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Two-weight norm inequalities for the Hardy–Littlewood maximal function for one-parameter rectangles

by

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Abstract. We give a Sawyer type characterization of the pairs of weights (v, w) for which the Hardy–Littlewood maximal function over one-parameter families of rectangles is bounded from $L^p(v)$ to $L^q(w)$, $1 < p \leqslant q \leqslant \infty$, $p < \infty$.

Introduction. In 1975 D. S. Kurtz ([K]) extended Muckenhoupt's theorem on weighted L^p boundedness of the Hardy–Littlewood maximal function to the case of one-parameter families of rectangles. In 1982 E. T. Sawyer ([S]) gave a characterization of the pairs of weights (v, w) for which the Hardy–Littlewood maximal function over cubes is bounded as an operator from $L^p(v)$ to $L^q(w)$, $1 < p \leqslant q \leqslant \infty$, $p < \infty$. Some of the main tools in Sawyer's proof are Calderón–Zygmund decomposition and the estimate of the Hardy–Littlewood maximal function by the dyadic maximal function given by Fefferman and Stein. On the other hand, an extension of the Calderón–Zygmund decomposition can be applied to one-weight problems, see for instance [AM]; however, this generalization does not work when the measures considered are different and do not satisfy a doubling condition or when the space is not of homogeneous type.

In this note we consider the two-weight problem for one-parameter families of rectangles. The main points are a generalization of Calderón–Zygmund decomposition and an extension of the Fefferman–Stein estimate.

In section 1 we give the statement of the result. In § 2 we introduce a family of rectangles of dyadic type and prove some technical lemmas. In § 3 we show how Sawyer's technique can be applied by giving an extension of the Fefferman–Stein estimate.

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§1. Given $i=1,\dots,n$, let $\varphi_i\colon \mathbf{R}^+\to\mathbf{R}^+$ be a nondecreasing function satisfying

(1.1) $$\lim_{x\to 0^+}\varphi_i(x)=0,$$

(1.2) $$\lim_{x\to\infty}\varphi_i(x)=\infty,$$

(1.3) there exists a sequence $\{r_k\}_{k\in\mathbf{Z}}$ such that $r_k<r_{k+1}$, $r_k\to 0$ $(k\to-\infty)$, $r_k\to\infty$ $(k\to\infty)$ and

$$\alpha\leqslant\varphi_i(r_k)/\varphi_i(r_{k+1})\leqslant 1,\quad i=1,\dots,n,\ k\in\mathbf{Z},$$

for some $\alpha\in(0,1)$.

Observe that if for every $i=1,\dots,n$ the function φ_i is continuous and satisfies (1.1) and (1.2), then we can get a sequence $\{r_k\}_{k\in\mathbf{Z}}$ such that (1.3) also holds. In fact, choose r_k such that

$$\tfrac{1}{2}=\prod_{j=1}^{n}\varphi_j(r_k)/\varphi_j(r_{k+1})\leqslant\varphi_i(r_k)/\varphi_i(r_{k+1})\leqslant 1,$$

for every $k\in\mathbf{Z}$ and every $i=1,\dots,n$.

Let $\mathscr{R}_\varrho$ be the family of all rectangles of the form

$$R_t(x,r)=\{y\colon |x_i-y_i|\leqslant t_i\varphi_i(r),\ i=1,\dots,n\}$$

where $t=(t_1,\dots,t_n)$, $1\leqslant t_i<4$, $x\in\mathbf{R}^n$ and $\varrho\geqslant r>0$. In the sequel we shall refer to $R_t(x,r)$ as a *rectangle of parabolic type*. When $t=(1,\dots,1)$ we shall say that $R(x,r)=R_t(x,r)$ is a *parabolic rectangle* with center x and radius r. Let $\mathscr{P}_\varrho$ be the family of all parabolic rectangles with $\varrho\geqslant r>0$, $\mathscr{R}=\bigcup_{\varrho>0}\mathscr{R}_\varrho$ and $\mathscr{P}=\bigcup_{\varrho>0}\mathscr{P}_\varrho$.

