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Analytic stochastic processes II

by

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Abstract. The paper is devoted to the study of Bessel analytic processes. It is proved that they form a basis in the space of all analytic stochastic processes.

1. Notation and preliminaries. For the terminology and notation used here, see [5]. Let us recall some concepts and definitions. Let (Ω, B_Ω, P) be a probability space. Throughout this paper $W = W(t,\omega)$ $(\omega \in \Omega)$ will denote the standard Brownian motion on the half-line $[0, \infty)$. For any $T \in (0, \infty]$ the space $\mathscr{A}_T$ consists of analytic stochastic processes. The topology in $\mathscr{A}_T$ is defined by the family of seminorms

$$\|X\|_t = \Big(t^{-1} \int_0^t \int_\Omega |X(u,\omega)|^2 P(d\omega)\, du\Big)^{1/2}$$

where $t \in (0, T)$. By I we denote the operation of *Itô integration*

$$(IX)(t,\omega) = \int_0^t X(u,\omega)\, dW(u,\omega).$$

The *Itô derivative*, denoted by D_I, is the left inverse of I. The *Hermite processes* H_n $(n = 0, 1, \ldots)$ are defined by the formula

$$H_n(t,\omega) = h_n\big(t, W(t,\omega)\big) \tag{1.1}$$

where h_n are Hermite polynomials of two variables. In particular, $H_0(t,\omega) = 1$ and $H_1(t,\omega) = W(t,\omega)$. The *exponential stochastic process* $E(c)$ is defined for any complex number c by the formula

$$E(c)(t,\omega) = \exp\big(cW(t,\omega) - \tfrac{1}{2}c^2 t\big) = \sum_{n=0}^{\infty} c^n H_n(t,\omega). \tag{1.2}$$

For any $T \in (0, \infty]$ the space A_T consists of all entire functions f with finite seminorms

$$s_t(f) = \Big(\int_{\mathbf{C}} |f(z)|^2 \lambda_t(dz)\Big)^{1/2}$$

where $t \in (0, T)$, $\mathbf{C}$ stands for the complex plane and the family of probability measures λ_t is defined by the formula

$$\lambda_t(B) = -(\pi t)^{-1} \int_0^{2\pi} \int_0^{\infty} 1_B(re^{i\theta}) \mathrm{Ei}(-r^2/t) r\, dr\, d\theta.$$

Here 1_B denotes the indicator of the set B and Ei is the integral exponential function. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the bilinear form $\langle f, g\rangle_t$ induced by the seminorm s_t is given by the formula

(1.3) $$\langle f, g\rangle_t = \sum_{n=0}^{\infty} \frac{n!}{n+1} a_n \bar{b}_n t^n.$$

Let λ be the standard Gaussian measure on the complex plane:

$$\lambda(B) = \pi^{-1} \int_0^{2\pi} \int_0^{\infty} 1_B(re^{i\theta}) e^{-r^2} r\, dr\, d\theta.$$

The *random Fourier transform* is an isomorphism from A_T onto $\mathscr{A}_T$ defined by

$$\hat{f}(t, \omega) = \lim_{r\to\infty} \int_{|z|\le r} \overline{E(z)}(t, \omega) f(z)\, dz$$

([5], Theorem 4.1). Using the random Fourier transform one can define a convolution $*$ of analytic stochastic processes. Namely, if $X \in \mathscr{A}_T$, $Y \in \mathscr{A}_U$, $X = \hat{f}$ and $Y = \hat{g}$ where $f \in A_T$, $g \in A_U$, then $X * Y = (fg)^\wedge$. Since $fg \in A_{\psi(T,U)}$, where $\psi(T, U)$ is the harmonic mean of T and U, we have $X * Y \in A_{\psi(T,U)}$. In particular, for $Y \in \mathscr{A}_\infty$ and $X \in \mathscr{A}_T$ we have $X * Y \in \mathscr{A}_T$.

