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On hyperreflexivity and rank one density for non-CSL algebras

by

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Abstract. If $\mathscr{A}$ is a reflexive algebra with invariant subspace lattice generated by two atoms, we prove that its rank one subalgebra is w^*-dense and that $\mathscr{A}$ is hyperreflexive precisely when the angle between the atoms is nonzero.

Introduction. In the study of nest algebras two properties have proved very useful, the density of the rank one subalgebra, first proved by Erdos in [5], and the existence of an estimate of the distance of an operator from the nest algebra in terms of the projections of the nest, obtained by Arveson [2] and Lance [8]. It is of interest to investigate whether these properties are valid in more general reflexive algebras.

Longstaff [11] has shown that a necessary condition for the first property is the complete distributivity of the lattice, while Laurie and Longstaff [10] have proved for commutative subspace lattices that this property is also sufficient. We are able to prove the density of the rank one subalgebra, in the w^*-topology, for the case of reflexive algebras whose invariant subspace lattice consists of two nontrivial complementary subspaces. This is, to our knowledge, the only noncommutative subspace lattice for which this property is known. The problem remains open even for an atomic Boolean lattice with three atoms, and also for a Boolean lattice with (infinitely many) one-dimensional atoms.

The density of the rank one subalgebra does not yield a distance estimate. Davidson and Power [4] have constructed a commutative subspace lattice which satisfies the first but not the second property. We show that our reflexive algebra fails to satisfy a distance estimate if and only if the angle between the two subspaces is zero. This is perhaps surprising since our lattice is finite and Boolean.

Kraus and Larson [7] have also given an example of two noncommuting projections whose invariant operator algebra does not satisfy a distance estimate. We generalize this example by proving that for a general two-

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atom lattice hyperreflexivity fails precisely when the angle between the two subspaces is zero.

Davidson and Harrison [3] have obtained formulae for the distance of a projection from a lattice. It is interesting to note that, in the case of a lattice consisting of two complementary subspaces, such a distance estimate may hold even when the angle between the subspaces is zero. Thus our result combined with those of Davidson and Harrison shows that the notion of a distance estimate for lattices has, in general, a wider applicability than the corresponding notion for algebras.

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1. Preliminaries. H will denote a complex Hilbert space and $L(H)$ the space of all bounded linear operators on H. $C(H)$, $C_1(H)$, or simply C, C_1, stand for the compact and trace class operators respectively. We will use the duality between $C_1(H)$ and $L(H)$. So for $T\in C_1$, $\operatorname{tr}(T)$, $\|T\|_1$ will denote the trace of T and its trace norm respectively, and the rank one operator $e\otimes f$ is defined by $e\otimes f(x) = \langle x, e\rangle f$ for all $x\in H$. Now if $\mathscr{A}\subseteq L(H)$, we write

$$^{\perp}\mathscr{A} = \{T\in C_1(H):\ \operatorname{tr}(TA) = 0 \text{ for all } A\in\mathscr{A}\}.$$

We will be concerned with two subspaces M, N of a Hilbert space H. Halmos [6] has given two useful geometric characterizations of the projections on these subspaces when they are *in generic position*, that is, when $M\cap N = M^{\perp}\cap N = M\cap N^{\perp} = M^{\perp}\cap N^{\perp} = \{0\}$:

1.1. THEOREM. *If M, N are subspaces of H in generic position, there exists a Hilbert space K and positive commuting injective contractions C, S on K with $C^2+S^2 = I$ such that P, Q (the orthogonal projections on M, N) are unitarily equivalent to*

$$\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix},\quad \begin{bmatrix} C^2 & CS\\ CS & S^2\end{bmatrix}$$

respectively.

1.2. THEOREM. *With M, N, H as in 1.1, there is a Hilbert space K and positive injective commuting contractions C, S such that $C^2+S^2 = I$, C is invertible and*

$$\begin{bmatrix} C^2 & CS\\ CS & S^2\end{bmatrix},\quad \begin{bmatrix} C^2 & -CS\\ -CS & S^2\end{bmatrix}$$

are unitarily equivalent to P, Q respectively.