Let us define the Hardy–Littlewood maximal functions for the families $\mathscr{R}_\varrho$, $\mathscr{R}$, $\mathscr{P}_\varrho$ and $\mathscr{P}$, i.e.

$$M_\varrho f(x)=\sup_{x\in R\in\mathscr{R}_\varrho}|R|^{-1}\int_R|f(y)|\,dy,\qquad Mf(x)=\sup_{x\in R\in\mathscr{R}}|R|^{-1}\int_R|f(y)|\,dy,$$

$$\mathscr{M}_\varrho f(x)=\sup_{x\in R\in\mathscr{P}_\varrho}|R|^{-1}\int_R|f(y)|\,dy,\qquad \mathscr{M}f(x)=\sup_{x\in R\in\mathscr{P}}|R|^{-1}\int_R|f(y)|\,dy.$$

Clearly M_ϱ is equivalent to $\mathscr{M}_\varrho$ and M is equivalent to $\mathscr{M}$.

The main result in this note is the following analogue to Sawyer's theorem ([S]).

(1.4) THEOREM. *Let $1<p\leqslant q\leqslant\infty$, $p<\infty$. Let v and ω be two positive Borel measures in $\mathbf{R}^n$. Then*

(1.5) $$\|Mf\|_{L^q(d\omega)}\leqslant C\|f\|_{L^p(dv)}$$

for every $f\in L^p(dv)$ if and only if $dv(x)=v(x)\,dx$ and

(1.6) $$\|\chi_R M(\chi_R v^{1-p'})\|_{L^q(d\omega)}\leqslant C\|\chi_R v^{1-p'}\|_{L^p(dv)}<\infty$$

for every $R\in\mathscr{R}$.

As a corollary of Theorem (1.4) we can get a characterization of pairs of measures for which (1.5) holds in terms of a condition like (1.6) for finite unions of parabolic rectangles. Let $\tilde{\mathscr{R}}$ be the family of finite unions of rectangles with fixed radius and pairwise disjoint interiors, i.e. $\tilde R\in\tilde{\mathscr{R}}$ iff there exist $r>0$ and $\{x_i\}_{i=1}^k\subset\mathbf{R}^n$ such that

$$\tilde R=\bigcup_{i=1}^k R(x_i,r),\qquad \mathring{R}(x_i,r)\cap\mathring{R}(x_j,r)=\emptyset,\quad i\neq j.$$

(1.7) LEMMA. *Let $1<p\leqslant q\leqslant\infty$, $p<\infty$. Let v be a nonnegative measurable function and let ω be a positive Borel measure in $\mathbf{R}^n$. Then (1.6) is equivalent to*

(1.8) $$\|\chi_R M(\chi_R v^{1-p'})\|_{L^q(d\omega)}\leqslant C\|\chi_{\tilde R} v^{1-p'}\|_{L^p(dv)}<\infty$$

for every $\tilde R\in\tilde{\mathscr{R}}$.

Proof. It is clear that (1.6) implies (1.8). Assume (1.8). Let $R\in\mathscr{R}$ be defined by the set of inequalities

$$|x_i-y_i|\leqslant s_i\varphi_i(\varrho),\quad 1\leqslant s_i<4,\ \varrho>0,\ x\in\mathbf{R}^n.$$

Observe now that R is contained in a rectangle R_m whose ith side length is

$$\left(\left[\frac{2s_i\varphi_i(\varrho)}{2\varphi_i(1/m)}\right]+1\right)2\varphi_i(1/m),$$

where $[\cdot]$ denotes integer part. $\{R_m\}$ converges to R as m tends to infinity. Let us prove that for m large enough we have $R_m\in\tilde{\mathscr{R}}$. It suffices to show that if

$$\big([s_i\varphi_i(\varrho)/\varphi_i(1/m)]+1\big)\varphi_i(1/m)=t_i\varphi_i(\varrho)$$

then $1\leqslant t_i<4$ for every m large enough. But

$$1\leqslant s_i=s_i\frac{\varphi_i(\varrho)}{\varphi_i(1/m)}\frac{\varphi_i(1/m)}{\varphi_i(\varrho)}\leqslant\frac{\big([s_i\varphi_i(\varrho)/\varphi_i(1/m)]+1\big)\varphi_i(1/m)}{\varphi_i(\varrho)}$$
$$=t_i\leqslant s_i+\frac{\varphi_i(1/m)}{\varphi_i(\varrho)},$$