Using the random Fourier transform one can also define for any complex number c the translation T_c by setting $T_c \hat{f} = (\tau_c f)^\wedge$ for $f \in A_T$ where

(1.4) $$(\tau_c f)(z) = f(z+c).$$

It is clear that $T_c(\mathscr{A}_T) = \mathscr{A}_T$ and, by (4.7) in [5], $T_c = \exp(cD_t)$.

PROPOSITION 1.1. *For any $f \in A_T$ and $t \in [0, T)$*

(1.5) $$\hat{f}(t, \omega) = (2\pi t)^{-1/2} \int_{-\infty}^{\infty} f(iu + W(t, \omega)) \exp(-u^2/2t)\, du.$$

For $t = 0$ the right-hand side of the above formula is assumed to be $f(0)$.

Proof. It is easy to check that the generating function of the sequence

$$(n!)^{-1}(2\pi t)^{-1/2} \int_{-\infty}^{\infty} (iu+x)^n \exp(-u^2/2t)\, du \qquad (n = 0, 1, \ldots)$$

coincides with the generating function (1.14) in [5] of the Hermite polynomials $h_n(t, x)$ $(n = 0, 1, \ldots)$. Thus, by (1.1), we have

(1.6) $$H_n(t, \omega) = (n!)^{-1} (2\pi t)^{-1/2} \int_{-\infty}^{\infty} (iu + W(t, \omega))^n \exp(-u^2/2t)\, du.$$

Since $H_n = (n!)^{-1}(z^n)^\wedge$ ([5], p. 279), this gives a special case of formula (1.5) for $f(z) = (n!)^{-1} z^n$ $(n = 0, 1, \ldots)$.

In the general case expanding a function f from A_T in a power series

(1.7) $$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

we conclude, by Proposition 4.1 in [5], that

(1.8) $$\limsup_{n\to\infty} n^{-1/2} |a_n|^{1/n} \le (eT)^{-1/2}.$$

Moreover,

(1.9) $$f(t, \omega) = \sum_{n=0}^{\infty} a_n H_n(t, \omega).$$

Given $t \in (0, T)$ we take a number s fulfilling the condition

(1.10) $$t < s < T.$$

From (1.8) it follows that

$$|a_n| \le a \left(\frac{n}{es}\right)^{n/2} \qquad (n = 0, 1, \ldots)$$

for some constant a. This leads at once to the existence of a constant b for which

$$\max\left(a_{2k}/(2k)!, a_{2k+1}/(2k+1)!\right) \le b 2^{-k} s^{-k} (k!)^{-1} \qquad (k = 0, 1, \ldots).$$

Thus

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n!} |z|^n \le b(1+|z|) \exp(|z|^2/2s).$$

Obviously, by (1.10)

$$\int_{-\infty}^{\infty} (1 + |iu+x|) \exp(|iu+x|^2/2s - u^2/2t)\, du < \infty.$$

Now formula (1.5) follows at once from (1.6), (1.7) and (1.9) by the dominated convergence theorem.

As an immediate consequence of the above proposition we get the following statement.

COROLLARY 1.1. *For any $f \in A_T$ and any complex number c*

$$T_c \hat{f}(t, \omega) = (2\pi t)^{-1/2} \int_{-\infty}^{\infty} f(iu + c + W(t, \omega)) \exp(-u^2/2t)\, du.$$

In particular, $T_c W = W + c$ and $T_c E(a) = e^{ac} E(a)$.

PROPOSITION 1.2. *For any $f \in A_T$ and any complex number a*

$$(E(a) * \hat{f})(t, \omega) = (2\pi t)^{-1/2} E(a)(t, \omega) \int_{-\infty}^{\infty} f(iu + W(t, \omega) - at) \exp(-u^2/2t)\, du.$$

