{n-1}(s) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta T(s)\, e_s^{\alpha-\beta}, \qquad \alpha \in \mathbf{M}.$$

For $s \in S(r^n)$ and $x \in I^d$ put

(c) $$A_n(s; x) := \begin{cases} \sum_{\alpha \in \mathbf{M}} \dfrac{(x-s)^\alpha}{\alpha!} e_s^\alpha & \text{if } s \in \mathbf{L}(r, n), \\ 0 & \text{if } s \in S(r^{n-1}), \end{cases} \qquad B_n(s; x) := T(x)\, A_n(s; x),$$

(d) $$A_n(x) := \sum_{s \in S(r^n)} \varphi(r^n, s; x)\, A_n(s; x), \qquad B_n(x) := \sum_{s \in S(r^n)} \varphi(r^n; s; x)\, B_n(s; x),$$

(e) $$C_n(x) := C_{n-1}(x) - B_n(x). \ \blacksquare$$

Proof of Theorem 5.1. We are going to show that, if $r$ is properly chosen, $e(x) := \sum_{n=1}^\infty A_n(x)$ satisfies the conclusions of the theorem. Note that

(5.1) $$D^\alpha B_n(s) = D^\alpha B_n(s; s) = D^\alpha C_{n-1}(s) \quad \text{and}$$
$$D^\alpha C_n(s) = 0 \quad \text{for } n \in \mathbf{N},\ s \in S(r^n),\ \alpha \in \mathbf{M}.$$

(Induction on $n$, using 5.4(b)–(e).) Our first task is to estimate $\omega_m(B_n; \varrho)$: If $s \in S(r^{n-1})$ then $B_n(s; x) \equiv 0$. If $s \in \mathbf{L}(r, n)$ the reader may use the identity

$$D^\alpha B_n(s;x)-D^\alpha B_n(s;y)=\sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\{[D^{\alpha-\beta}T(x)-D^{\alpha-\beta}T(y)]D^\beta A_n(s;x)$$
$$+D^{\alpha-\beta}T(y)[D^\beta A_n(s;x)-D^\beta A_n(s;y)]\}$$

to verify that

$$\omega_m(B_n(s;\cdot);\varrho)\leqslant c\{\max_{\beta\in\mathbf{M}}\omega(D^\beta T;\varrho)+\varrho\|T\|_m\}\sum_{\beta\in\mathbf{M}}\|e_s^\beta\|.$$

With the notation

$$\tau(\varrho):=\Gamma_m(T)\{\max_{\beta\in\mathbf{M}}\omega(D^\beta T;\varrho)+\varrho\|T\|_m\}$$

we obtain from 5.4(a)

$$\omega_m(B_n(s;\cdot);\varrho)\leqslant c\tau(\varrho)\|C_{n-1}\|_m,\quad n\in\mathbf{N},\ s\in S(r^n).$$

On applying Lemma 1.1(c) this gives us in view of (5.1)

(5.2) $\omega_m(B_n;\varrho)$

$$\leqslant c\varrho r^n[\tau(r^{-n})\|C_{n-1}\|_m+\omega_m(C_{n-1};r^{-n})]+c\tau(\varrho)\|C_{n-1}\|_m,\quad \varrho\leqslant r^{-n}.$$

From the second line of (5.1) and Taylor's formula we get

(5.3) $\|D^\alpha C_{n-1}\|_0\leqslant cr^{(n-1)(|\alpha|-m)}\omega_m(C_{n-1};r^{1-n})$, hence

$$\|C_{n-1}\|_m\leqslant c\omega_m(C_{n-1};r^{1-n})\leqslant cr\omega_m(C_{n-1};r^{-n}),\quad n\geqslant 2.$$

With a constant $c_2=c_2(m,d)>0$ formulae 5.4(e), (5.2) and (5.3) yield for $n\geqslant 2$, $\varrho\leqslant r^{-n}$