and applying (1.1) and taking m large we see that the last term is less than 4. Now (1.6) follows from (1.8) in the following way:

$$\|\chi_R M(\chi_R v^{1-p'})\|_{L^q(d\omega)}\leqslant\liminf_{m\to\infty}\|\chi_{R_m}M(\chi_{R_m}v^{1-p'})\|_{L^q(d\omega)}$$
$$\leqslant\liminf_{m\to\infty}\|\chi_{R_m}v^{1-p'}\|_{L^p(dv)}=\|\chi_R v^{1-p'}\|_{L^p(dv)}.\ \blacksquare$$

(1.9) Corollary. *Let $1 < p \leqslant q \leqslant \infty$, $p < \infty$. Let v and ω be two positive Borel measures on $\mathbf{R}^n$. Then*

$$\|\mathscr{M}f\|_{L^q(d\omega)} \leqslant C\|f\|_{L^p(dv)} \tag{1.10}$$

for every $f \in L^p(dv)$ if and only if $dv(x) = v(x)\,dx$ and (1.8) is true for every $\tilde{R} \in \tilde{\mathscr{R}}$.

Proof. Follows from Lemma (1.7) and Theorem (1.4). ■

Even though (1.8) resembles condition (3.2) in Theorem (3.1) of B. Jawerth ([J]), where more general geometrical shapes are allowed for $q = p$, we would like to point out that Corollary (1.9) is not a particular case of that theorem, because condition (3.3) in [J] is not required here.

(1.11) Remark. As a corollary of the proof of Theorem (1.4) we shall see in Section 3 that if

$$\{\varphi_i(r_{k+1})/\varphi_i(r_k):\ i = 1, \ldots, n,\ k \in \mathbf{Z}\}$$

is a bounded set of integer numbers greater than one, then, with the same assumptions as in (1.9), inequality (1.10) holds if and only if $dv(x) = v(x)\,dx$ and

$$\|\chi_R \mathscr{M}(\chi_R v^{1-p'})\|_{L^q(d\omega)} \leqslant C\|\chi_R v^{1-p'}\|_{L^p(dv)} < \infty \tag{1.12}$$

for every $R \in \mathscr{P}$.

Observe now that if $\varphi_i(r) = r^{p_i/q_i}$ with $p_i, q_i \in \mathbf{N}$, taking $r_k = 2^{k\prod_{i=1}^n q_i}$ we have $\varphi_i(r_{k+1})/\varphi_i(r_k) = 2^{\prod_{i=1}^n q_i}$. So (1.12) holds for these rational parabolic rectangles.

(1.13) Remark. If $q = p$ and $\omega = v$ inequalities (1.6), (1.8) and (1.12) are all equivalent to Muckenhoupt's A_p type condition for parabolic rectangles:

$$\Big(\int_R v\Big)\Big(\int_R v^{-1/(p-1)}\Big)^{p-1} \simeq |R|^p$$

for every $R \in \mathscr{P}$. The proof follows the lines of [HKN]. And so our result contains that of D. S. Kurtz ([K]) in which only continuous φ_i's are considered.

(1.14) Remark. It is worth pointing out that similar results are valid for fractional maximal operators, and for maximal operators when the averages are taken with respect to a Borel measure μ satisfying the doubling condition

$$0 < \mu(2R) \leqslant C\mu(R) < \infty \quad \text{for every } R \in \mathscr{P}.$$

Here $2R$ is the rectangle concentric with R and with side lengths twice those of R.

§ 2. Let $k \in \mathbf{Z}$. We shall construct a family $\mathscr{D}_k$ of rectangles in $\mathscr{R}_{r_k}$ which share with the dyadic cubes of $\mathbf{R}^n$ the following property:

$$\text{if } \mathring{R}_i \cap \mathring{R}_j \neq \emptyset \text{ then either } R_i \subseteq R_j \text{ or } R_j \subseteq R_i. \tag{2.1}$$

This construction will be carried out in such a way that the maximal function M_{r_k} will be bounded in terms of maximal functions over translations of $\mathscr{D}_k$.

The next result is an immediate consequence of (2.1).