Proof. Taking into account the definition of convolution and the formula $E(a) = (e^{az})^\wedge$ we have, be Proposition 1.1,

$$(1.11)\quad \big(E(a)*f\big)(t,\omega) = (2\pi t)^{-1/2}\int_{-\infty}^{\infty} f\big(iu+W(t,\omega)\big)\exp\big(iua+aW(t,\omega)-u^2/2t\big)\,du$$

for $t\in[0,T)$. Since, by Proposition 4.2 in [5],

$$\limsup_{|z|\to\infty}\frac{\log|f(z)|}{|z|^2}\leqslant (2T)^{-1},$$

by an obvious application of Cauchy's Theorem we may take the right-hand side integral in (1.11) along any line parallel to the real axis. Thus, by a change of variable,

$$\big(E(a)*f\big)(t,\omega) = (2\pi t)^{-1/2}\int_{-\infty}^{\infty} f\big(iu+W(t,\omega)-at\big)\exp(-u^2/2t)\,du\exp\big(aW(t,\omega)-\tfrac{1}{2}a^2t\big),$$

which, by (1.2), completes the proof.

Our goal is to define and study Bessel analytic processes. In order to realize this we need a few results on Bessel functions regarded as elements of the space A_T.

2. Bessel functions in A_T. In this chapter we shall study the properties of Bessel functions that can be described in the context of the space A_T.

We set for any $f\in A_T$

$$(2.1)\quad (Kf)(z) = \int_0^{\pi} e^{iz\cos\varphi} f(z\sin^2\varphi)\,d\varphi.$$

LEMMA 2.1. *K is a continuous linear mapping from A_T into A_T.*

Proof. The linearity of K is obvious. Moreover, it is clear that for any $f\in A_T$ the function Kf is entire. Introducing the notation

$$g_\varphi(z) = f(z\sin^2\varphi),\qquad h_\varphi(z) = e^{iz\cos\varphi}$$

we have, by Proposition 4.1 in [5], for any $\varphi\in[0,\pi]$

$$(2.2)\quad s_u(g_\varphi)\leqslant s_u(f)\qquad (u\in(0,T)),$$

$$(2.3)\quad s_v(h_\varphi)\leqslant s_v(h_0)\qquad (v\in(0,\infty)).$$

Moreover, the order of the function h_φ is not greater than 1, which, by Proposition 4.2 in [5], yields $h_\varphi\in A_\infty$ $(\varphi\in[0,\pi])$. Let $\psi(u,v)$ denote the harmonic mean of u and v. Given $t\in(0,T)$ we can find a pair u,v of positive numbers fulfilling the conditions $u<T$ and $t<\psi(u,v)$. We can now appeal to the proof of Proposition 4.4 in [5] and obtain the inequality

$$s_t(h_\varphi g_\varphi)\leqslant c(t,u,v)s_u(g_\varphi)s_v(h_\varphi)\qquad (\varphi\in[0,\pi])$$

where $c^2(t,u,v) = \sum_{n=0}^{\infty}(n+1)\big(t/\psi(u,v)\big)^n$. Thus, by (2.2) and (2.3),

$$s_t(Kf)\leqslant\int_0^{\pi} s_t(h_\varphi g_\varphi)\,d\varphi\leqslant \pi c(t,u,v)s_u(f)s_v(h_0)\qquad (t\in(0,T)),$$

which shows that $Kf\in A_T$ and the mapping K is continuous. This completes the proof.

In what follows we shall use the following formulae for the Bessel functions J_n $(n=0,1,\ldots)$:

$$(2.4)\quad J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\varphi-in\varphi}\,d\varphi,$$

$$(2.5)\quad J_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+n}}{2^{2k+n}k!\,(n+k)!},$$

$$(2.6)\quad J_n(z) = \frac{z^n}{2^n\sqrt{\pi}\,\Gamma(n+1/2)}\int_0^{\pi} e^{iz\cos\varphi}(\sin\varphi)^{2n}\,d\varphi$$

([1], Chapter 7). Since the order of J_n is equal to 1, we conclude, by Proposition 4.2 in [5], that $J_n\in A_\infty$.