(5.4) $\omega_m(C_n;\varrho)$

$$\leqslant\omega_m(C_{n-1};\varrho)+c_2[\varrho r^n\tau(r^{-n})+\tau(\varrho)]\,\omega_m(C_{n-1};r^{1-n})+c_2\varrho r^n\omega_m(C_{n-1};r^{-n})$$
$$\leqslant\omega_m(C_{n-1};\varrho)+c_2[\varrho r^{n+1}\tau(r^{-n})+\varrho r^n+r\tau(\varrho)]\,\omega_m(C_{n-1};r^{-n}).$$

In order to solve this recursion formula we need a certain a priori estimate. By assumption $\tau(\varrho)\leqslant K_T\varkappa(\varrho)$, where $K_T>0$ depends on $T$ only. From (5.4) it follows that

$$\omega_m(C_n;\varrho)\leqslant\omega_m(C_{n-1};\varrho)+c_2\frac{\varkappa(\varrho)}{\varkappa(r^{-n})}$$
$$\times\left\{K_T r\frac{\varrho}{\varkappa(\varrho)}\frac{\varkappa(r^{-n})}{r^{-n}}\varkappa(r^{-n})+\frac{\varrho}{\varkappa(\varrho)}\frac{\varkappa(r^{-n})}{r^{-n}}+K_T r\varkappa(r^{-n})\right\}\omega_m(C_{n-1};r^{-n}).$$

Since $\varrho\mapsto\varrho/\varkappa(\varrho)$ is nondecreasing and $\lim_{n\to\infty}\varkappa(r^{-n})=0$ we get with $c_3:=2c_2$

$$\omega_m(C_n;\varrho)\leqslant\omega_m(C_{n-1};\varrho)+c_2\frac{\varkappa(\varrho)}{\varkappa(r^{-n})}\{1+2K_T r\varkappa(r^{-n})\}\,\omega_m(C_{n-1};r^{-n})$$



$$\leqslant\omega_m(C_{n-1};\varrho)+c_3\frac{\varkappa(\varrho)}{\varkappa(r^{-n})}\omega_m(C_{n-1};r^{-n}),\quad \varrho\leqslant r^{-n},$$

for $n$ sufficiently large, say $n\geqslant N\geqslant 2$. By induction we obtain

(5.5) $\omega_m(C_n;\varrho)$

$$\leqslant\omega_m(C_N;\varrho)+c_3\varkappa(\varrho)\sum_{k=N+1}^{n}(c_3+1)^{n-k}\frac{\omega_m(C_N;r^{-k})}{\varkappa(r^{-k})},\quad n\geqslant N,\ \varrho\leqslant r^{-n}.$$

Now recall that $C_N=f-\sum_{k=1}^N B_k$, where $f, B_k\in\Lambda^{m,\varkappa}(I^d,F)$. It follows that $C_N\in\Lambda^{m,\varkappa}(I^d,F)$, i.e. $\omega_m(C_N;\varrho)\leqslant K\varkappa(\varrho)$. Note that by Lemma 3.2 we have $\varkappa\in(\mathrm{A}_\varepsilon\mathrm{B}_\zeta)$Lf for some $0<\varepsilon<\zeta<1$.

Choose $r$ so large that

(I) $$r^{\varepsilon/2}\geqslant c_3+1.$$

Then from (5.5) it follows that

(5.6) $$\omega_m(C_n;r^{-n})\leqslant K\varkappa(r^{-n})+c_3K\varkappa(r^{-n})\sum_{k=N+1}^{n}r^{(n-k)\varepsilon/2}$$
$$\leqslant K_* r^{n\varepsilon/2}\varkappa(r^{-n}),\quad n\geqslant N,$$

since $\sum_{k=N+1}^{\infty}r^{-k\varepsilon/2}<\infty$. Here $K_*>0$ depends on $T, f, r, m, d, \varkappa$ and $\varepsilon$. We go back to the first inequality of formula (5.4) which, in view of (5.6), now reads

$$\omega_m(C_n;\varrho)\leqslant\omega_m(C_{n-1};\varrho)+K_*[\varrho r^n\tau(r^{-n})+\tau(\varrho)]r^{(n-1)\varepsilon/2}\varkappa(r^{1-n})$$
$$+c_2\varrho r^n\omega_m(C_{n-1};r^{-n}),\quad n\geqslant N,\ \varrho\leqslant r^{-n}.$$