(2.2) Lemma. *Let $\mathscr{F} \subset \mathscr{D}_k$. Then there exists a sequence $\{R_i\} \subset \mathscr{F}$ with pairwise disjoint interiors such that for every $R \in \mathscr{F}$ there is an i for which $R \subset R_i$.*

Write $\varphi = \varphi_i$ and $e_j = [\varphi(r_{k-j+1})/\varphi(r_{k-j})]$, $j \in \mathbf{N}$. Let

$$s_0 = 2\varphi(r_k), \qquad s_j = \frac{s_{j-1}}{e_j a_j}, \quad j \geqslant 1,$$

where

$$a_j = \begin{cases} 1 & \text{if } 2 \leqslant \dfrac{s_{j-1}}{e_j\varphi(r_{k-j})} < 4, \\ 2 & \text{if } 4 \leqslant \dfrac{s_{j-1}}{e_j\varphi(r_{k-j})}. \end{cases}$$

The following elementary properties of the sequence s_j will be basic for the construction of $\mathscr{D}_k$.

(2.3) Lemma.

$$s_j = 2\varphi(r_k)\Big/\prod_{i=1}^{j} e_i a_i, \quad j \geqslant 1, \tag{2.4}$$

$$2 \leqslant s_j/\varphi(r_{k-j}) < 4, \quad j \geqslant 0, \tag{2.5}$$

$$\big(s_j - \varphi(r_{k-j})\big)\prod_{i=1}^{j} e_i a_i \geqslant \varphi(r_k), \quad j \geqslant 1. \tag{2.6}$$

Proof. (2.4) follows immediately from the definition of s_j. Let us prove (2.5) inductively. Clearly (2.5) holds for $j = 0$. Assume (2.5) for $j = m$. Now,

$$\frac{s_{m+1}}{\varphi(r_{k-(m+1)})} = \frac{s_m}{e_{m+1}a_{m+1}}\,\frac{1}{\varphi(r_{k-m-1})}. \tag{2.7}$$

If $2 \leqslant s_m/(e_{m+1}\varphi(r_{k-m-1})) < 4$, then $a_{m+1} = 1$ and (2.5) follows from (2.7). Assume now that $4 \leqslant s_m/(e_{m+1}\varphi(r_{k-m-1}))$. Then $a_{m+1} = 2$ and by induction hypothesis we have

$$2 \leqslant \frac{s_m}{e_{m+1}a_{m+1}\varphi(r_{k-m-1})}\,\frac{\varphi(r_{k-m})}{\varphi(r_{k-m})} = \frac{s_m}{\varphi(r_{k-m})}\,\frac{\varphi(r_{k-m})}{e_{m+1}a_{m+1}\varphi(r_{k-m-1})}$$
$$< 4\frac{\varphi(r_{k-m})}{e_{m+1}a_{m+1}\varphi(r_{k-m-1})} \leqslant 4.$$

Inequality (2.6) follows now from (2.4) and (2.5) in the following way:

$$(s_j-\varphi(r_{k-j}))\prod_{i=1}^{j} e_i a_i = 2\varphi(r_k)-\varphi(r_{k-j})\prod_{i=1}^{j} e_i a_i$$

$$= 2\varphi(r_k)-\frac{\varphi(r_{k-j})}{s_j}2\varphi(r_k) \geqslant 2\varphi(r_k)-\varphi(r_k) = \varphi(r_k). \blacksquare$$

For $j\in \mathbf{N}\cup\{0\}$ and $h\in\mathbf{Z}$, set

$$I_{h,j} = \{x\in\mathbf{R}\colon |x-(2h+1)s_j| \leqslant s_j\}.$$

Let $\{I^i_{h,j}\colon j\in\mathbf{N}\cup\{0\},\ h\in\mathbf{Z}\}$ be the sequence of intervals in the ith axis associated to $\varphi=\varphi_i$, $i=1,\ldots,n$. Let $h=(h_1,\ldots,h_n)\in\mathbf{Z}^n$ and

$$R_{h,j} = \prod_{i=1}^{n} I^i_{h_i,j}.$$

Applying (2.5) we see that $R_{h,j}$ belongs to $\mathscr{R}_{r_k}$. We now define

$$\mathscr{D}_k = \{R_{h,j}\colon h\in\mathbf{Z}^n \text{ and } j\in\mathbf{N}\cup\{0\}\}, \tag{2.8}$$

which clearly satisfies (2.1).