We say that a function f from A_T *has Neumann expansion in A_T* if for some sequence of coefficients $b_0, b_1, \ldots$ it can be represented by a series

$$(2.7)\quad f(z) = \sum_{n=0}^{\infty} b_n J_n(z)$$

convergent in A_T. Notice that the convergence in A_T yields the uniform convergence on every compact subset of the complex plane ([5], p. 276). Consequently, by the Nielsen formulae the coefficients $b_0, b_1, \ldots$ in (2.7) are uniquely determined. More precisely, taking the power series expansion of f,

$$(2.8)\quad f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we have the formulae

$$(2.9)\quad b_0 = a_0,\qquad b_n = n2^n\sum_{k=0}^{[n/2]}\frac{(n-k-1)!}{4^k k!}a_{n-2k}\qquad (n=1,2,\ldots),$$

and

$$a_n = \frac{1}{2^n n!}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{k}b_{n-2k}\qquad (n=0,1,\ldots)$$

where the square brackets denote the integral part ([1], Chapter 7.10).

LEMMA 2.2. *An entire function f belongs to the range $K(A_T)$ if and only if it has Neumann expansion in A_T*

$$f(z) = \sum_{n=0}^{\infty} b_n J_n(z)$$

with coefficients fulfilling the condition

$$\limsup_{n\to\infty} n^{-1/2}|b_n|^{1/n} \leqslant 2(eT)^{-1/2}. \tag{2.10}$$

This condition guarantees the convergence of the series in A_T.

Proof. Starting from formula (2.6) we have, by (2.1),

$$J_n(z) = \frac{K(z^n)}{2^n\sqrt{\pi}\,\Gamma(n+1/2)} \quad (n = 0, 1, \ldots). \tag{2.11}$$

Given a sequence $b_0, b_1, \ldots$ of coefficients fulfilling condition (2.10) we put

$$c_n = \frac{b_n}{2^n\sqrt{\pi}\,\Gamma(n+1/2)} \quad (n = 0, 1, \ldots).$$

It is easy to check the inequality

$$\limsup_{n\to\infty} n^{1/2}|c_n|^{1/n} \leqslant (T^{-1}e)^{1/2},$$

which, by Proposition 4.1 in [5], shows that the series $g(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in A_T and, consequently, $g \in A_T$. By (2.11) and Lemma 2.1

$$(Kg)(z) = \sum_{n=0}^{\infty} b_n J_n(z)$$

where the series converges in A_T. In other words, $\sum_{n=0}^{\infty} b_n J_n \in K(A_T)$.

Suppose now that a function f belongs to A_T and has power series expansion (2.8). By Proposition 4.1 in [5] the coefficients $a_0, a_1, \ldots$ fulfil the condition

$$\limsup_{n\to\infty} n^{1/2}|a_n|^{1/n} \leqslant (T^{-1}e)^{1/2}. \tag{2.12}$$

Setting $b_n = 2^n\sqrt{\pi}\,\Gamma(n+1/2)a_n$ $(n = 0, 1, \ldots)$ we can easily check condition (2.10). Moreover, by (2.11) and Lemma 2.1,

$$(Kf)(z) = \sum_{n=0}^{\infty} b_n J_n(z)$$

where the series converges in A_T. This completes the proof.

LEMMA 2.3. $K(A_T) = A_T$.

Proof. Suppose that f belongs to A_T and has power series expansion (2.8). By Proposition 4.1 in [5], the coefficients $a_0, a_1, \ldots$ fulfil condition (2.12). Define $b_0, b_1, \ldots$ by (2.9). Then f has Neumann expansion in A_T provided $\sum_{n=0}^{\infty} b_n J_n(z)$ converges in A_T. To prove this it suffices, by Lemma 2.2, to show that the coefficients $b_0, b_1, \ldots$ fulfil condition (2.10). By (2.12) for any $q > (T^{-1}e)^{1/2}$ we can find a positive number b such that

$$|a_n| \leqslant b e^{-n/2}(n!)^{-1/2}q^n \quad (n = 0, 1, \ldots). \tag{2.13}$$