By assumption we have $\tau(\varrho)\leqslant\varkappa(\varrho)h_T(\varrho)$, where $h_T\colon[0,1]\to[0,\infty)$ is a nondecreasing function, and $\lim_{\varrho\to 0}h_T(\varrho)=0$ if $T\in\lambda^{m,\varkappa}(I^d,\mathrm{L}(E,F))$. Note also that, since $\varkappa$ satisfies $(\mathrm{B}_\zeta)$,

$$b\frac{\varkappa(u)}{u^\zeta}\geqslant\varkappa(v)\int_v^1\frac{1}{t^{1+\zeta}}dt\geqslant\frac{1-2^{-\zeta}}{\zeta}\frac{\varkappa(v)}{v^\zeta},\quad 0<u\leqslant v\leqslant\tfrac{1}{2}.$$

Thus, for $n\geqslant N$ and $\varrho=u\leqslant v=r^{-n}$,

$$\varrho r^n\varkappa(r^{-n})=\varrho^{1-\zeta}r^n\varkappa(\varrho)\frac{\varrho^\zeta}{\varkappa(\varrho)}\varkappa(r^{-n})$$
$$\leqslant\varrho^{1-\zeta}r^n\varkappa(\varrho)\frac{\zeta b}{1-2^{-\zeta}}r^{-n\zeta}\leqslant\frac{\zeta b}{1-2^{-\zeta}}r^{n\varepsilon/2}\varrho^{\varepsilon/2}\varkappa(\varrho),$$

if $1-\zeta\geqslant\varepsilon/2$, which we may assume without loss of generality. This yields for $\varrho\leqslant r^{-n}$

$$\omega_m(C_n;\varrho)\leqslant\omega_m(C_{n-1};\varrho)+K_*[\varrho r^n\varkappa(r^{-n})h_T(r^{-n})+\varkappa(\varrho)h_T(\varrho)]\,r^{(n-1)\varepsilon/2}\varkappa(r^{1-n})$$
$$+c_2\varrho r^n\omega_m(C_{n-1};r^{-n})$$

$$\leqslant \omega_m(C_{n-1};\varrho)+K_*\varkappa(\varrho)\,[r^{n\varepsilon/2}\varrho^{\varepsilon/2}h_T(r^{-n})+h_T(\varrho)]\,r^{(n-1)\varepsilon/2}\varkappa(r^{1-n}) + c_2\,\varrho r^n\,\omega_m(C_{n-1};\,r^{-n})$$

$$\leqslant \omega_m(C_{n-1};\varrho)+K_*\varkappa(\varrho)\,h_1(\varrho)\,r^{(n-1)\varepsilon}\varkappa(r^{1-n})+c_2\,\varrho\,r^n\,\omega_m(C_{n-1};\,r^{-n}),$$

with $h_1(\varrho) = \varrho^{\varepsilon/2}h_T(1)+h_T(\varrho)$. $K_*$ now depends on $b$, $\zeta$ also. By induction we obtain

$$\omega_m(C_n;\varrho) \leqslant \omega_m(C_N;\varrho)+K_*\varkappa(\varrho)\,h_1(\varrho)\sum_{k=N+1}^{n} r^{(k-1)\varepsilon}\varkappa(r^{1-k})$$
$$+c_2\,\varrho\sum_{k=N+1}^{n}(c_2+1)^{n-k}r^k\,\omega_m(C_N;r^{-k})$$
$$+c_2\,K_*\,\varrho\sum_{k=N+2}^{n}(c_2+1)^{n-k}r^k\varkappa(r^{-k})\,h_1(r^{-k})\sum_{i=N+1}^{k-1}r^{(i-1)\varepsilon}\varkappa(r^{1-i}).$$