A weakly closed unital subalgebra $\mathscr{A}$ of $L(H)$ is called *hyperreflexive* — a term first introduced by Arveson [1] — if and only if every operator in $^{\perp}\mathscr{A}$ can be written as the sum of an absolutely convergent series of rank one operators in $^{\perp}\mathscr{A}$. This is equivalent [1] to the validity of a *distance estimate*, i.e. the existence of a positive constant $\mathscr{K}(\mathscr{A})$, called the *hyperreflexivity constant* of $\mathscr{A}$, such that

$$\sup\{\|P^{\perp}TP\|:\ P\in\operatorname{Lat}\mathscr{A}\}\leqslant d(T,\mathscr{A})\leqslant\mathscr{K}(\mathscr{A})\sup\{\|P^{\perp}TP\|:\ P\in\operatorname{Lat}\mathscr{A}\}$$

for all $T\in L(H)$. If $\mathscr{A}$ is a nest algebra, then $\mathscr{K}(\mathscr{A}) = 1$ as shown by Arveson [2] and Lance [8]. Note that hyperreflexivity always implies reflexivity. Now if $X\in{}^{\perp}\mathscr{A}$ take $k(X, {}^{\perp}\mathscr{A})$ to be the infimum of all sums $\sum_{n=1}^{\infty}\|X_n\|_1$ where each $X_n\in{}^{\perp}\mathscr{A}$ is of rank one and $X = \sum_{n=1}^{\infty}X_n$. Arveson [1] proved that

$$\mathscr{K}(\mathscr{A}) = \sup\{k(X, {}^{\perp}\mathscr{A}):\ X\in{}^{\perp}\mathscr{A} \text{ with } \|X\|_1\leqslant 1\}. \tag{1.3}$$

We conclude this section with some useful lemmas.

1.4. LEMMA. *If $\mathscr{L}$ is a reflexive lattice of subspaces of H and $\operatorname{Alg}\mathscr{L}$ denotes the algebra of all operators on H leaving each element of $\mathscr{L}$ invariant, then*

(a) *$e\otimes f\in{}^{\perp}(\operatorname{Alg}\mathscr{L})$ if and only if there exists $P\in\mathscr{L}$ such that $f\in P$ and $e\in P^{\perp}$.*

(b) *$e\otimes f\in\operatorname{Alg}\mathscr{L}$ if and only if there exists $L\in\mathscr{L}$ such that $f\in L$ and $e\in(L_{-})^{\perp}$ where $L_{-} = \bigvee\{P\in\mathscr{L}:\ L\nsubseteq P\}$.*

(a) has been proved in [9] and (b) in [11]. For (b) we only need a completeness hypothesis for $\mathscr{L}$.

If we denote by R the subalgebra of $\operatorname{Alg}\mathscr{L}$ generated by its rank one operators then a trivial argument using the previous lemma yields the following:

1.5. PROPOSITION. *Let $\mathscr{L}$ be a complete lattice of subspaces of H. Then $^{\perp}R = \{X\in C_1:\ X(L)\subseteq L_{-} \text{ for all } L\in\mathscr{L}\}$.*

1.6. LEMMA (Katavolos). *Set $\mathscr{A} = \operatorname{Alg}\mathscr{L}$. If $\operatorname{tr}(X) = 0$ for every $X\in{}^{\perp}R$, then $\mathscr{A} = \bar{R}$ where the closure is taken in the w^*-topology of $L(H)$.*

Proof. Trivially $^{\perp}\mathscr{A}\subseteq{}^{\perp}R$. Since $\operatorname{tr}(X) = 0$ for all $X\in{}^{\perp}R$, $I\in\bar{R}$. Since $\bar{R}$ is an ideal of $\mathscr{A}$, the lemma follows. ∎

2. A w^*-density theorem. For the rest of this paper $\mathscr{L} = \{0, M, N, H\}$ where M, N are nontrivial complementary subspaces of H ($M\cap N = 0$, and $M\vee N = H$).

2.1. THEOREM. *For all M, N as above we have $\bar{R} = \operatorname{Alg}\mathscr{L}$ (closure in the w^*-topology).*

Proof. We first assume that M, N are in generic position and use Theorem 1.2. Take

$$T=\begin{bmatrix} X & Y \\ Z & \Omega \end{bmatrix}, \qquad T\in {}^{\perp}R.$$

By 1.6 it is sufficient to show that $\operatorname{tr}(T)=0$. Since $M_{-}=N$ and $N_{-}=M$ from 1.5 it follows that

(1) $TP=QTP$,
(2) $TQ=PTQ$,

where P, Q are the orthogonal projections on M, N respectively. Writing (1) in matrix form we obtain

$$\begin{bmatrix} C^2XC^2-C^2YCS+CSZC^2-CS\Omega CS & * \\ * & * \end{bmatrix}=\begin{bmatrix} XC^2-YCS & * \\ * & * \end{bmatrix}.$$

Using the facts that C is invertible, S has dense range, therefore is injective and $C^2+S^2=I$ we obtain

(3) $CZC=-SXC-SYS-C\Omega S$.