Observe that for $k = 0, 1, \ldots, [n/2]$

$$\frac{((n-k)!)^2}{(n-2k)!\,n!} = \frac{(n-2k+1)(n-2k+2)\ldots(n-2k+k)}{(n-k+1)(n-k+2)\ldots(n-k+k)} \leqslant 1$$

and, consequently,

$$\frac{(n-k-1)!}{((n-2k)!)^{1/2}} \leqslant (n!)^{1/2} \quad (k = 0, 1, \ldots, [n/2]).$$

Combining this with (2.9) and (2.13) we get the inequality

$$|b_n| \leqslant nb(n!)^{1/2}(e^{-1/2}2q)^n \sum_{k=0}^{[n/2]} \frac{(e^{1/2}(2q)^{-1})^{2k}}{k!}.$$

Now a standard calculation shows that

$$\limsup_{n\to\infty} n^{-1/2}|b_n|^{1/n} \leqslant 2qe^{-1},$$

which, by the arbitrariness of $q > (T^{-1}e)^{1/2}$, yields (2.10). The lemma is thus proved.

As an immediate consequence of Lemmas 2.2 and 2.3 we get the following statement which can be viewed as a representation theorem.

THEOREM 2.1. *An entire function f belongs to A_T if and only if it has Neumann expansion in A_T*

$$f(z) = \sum_{n=0}^{\infty} b_n J_n(z)$$

where $\limsup_{n\to\infty} n^{-1/2}|b_n|^{1/n} \leqslant 2(eT)^{-1/2}$.

The above theorem and the uniqueness of Neumann expansion yield the following corollary.

COROLLARY 2.1. *Suppose that an entire function f admits the classical Neumann expansion*

$$f(z) = \sum_{n=0}^{\infty} b_n J_n(z)$$

where the series converges uniformly on every compact subset of the complex plane. If in addition $f \in A_T$, then this series converges in A_T and, consequently, gives the Neumann expansion of f in A_T.

Applying the above rule to the classical formulae ([1], Chapter 7) we get examples of Neumann expansions in A_∞:

(2.14) $$1 = J_0(z)+2\sum_{k=1}^{\infty} J_{2k}(z),$$

(2.15) $$z^n = 2^n \sum_{k=0}^{\infty} \frac{(n+k-1)!\,(n+2k)}{k!} J_{2k+n}(z) \qquad (n = 1,2,\ldots),$$

and for any complex number c

(2.16) $$e^{cz} = J_0(z)+\sum_{k=1}^{\infty} \big((c+\sqrt{c^2+1})^k+(c-\sqrt{c^2+1})^k\big)J_k(z).$$

For the further discussion we need an estimate for the seminorms of Bessel functions.

LEMMA 2.4. *For any $t\in(0,\infty)$ we have the inequality*

$$s_t(J_n) \leqslant 2^{-n}\big((n+1)!\big)^{-1/2}t^{n/2}a(t)$$

for some function $a(t)$.

Proof. From (4.1) in [5] and (2.5) we get the formula

$$s_t^2(J_n) = \sum_{k=0}^{\infty} \frac{(2k+n)!\,t^{2k+n}}{4^{2k+n}(2k+n+1)(k!)^2\big((k+n)!\big)^2}.$$

Using induction on k one can prove the inequality

$$\frac{(2k+n)!}{4^k(2k+n+1)\big((k+n)!\big)^2} \leqslant \frac{1}{(n+1)!} \qquad (k = 0, 1, \ldots),$$

which yields

$$s_t^2(J_n) \leqslant 4^{-n}\big((n+1)!\big)^{-1} t^n a^2(t)$$

with $a^2(t) = \sum_{k=0}^{\infty} 4^{-k}(k!)^{-2}t^{2k}$. This completes the proof.