Let us now prove a technical lemma related to translations of $\mathscr{D}_k$ which will be useful for the proof of Theorem (1.4).

Let R_0 be a parabolic rectangle with radius $r\leqslant r_k$ and center at $x\in\mathbf{R}^n$. Pick $j\geqslant 1$ such that $r_{k-j}<r\leqslant r_{k-j+1}$. Now let $\mathscr{F}_{R_0,k}$ be the family of rectangles $R\in\mathscr{D}_k$ with ith side length $2s^i_{j-1}$ such that there exists $\tau\in T_k=\prod_{i=1}^{n}[-5\varphi_i(r_k),5\varphi_i(r_k)]$ for which $R_0\subset\tau+R$. Finally, we set

$$\Omega_{R_0,k} = \{\tau\in T_k\colon \exists\, R\in\mathscr{F}_{R_0,k}\colon R_0\subset\tau+R\}.$$

(2.9) Lemma. *There exists a constant C depending only on n such that*

$$|\Omega_{R_0,k}| \geqslant C|T_k|. \tag{2.10}$$

Proof. The construction of $\mathscr{D}_k$ and the fact that every n-dimensional translation can be expressed as an iteration of 1-dimensional translations allow us to consider only the case $n=1$. Clearly there is an $R\in\mathscr{D}_k$ of length $2s_k$ such that $R\cap R_0\neq\emptyset$. On the other hand, R can be written as the union of $\prod_{i=1}^{j-1} e_i a_i$ intervals belonging to $\mathscr{F}_{R_0,k}$. Let I be one of these intervals. The measure of the set of those τ's for which $R_0\subset\tau+I$ is $2(s_{j-1}-\varphi(r))$. Let $\Omega'_{R_0,k}$ be the set of translations of all of these intervals I; then $\Omega'_{R_0,k}\subset\Omega_{R_0,k}$. Since the sets of translations corresponding to two different intervals I are disjoint, applying (2.6), we get the desired result in the following way:

$$|\Omega_{R_0,k}| \geqslant |\Omega'_{R_0,k}| = 2(s_{j-1}-\varphi(r))\prod_{i=1}^{j-1} e_i a_i$$

$$\geqslant 2(s_{j-1}-\varphi(r_{k-(j-1)}))\prod_{i=1}^{j-1} e_i a_i \geqslant 2\varphi(r_k) = \tfrac{1}{5}|T_k|. \blacksquare$$

§ 3. Let $k\in\mathbf{Z}$ and let $\mathscr{D}_k$ be the family of rectangles defined in (2.8). Define

$$M^\tau_{\mathscr{D}_k} f(x) = \sup_{x-\tau\in R\in\mathscr{D}_k} |R|^{-1}\int_R |f(y+\tau)|\,dy.$$

Because of (2.1), the method introduced by E. Sawyer ([S]) in order to obtain the boundedness of the maximal function taken over dyadic cubes of $\mathbf{R}^n$ with bounded sides can be extended to the present geometrical situation.

(3.1) Lemma. *Let $1<p\leqslant q\leqslant\infty$, $p<\infty$. Let v be a nonnegative measurable function and ω a positive Borel measure in $\mathbf{R}^n$. Then there exists $C>0$ such that for each $\tau\in\mathbf{R}^n$*

$$\|M^\tau_{\mathscr{D}_k} f\|_{L^q(d\omega)} \leqslant C\|f\|_{L^p(v\,dx)} \tag{3.2}$$

is equivalent to

$$\|\chi_R(M^\tau_{\mathscr{D}_k}(\chi_R v^{1-p'}))\|_{L^q(d\omega)} \leqslant C\|\chi_R v^{1-p'}\|_{L^p(v\,dx)} < \infty, \tag{3.3}$$

for every $R\in\mathscr{D}_k$.

Proof. Follows from Lemma (2.2) applying Sawyer's technique. ■

The following lemma provides the desired estimate of $\mathscr{M}_{r_k}$ in terms of $M^\tau_{\mathscr{D}_k}$.