After increasing $K_*$ again, because of $\sum_{i=N+1}^{\infty} r^{(i-1)\varepsilon}\varkappa(r^{1-i}) < \infty$, this yields

$$(5.7)\quad \omega_m(C_n;\varrho) \leqslant \omega_m(C_N;\varrho)+K_*\varkappa(\varrho)\,h_1(\varrho)$$
$$+c_2\,\varrho\sum_{k=N+1}^{n}(c_2+1)^{n-k}r^k\,\omega_m(C_N;\,r^{-k})$$
$$+K_*\,\varrho\sum_{k=N+2}^{n}(c_2+1)^{n-k}r^k\varkappa(r^{-k})\,h_1(r^{-k}),\qquad n\geqslant N,\ \varrho\leqslant r^{-n}.$$

Now fix the constant $c_2 = c_2(m,d) > 0$ in (5.7) and choose

$$\text{(II)}\qquad r \geqslant (c_2+1)^{1/(1-\zeta)}.$$

Let $h_2(\varrho) := \sup_{0<t\leqslant\varrho}\omega_m(C_N;t)/\varkappa(t)$. From (5.7) it follows that

$$\omega_m(C_n;\,r^{-n}) \leqslant \varkappa(r^{-n})\,h_2(r^{-n})+K_*\varkappa(r^{-n})\,h_1(r^{-n})$$
$$+c_2\sum_{k=N+1}^{n}r^{(k-n)\zeta}\varkappa(r^{-k})\,h_2(r^{-k})+K_*\sum_{k=N+2}^{n}r^{(k-n)\zeta}\varkappa(r^{-k})\,h_1(r^{-k}),\qquad n\geqslant N.$$

Condition ($\mathrm{B}'_\zeta$) yields (cf. the proof of Lemma 3.3)

$$(5.8)\qquad \omega_m(C_n;\,r^{-n}) \leqslant \varkappa(r^{-n})\,H(n),\qquad n\geqslant N,$$

where the function $H\colon \mathbf{N}_{\geqslant N}\to[0,\infty)$ is bounded, and $\lim_{n\to\infty}H(n) = 0$ if $C_N, T\in\lambda^{m,\varkappa}$. Note that $C_N\in\Lambda^{m,\varkappa}$ (resp. $\lambda^{m,\varkappa}$) if $f, T\in\Lambda^{m,\varkappa}$ (resp. $\lambda^{m,\varkappa}$)! $H$ depends on all the data $T, f, r, m, d, \varkappa, \varepsilon, \zeta$. Combining now 5.4(a), (5.3) and (5.8) we obtain

$$(5.9)\quad \|e_s^\alpha\| \leqslant c\Gamma_{|\alpha|}(T)\,r^{n(|\alpha|-m)}\varkappa(r^{-n})\,H(n),\qquad n\geqslant N,\ s\in\mathbf{L}(r,\,n+1),\ \alpha\in\mathbf{M}.$$

Recall that $\mathbf{K}(r) = \bigcup_{n=1}^{\infty}\mathbf{L}(r,n)\times\mathbf{M}$ (disjoint union) and put

$$\hat{e}_{(s,\alpha)} := r^{n(m-|\alpha|)}\,e_s^\alpha\qquad\text{for } n\in\mathbf{N}_0,\ s\in\mathbf{L}(r,\,n+1),\ \alpha\in\mathbf{M}.$$



Then (5.9) says that $\hat{e}\in l_\infty^\varkappa(\mathbf{K}(r), E)$, and even $\hat{e}\in c_0^\varkappa(\mathbf{K}(r), E)$ if $f, T\in\lambda^{m,\varkappa}$. Let $c_1 = c_1(m, d)$ be the constant in (1.1) and let $r$ satisfy

$$\text{(III)}\qquad r^{1-\zeta}\geqslant c_1+1$$

in addition to (I), (II). Then by Theorem 3.5 the function

$$e := (\Phi_E^r)^{-1}(\hat{e})$$

belongs to $\Lambda^{m,\varkappa}(I^d, E)$, and $e\in\lambda^{m,\varkappa}(I^d, E)$ if $f, T\in\lambda^{m,\varkappa}$. By 5.4(c), (d) we have