Given $a\in(0,\infty)$ we denote by B_a the Hilbert space of entire functions g with $s_a(g)<\infty$. It is clear that $B_a\subset A_a$. We keep a nonnegative integer r fixed. For any $g\in B_a$ we define the sequence $c_0(g), c_1(g), \ldots$ by setting

(2.17) $$c_0(g) = \langle J_r, g\rangle_a$$

(2.18) $$c_k(g) = \langle J_{r-k}, g\rangle_a+(-1)^k\langle J_{r+k}, g\rangle_a \qquad (k = 1, 2, \ldots, r),$$

(2.19) $$c_k(g) = (-1)^{k+r}\langle J_{k-r}, g\rangle_a+(-1)^k\langle J_{r+k}, g\rangle_a \qquad (k = r+1, r+2, \ldots)$$

where the bilinear form $\langle f, g\rangle_a$ is given by (1.3).

LEMMA 2.5. *For any $g\in B_a$ the series*

(2.20) $$(Lg)(z) = \sum_{k=0}^{\infty} c_k(g)J_k(z)$$

converges in A_∞ and the order of the function Lg is not greater than 1.

Proof. Starting from the Schwarz inequality

$$|c_k(g)| \leqslant |\langle J_{k-r}, g\rangle_a|+|\langle J_{r+k}, g\rangle| \leqslant s_a(g)\big(s_a(J_{k-r})+s_a(J_{r+k})\big)$$

for $k>r$ and applying Lemma 2.4 we get the estimate

$$|c_k(g)| \leqslant b2^{-k}a^{k/2}s_a(g) \qquad (k = 0, 1, \ldots)$$

with some constant b, which together with the well-known estimate

(2.21) $$|J_k(z)| \leqslant 2^{-k}(k!)^{-1}|z|^k\exp|z| \qquad (k = 0, 1, \ldots)$$

([1], Chapter 7) yields

$$\sum_{k=0}^{\infty} |c_k(g)|\,|J_k(z)| \leqslant bs_a(g)\exp\big((1+4^{-1}a^{1/2})|z|\big).$$

Consequently, the series (2.20) converges uniformly on every compact subset of the complex plane and the order of its sum Lg is not greater than 1. Applying Proposition 4.2 in [5] we conclude that $Lg\in A_\infty$, which, by Corollary 2.1, shows that the Neumann expansion (2.20) converges in A_∞. This completes the proof.

LEMMA 2.6. *The mapping L from B_a into A_∞ is one-to-one.*

Proof. Suppose that $g\in B_a$ and $Lg = 0$. Then, by the uniqueness of the Neumann expansion we have $c_k(g) = 0$ $(k = 0, 1, \ldots)$. To prove that $g = 0$ it suffices to show that $\langle J_k, g\rangle_a = 0$ for all $k = 0, 1, \ldots$ because $g\in A_a$ and, by Theorem 2.1, the above equalities yield $s_a(g) = \langle g, g\rangle_a = 0$, which, by (1.3), implies $g = 0$.

First consider the case $r = 0$. Then, by (2.17), $\langle J_0, g\rangle_a = 0$ and, by (2.19), $\langle J_k, g\rangle_a = 0$ for $k\geqslant 1$.

Suppose now that $r\geqslant 1$. By (2.18) the equality $c_r(g) = 0$ yields

(2.22) $$|\langle J_0, g\rangle_a| = |\langle J_{2r}, g\rangle_a|.$$

Further, for $n = 1, 2, \ldots$ and $m = 0, 1, \ldots$, setting $k = n+(2m+1)r$ in (2.19) we get

(2.23) $$|\langle J_n, g\rangle_a| = |\langle J_{n+2r}, g\rangle_a| = \ldots = |\langle J_{n+2mr}, g\rangle_a|.$$

Observe that, by Lemma 2.4, $\lim_{k\to\infty}s_a(J_k) = 0$, which yields

$$\lim_{m\to\infty} \langle J_{n+2mr}, g\rangle_a = 0.$$

Consequently, from (2.23) we get $\langle J_n, g\rangle_a = 0$ for $n\geqslant 1$, which, by (2.22), gives $\langle J_0, g\rangle_a = 0$. The lemma is thus proved.

We are now in a position to prove the following theorem.