(3.4) Lemma. *There exists a constant C depending only on n and α such that*

$$\mathscr{M}_{r_k} f(x) \leqslant C|T_k|^{-1}\int_{T_k} M^\tau_{\mathscr{D}_k} f(x)\,d\tau \tag{3.5}$$

for every $x\in\mathbf{R}^n$ and every $f\in L^1_{\rm loc}(\mathbf{R}^n)$.

Proof. Let $x\in\mathbf{R}^n$ and $R_0\in\mathscr{P}$ such that $x\in R_0$, the radius of R_0 is $\leqslant r_k$ and

$$\mathscr{M}_{r_k} f(x) \leqslant 2|R_0|^{-1}\int_{R_0}|f(y)|\,dy.$$

Let $j\geqslant 1$ be such that $r_{k-j}<r\leqslant r_{k-j+1}$ and take $R\in\mathscr{F}_{R_0,k}$. Then there exists a $\tau\in T_k$ such that

$$\mathscr{M}_{r_k} f(x) \leqslant 2\frac{|R|}{|R_0|} M^\tau_{\mathscr{D}_k} f(x),$$

which, from (2.5) and (1.3), is bounded by

$$2\prod_{i=1}^{n}\frac{s^i_{j-1}}{\varphi_i(r_{k-j+1})}\frac{\varphi_i(r_{k-j+1})}{\varphi_i(r_{k-j})} M^\tau_{\mathscr{D}_k} f(x) \leqslant 2\left(\frac{4}{\alpha}\right)^n M^\tau_{\mathscr{D}_k} f(x).$$

Finally, integrating with respect to $\tau\in\Omega_{R_0,k}$ and using inequality (2.10) we get (3.5). ■

Proof of Theorem (1.4). The "only if" part is similar to that of Sawyer. For the converse, applying (3.5), Minkowski's inequality and (3.2) we get

$$\|\mathcal{M}_{r_k} f\|_{L^q(d\omega)} \leqslant C\,|T_k|^{-1} \int_{T_k} \|M^{\tau}_{\mathcal{D}_k} f\|_{L^q(d\omega)}\, d\tau \leqslant C\,\|f\|_{L^p(v\,dx)}.$$

Letting $k \to \infty$ gives $\|\mathcal{M} f\|_{L^q(d\omega)} \leqslant C\,\|f\|_{L^p(v\,dx)}$. Now (1.5) follows from the equivalence of M and $\mathcal{M}$. ∎

Proof of Remark (1.11). If $F = \{\varphi_i(r_{k+1})/\varphi_i(r_k):\ i = 1, \ldots, n,\ k \in \mathbf{Z}\}$ is a bounded set of integers greater than one, then we can construct $\mathcal{D}_k$ using the sequence $\{s^i_j\}_{j=0}^{\infty}$ defined by

$$s^i_0 = \varphi_i(r_{k+1}), \qquad s^i_{j-1} = \frac{s^i_j}{\varphi_i(r_{k-j+2})/\varphi_i(r_{k-j+1})}, \quad j \geqslant 1.$$

The following analogues of (2.4)–(2.6) are valid:

$$(2.4)' \qquad s^i_j = \frac{\varphi_i(r_{k+1})}{\prod_{m=1}^{j} \varphi_i(r_{k-m+2})/\varphi_i(r_{k-m+1})} = \varphi_i(r_{k-j+1}), \qquad i = 1, \ldots, n,$$

$$(2.5)' \qquad 2 \leqslant s^i_j/\varphi_i(r_{k-j}) < K_1 + 1, \quad \text{where} \quad K_1 = \max F,$$

$$(2.6)' \qquad (s^i_j - \varphi_i(r_{k-j})) \prod_{m=1}^{j} \varphi_i(r_{k-m+2})/\varphi_i(r_{k-m+1}) \geqslant (1 - K_2^{-1})\,\varphi_i(r_{k+1}),$$

where $K_2 = \min F$.

Inequalities (2.10) and (3.5) hold with $T_k = \prod_{i=1}^{n} [-5\varphi_i(r_{k+1}),\ 5\varphi_i(r_{k+1})]$. Remark (1.11) now follows from the fact that $\mathcal{D}_k \subset \mathcal{P}$.

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