In the same manner (2) gives $CZC-C\Omega S=SXC-SYS$, which together with (3) yields

(4) $-C\Omega S=SXC$,

which is the key relation for the proof.

Multiplying (4) on the left by $C^{-1}S^{k-1}$ yields $S^{k-1}\Omega S=-C^{-1}S^kXC$ and so $\operatorname{tr}([X+\Omega]S^k)=0$, which gives for all polynomials p with $p(0)=0$

$$\operatorname{tr}\big((X+\Omega)\,p(S)\big)=0.$$

By using an approximation argument we have

$$\operatorname{tr}\big((X+\Omega)S^{1/n}\big)=0 \quad \text{for all } n\in\mathbf{N}.$$

Since $S^{1/n}\to I$ in the w^*-topology, we finally get

$$\operatorname{tr}(X+\Omega)=\operatorname{tr}(T)=0,$$

which completes the proof when M, N are in generic position.

For the general case, decompose $H=(M^{\perp}\cap N)\oplus(M\cap N^{\perp})\oplus H_0$. If $T\in {}^{\perp}R$, it is easy to see that its compressions to $M^{\perp}\cap N$, $M\cap N^{\perp}$ vanish. Hence if T' is the compression of T to H_0 we have $\operatorname{tr}(T)=\operatorname{tr}(T')=0$ by the first part of the proof, since $M\cap H_0$, $N\cap H_0$ are in generic position. ∎

It is important to realize that if the angle between M and N is nonzero, then we can find a Hilbert space H' and M', N' all similar to H, M, N, where M', N' are complementary and orthogonal. This fact automatically yields Theorem 2.1 since $T\in {}^{\perp}R$ is then similar to $\left[\begin{smallmatrix} 0 & Y \\ Z & 0 \end{smallmatrix}\right]$.

3. Hyperreflexivity of $\operatorname{Alg}\mathscr{L}$. Before proceeding, it is important to present some facts concerning $\sigma(C)$. In Theorem 1.1, M and N are represented as $\{(t,0)\colon t\in K\}$ and $\{(Cw,Sw)\colon w\in K\}$ respectively. So if $\langle M,N\rangle$ stands for the angle between the two subspaces, we have

$$\cos\langle M,N\rangle=\sup\{|\langle Cw,t\rangle|\colon \|w\|,\|t\|\leqslant 1\}=\|C\|.$$

So if $\langle M,N\rangle=0$ we have $\|C\|=1$, hence $1\in\sigma(C)$ and 1 is a cluster point of $\sigma(C)$, since S is injective and $C^2+S^2=I$.

3.1. Lemma. *$\operatorname{Alg}\mathscr{L}$ is hyperreflexive if and only if every T in ${}^{\perp}(\operatorname{Alg}\mathscr{L})$ decomposes as $T=T_1+T_2$ with $T\in {}^{\perp}(\operatorname{Alg}\mathscr{N}_i)$, $i=1,2$, where $\mathscr{N}_1=\{0,M,H\}$ and $\mathscr{N}_2=\{0,N,H\}$.*

Proof. Suppose that $\mathscr{A}=\operatorname{Alg}\mathscr{L}$ is hyperreflexive and $T\in {}^{\perp}\mathscr{A}$. Then $T=\sum_{n=1}^{\infty}e_n\otimes f_n$, $\sum_{n=1}^{\infty}\|e_n\|\,\|f_n\|<\infty$, where for each n, $e_n\otimes f_n\in {}^{\perp}\mathscr{A}$, hence either $f_n\in M$ and $e_n\in M^{\perp}$, in which case $e_n\otimes f_n\in {}^{\perp}(\operatorname{Alg}\mathscr{N}_1)$, or $f_n\in N$ and $e_n\in N^{\perp}$, in which case $e_n\otimes f_n\in {}^{\perp}(\operatorname{Alg}\mathscr{N}_2)$. The lemma now follows by writing

$$T_1=\sum\{e_n\otimes f_n\colon e_n\otimes f_n\in {}^{\perp}(\operatorname{Alg}\mathscr{N}_1)\}$$

and $T_2=T-T_1$. The converse follows by hyperreflexivity of nest algebras. ∎

Observe that if M, N are orthogonal, T has the form

$$\begin{bmatrix} 0 & T_1 \\ T_2 & 0 \end{bmatrix}$$

and the hyperreflexivity of $\operatorname{Alg}\mathscr{L}$ follows from the lemma. Since hyperreflexivity is preserved by similarity, $\operatorname{Alg}\mathscr{L}$ is hyperreflexive whenever $\langle M,N\rangle>0$. Our second main theorem shows that this is in fact the only case.