$$e(x) = \sum_{n=1}^{\infty}A_n(x),$$

the series converging in the topology of $C^m(I^d, E)$ (cf. the proof of Th. 3.5). Suppose now that $\xi\in S(r^1) = \mathbf{L}(r, 1)$ and $D^\alpha f(\xi) = 0$ for $\alpha\in M$. By 5.4(a) we then have $e_\xi^\alpha = 0$, $\alpha\in M$. Note that $D^\alpha A_1(\xi) = e_\xi^\alpha$ and $D^\alpha A_n(\xi) = 0$ for $n\geqslant 2$ (since $\xi\in S(r^{n-1})$). Thus, $D^\alpha e(\xi) = D^\alpha A_1(\xi) = 0$. It remains to show that $T(x)\,e(x)\equiv f(x)$. But this follows from the identity

$$f(x)-T(x)\sum_{k=1}^{n}A_k(x) = f(x)-\sum_{k=1}^{n}B_k(x) = C_n(x)$$

and the estimate (cf. (5.3), (5.8))

$$\|C_n\|_m \leqslant c\,\omega_m(C_n;\,r^{-n}) \leqslant c\,\varkappa(r^{-n})\,H(n)\to 0,\qquad n\to\infty.\ ■$$

Remarks. (i) Theorem 5.1 has first been proved in [15] for $m = 0$. The proof is considerably less complicated in this special case.

(ii) In Theorem 5.1 it is possible to prescribe initial values of $e$ and its derivatives at the point $\xi$, namely: Let $M$ be a subset of $\mathbf{M}$ such that $\alpha\in M$, $\beta\leqslant\alpha$ imply $\beta\in M$. Let there be given a set $(p^\alpha)_{\alpha\in M}\subset E$ with

$$D^\alpha f(\xi) = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}D^\beta\,T(\xi)\,e^{\alpha-\beta},\qquad \alpha\in M.$$

Consider the function

$$\tilde{f}(x) := f(x)-T(x)\sum_{\alpha\in M}\frac{(x-\xi)^\alpha}{\alpha!}\,p^\alpha,\qquad x\in I^d.$$

Then $\tilde{f}$ meets the requirements of Theorem 5.1 and $D^\alpha\tilde{f}(\xi) = 0$ for $\alpha\in M$. Thus, there is a solution $\tilde{e}\in\Lambda^{m,\varkappa}(I^d, E)$ ($\tilde{e}\in\lambda^{m,\varkappa}(I^d, E)$) of $T(x)\,\tilde{e}(x)\equiv\tilde{f}(x)$ with $D^\alpha\tilde{e}(\xi) = 0$, $\alpha\in M$. Now the function

$$e(x) := \tilde{e}(x)+\sum_{\alpha\in M}\frac{(x-\xi)^\alpha}{\alpha!}\,p^\alpha,\qquad x\in I^d,$$

satisfies $T(x)\,e(x)\equiv f(x)$ and $D^\alpha e(\xi) = p^\alpha$, $\alpha\in M$. This observation is essential in deriving a lifting theorem for $C^\infty$-functions from 5.1 (see [15], §5).