THEOREM 2.2. *Suppose that a sequence $c_1, c_2, \ldots$ of distinct nonzero complex numbers fulfils the condition*

$$\sum_{k=1}^{\infty} |c_k|^{-p} = \infty$$

for some $p > 1$. Then for every $r = 0, 1, \ldots$ and $T \in (0, \infty]$ the linear span of the functions $J_r(z+c_k)$ $(k = 1, 2, \ldots)$ is dense in A_T.

Proof. Let l be a continuous linear functional on A_T vanishing on all functions $J_r(z+c_k)$ $(k = 1, 2, \ldots)$. To prove the assertion it suffices to show that l vanishes identically on A_T. Using notation (1.4) we have

$$l(\tau_{c_k} J_r) = 0 \quad (k = 1, 2, \ldots). \tag{2.24}$$

By the Mazur–Orlicz Theoem ([4], p. 119) the functional l is of the form

$$l(f) = \langle f, g \rangle_a \tag{2.25}$$

where $a \in (0, T)$ and $g \in B_a$. We start from the Neumann–Schläfli identity ([6], p. 357)

$$(\tau_y J_r)(z) = J_0(y) J_r(z) + \sum_{k=1}^{r} J_k(y)\big(J_{r-k}(z) + (-1)^k J_{r+k}(z)\big)$$
$$+ \sum_{k=r+1}^{\infty} J_k(y)(-1)^k\big((-1)^r J_{k-r}(z) + J_{k+r}(z)\big)$$

and observe that, by (2.21), the sequence of coefficients $J_k(y)$ $(k = 0, 1, \ldots)$ fulfils condition (2.10) with $T = \infty$. Thus the above series converges in A_∞. Consequently, by (2.20) and (2.25),

$$l(\tau_y J_r) = (Lg)(y) \tag{2.26}$$

for any complex number y. Suppose that the entire function $(Lg)(z)$ does not vanish identically. By (2.24) the numbers $c_1, c_2, \ldots$ are its zeros. By Lemma 2.5 the order of $(Lg)(z)$ is at most 1. Taking into account the relation between the order and the convergence exponent of zeros of an entire function ([2], Theorem 2.5.18) we get the inequality $\sum_{k=1}^{\infty} |c_k|^{-q} < \infty$ for every $q > 1$. But this contradicts the assumption. Consequently, $(Lg)(z)$ vanishes identically. Applying Lemma 2.6 we conclude that g vanishes identically too, which, by (2.25), shows that $l = 0$ on A_T. The theorem is thus proved.

Notice that the assumption $p > 1$ in the above theorem is essential. In fact, setting $r = 0$ and taking as $c_1, c_2, \ldots$ the zeros of the Bessel function J_0 we have $|c_k|^{-1} = \infty$ ([1], Chapter 7). Taking as g in (2.20) the constant function equal to 1 we get the formula $(L1)(z) = J_0(z)$. Put $l(f) = \langle f, 1 \rangle_a$ where $a \in (0, T)$. The functional l does not vanish identically because $l(1) = \langle 1, 1 \rangle_a = 1$. On the other hand, by (2.26), $l(\tau_y J_0) = J_0(y)$ and, consequently, $l(\tau_{c_k} J_0) = 0$ $(k = 1, 2, \ldots)$, which shows that the linear span of the translates $J_0(z+c_k)$ $(k = 1, 2, \ldots)$ is not dense in A_T.

With the above background prepared we can now proceed to the study of analytic stochastic processes.

3. Bessel analytic processes. In the literature there exists the term "the Bessel process" referring to the radial motion associated with a multidimensional Brownian motion ([3], Chapter 2.7). Here we shall give a definition of a different kind of processes which can also be regarded as a random analogue of the Bessel functions. Namely, the stochastic processes defined by the formula

$$J_n(t, \omega) = (2\pi)^{-1} \int_0^{2\pi} E(i \sin\varphi)(t, \omega) e^{-in\varphi}\, d\varphi \quad (n = 0, 1, \ldots)$$

are called *Bessel analytic processes*. It is clear that they belong to $\mathscr{A}_\infty$. Using representation (1.2) we have the formula

$$J_n(t, \omega) = (2\pi)^{-1} \int_0^{2\pi} \exp\big(iW(t, \omega)\sin\varphi + \tfrac{1}{2} t \sin^2\varphi - in\varphi\big)\, d\varphi \quad (n = 0, 1, \ldots).$$