3.2. Theorem. *$\operatorname{Alg}\mathscr{L}$ is hyperreflexive if and only if $\langle M,N\rangle>0$.*

Proof. It is enough to prove that $\mathscr{A}=\operatorname{Alg}\mathscr{L}$ is not hyperreflexive when $\langle M,N\rangle=0$. Suppose that $\operatorname{Alg}\mathscr{L}$ is hyperreflexive and that M, N are in generic position. We will construct an operator $T\in {}^{\perp}\mathscr{A}$ which cannot be decomposed as in Lemma 3.1.

In Theorem 2.1 we have proved that ${}^{\perp}\mathscr{A}={}^{\perp}R$, so $T\in {}^{\perp}\mathscr{A}$ iff $QTP=TP$, $PTQ=TQ$ with P, Q as in 1.1.

Write T in 2×2 matrix form as

$$\begin{bmatrix} X & Y \\ Z & \Omega \end{bmatrix}.$$

The previous relations yield

(1) $CZ=SX$,

(2) $ZC=-\Omega S$,

which are also sufficient for T to be in ${}^{\perp}\mathscr{A}$.

If $T=T_1+T_2$ as in 3.1 we have

$$T_1 P = 0, \quad PT_1 = T, \quad T_2 Q = 0, \quad QT_2 = T.$$

The above equalities yield the following matrix forms for T_1, T_2 respectively:

(3) $\begin{bmatrix} 0 & Y_1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} X & Y_2 \\ Z & \Omega \end{bmatrix}$, with

(4) $XC + Y_2 S = 0$ and $Y_1 + Y_2 = Y$.

From the observation at the beginning of this section, 0 is a cluster point of $\sigma(S)$.

Let $\{E_\lambda\}$ be a spectral resolution of S. Choose a decreasing sequence $\{\lambda_n\}$ such that $1/n^2 \leqslant \lambda_n < 1/n$ and (writing E_n for the projection corresponding to λ_n) $E_{n-1} - E_n \neq 0$. Let u_n be a unit vector in the range of $E_{n-1} - E_n$. Note that the restriction of $(I-S^2)^{1/2}$ to the range of E_1 has an inverse on this space which has norm $\leqslant 1/\sqrt{1-\lambda_1^2}$. Let D be this inverse on $\operatorname{im}(E_1)$ and 0 on $\operatorname{im}(I-E_1)$. Note that, for x in the range of E_1, $CDx = DCx = x$. Let

$$W = \sum_{n=2}^{\infty} \frac{1}{\lambda_{n-1} n^2} u_n \otimes u_n.$$

Then since $1/n^2 \leqslant \lambda_n < 1/n$ and $\{\lambda_n\}$ is decreasing W is bounded but not trace class. However, $\|Su_n\| \leqslant \lambda_{n-1}$ so WS is trace class and hence so are X, Z, Ω defined by $X = -WSD$, $Z = -DSWSD$, $\Omega = DSW$. If we take $Y = 0$ and X, Z, Ω as above, then $T \in {}^{\perp}\mathscr{A}$. But T is decomposed as in (3) and (4). (4) is also satisfied by W in place of Y_2 and, using the fact that S has dense range,

$$XC + Y_2 S = 0 = XC + WS$$

yields $Y_2 = W$, which is absurd since Y_2 has to be trace class.

For the general case if $\langle M, N \rangle = 0$ argue as in the proof of 2.1 considering H_0. If $\mathscr{L}' = \{0, M \cap H_0, N \cap H_0, H_0\}$, we have constructed a $T' \in {}^{\perp}(\operatorname{Alg} \mathscr{L}')$ which cannot be decomposed as in Lemma 3.1. Then the operator $T' \oplus 0 \oplus 0$ belongs to ${}^{\perp}(\operatorname{Alg} \mathscr{L})$ and cannot be decomposed. The proof is now complete. ■

Addendum. We have recently been informed that a result analogous to 2.1 has been obtained by Lambrou and Longstaff in *Unit ball density and the equation* $AX = \Psi B$ (preprint).

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