(iii) Let $\Omega$ be a compact $C^\infty$-manifold with (or without) boundary. Then by means of local coordinates we may define the function spaces $\Lambda^{m,\varkappa}(\Omega, F)$, $\lambda^{m,\varkappa}(\Omega, F)$. It should be clear that the main results of this article (4.4, 5.1, 5.2) remain valid in this more general situation. For example, the analogue of Theorem 4.4 can be obtained by imbedding $\Omega$ into $\mathbf{R}^n$, $n$ sufficiently large, and considering a $C^\infty$-retraction $\eta: U \to \Omega$, where $U$ is a compact neighbourhood of $\Omega \setminus \partial\Omega$ in $\mathbf{R}^n$. Then $\Lambda^{m,\varkappa}(\Omega, F)$ becomes a complemented subspace of $\Lambda^{m,\varkappa}(U, F)$, hence of $l_\infty(\mathbf{N}, F)$ by Th. 4.4. (The imbedding $C^m(\Omega, F) \to C^m(U, F)$ is just $f \mapsto f \circ \eta$.) Our considerations at the end of § 4 show that also $\Lambda^{m,\varkappa}(\Omega, F)$ contains a complemented subspace isomorphic to $l_\infty(\mathbf{N}, F)$, and that the same is true if $\Lambda$ is replaced by $\lambda$, $l_\infty$ by $c_0$. Thus Pełczyński's decomposition method [18] applies.

(iv) P. Furlan ([5], 6.10) pointed out that the isomorphisms $\lambda^{1,\varkappa}(S^2, \mathbf{R}) \cong c_0$, $\Lambda^{1,\varkappa}(S^2, \mathbf{R}) \cong l_\infty$ and our lifting theorem 4.1 do not hold for any Lf $\varkappa$ which does not belong to $(\mathrm{A}_0\mathrm{B}_1)$Lf (when $\lim_{x\to 0} x/\varkappa(x) = 0$). Thus, the results of this paper cannot be generalized to weight functions $\varkappa$ in a larger class than $(\mathbf{A}_0\mathbf{B}_1)$Lf!

## References


[1] R. Bonic, J. Frampton and A. Tromba, *Λ-manifolds*, J. Funct. Anal. 3 (1969), 310–320.

[2] Z. Ciesielski, *Some properties of Schauder bases of the space C ⟨0, 1⟩*, Bull. Acad. Pol. Sci. 8 (1960), 141–144.

[3] —, *On the isomorphisms of the spaces $H_\alpha$ and m*, ibid., 217–222.

[4] J. Frampton and A. Tromba, *On the classification of spaces of Hölder continuous functions*, J. Funct. Anal. 10 (1972), 336–345.

[5] P. Furlan, *Isomorphie- und Faktorisationssätze für Räume verallgemeinert Lipschitzstetiger Funktionen*, Dissertation, Dortmund 1984.

[6] G. Glaeser, *Etude de quelques algèbres Tayloriennes*, J. Analyse Math. 6 (1958), 1–125.

[7] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo 7 (1952).

[8] —, *Sur les espaces (F) et (DF)*, Summa Brasil. Math. 3 (1954), 57–123.

[9] —, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[10] G. M. Henkin, *Impossibility of a uniform homeomorphism between spaces of smooth functions of one and of n variables ($n \geq 2$)*, Math. USSR-Sb. 3 (1967), 551–561.

[11] J. Hilsmann, *Struktursätze für Banachräume holomorpher Funktionen in mehreren Variablen*, Dissertation, Dortmund 1985.

[12] W. Kaballo, *Lifting theorems for vector valued functions and the ε-tensor product*; in: Functional Analysis: Surveys and Recent Results, North-Holland Math. Stud. 27, 1977, 149–166.

[13] —, *Lifting Sätze für Vektorfunktionen und (εL)-Räume*, J. Reine Angew. Math. 309 (1979), 55–85.

[14] W. Kaballo und D. Vogt, *Lifting-Probleme für Vektorfunktionen und ⊗-Sequenzen*, Manuscripta Math. 32 (1980), 1–27.

[15] F. Mantlik, *Parameterabhängige lineare Gleichungen in Banach- und in Fréchetträumen*, Dissertation, Dortmund 1988.





[16] —, *Linear equations depending differentiably on a parameter*, Integral Equations Operator Theory 13 (1990), 231–250.

[17] A. E. Merzon, *On the covering of bounded sets in locally convex spaces*, Funct. Anal. Appl. 6 (6) (1972), 335–336.

[18] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), 209–228.

[19] J. Ryll, *Interpolating bases for spaces of differentiable functions*, ibid. 63 (1978), 125–144.



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