Since $E(i \sin\varphi) = (e^{iz\sin\varphi})^\wedge$, we have, by (2.4), a very useful formula connecting Bessel analytic processes and Bessel functions:

$$J_n(t, \omega) = \big(J_n(z)\big)^\wedge \quad (n = 0, 1, \ldots). \tag{3.1}$$

Using this formula we get from (2.5) the expansion of Bessel analytic processes in series of Hermite processes:

$$J_n(t, \omega) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+n}} \binom{n}{k} H_{2k+n}(t, \omega) \quad (n = 0, 1, \ldots).$$

Moreover, by Proposition 1.1,

$$J_n(t, \omega) = (2\pi t)^{-1/2} \int_{-\infty}^{\infty} J_n\big(iu + W(t, \omega)\big) \exp(-u^2/2t)\, du \quad (n = 0, 1, \ldots).$$

Further using the formula $n! H_n * E(i\cos\varphi) = (z^n e^{iz\cos\varphi})^\wedge$ we get, by (2.6) and Proposition 1.2, another expression:

$$J_n(t, \omega) =$$
$$\frac{n!}{2^n \sqrt{\pi}\, \Gamma(n+1/2)} \int_0^{\pi} h_n\big(t, W(t, \omega) - it\cos\varphi\big) \sin^{2n}\varphi \exp\big(iW(t, \omega)\cos\varphi + \tfrac{1}{2}\cos^2\varphi\big)\, d\varphi.$$

It is easy to check using (4.7) in [5] and (3.1) that the process $J_n(t, \omega)$ fulfils the equation

$$W^{*2} * D_I^2 X + W * D_I X + (W^{*2} - n^2) * X = 0$$

and each analytic process fulfilling the above equation is proportional to $J_n(t, \omega)$.

We say that a stochastic process X from $\mathscr{A}_T$ *has random Neumann expansion* if for some sequence of coefficients $b_0, b_1, \ldots$ it can be represented by a series

$$X(t,\omega) = \sum_{n=0}^{\infty} b_n J_n(t,\omega) \tag{3.2}$$

convergent in $\mathscr{A}_T$. We can now appeal to Theorem 4.1 in [5], Theorem 2.1 and formula (3.1) and obtain the following statement which can be viewed as a representation theorem for analytic stochastic processes.

THEOREM 3.1. *A stochastic process X belongs to $\mathscr{A}_T$ if and only if it has random Neumann expansion* (3.2) *with coefficients $b_0, b_1, \ldots$ fulfilling the condition*

$$\limsup_{n\to\infty} n^{-1/2}|b_n|^{1/n} \leqslant 2(eT)^{-1/2}.$$

The coefficients $b_0, b_1, \ldots$ are uniquely determined.

From (2.14)–(2.16) we get the following examples of random Neumann expansions:

$$1 = J_0(t,\omega) + 2\sum_{k=1}^{\infty} J_{2k}(t,\omega),$$

$$W(t,\omega) = 2\sum_{k=0}^{\infty} (2k+1) J_{2k+1}(t,\omega),$$

$$H_n(t,\omega) = 2^n \sum_{k=0}^{\infty} \frac{(2k+n)}{n+k}\binom{n+k}{k} J_{2k+n}(t,\omega) \qquad (n = 2, 3, \ldots),$$

$$E(c)(t,\omega) = J_0(t,\omega) + \sum_{n=1}^{\infty} \big((c+\sqrt{c^2+1})^n + (c-\sqrt{c^2+1})^n\big) J_n(t,\omega).$$

Further, from the formulae

$$\frac{d}{dz}J_0(z) = -J_1(z), \quad \frac{d}{dz}J_n(z) = \tfrac{1}{2}\big(J_{n-1}(z) - J_{n+1}(z)\big) \quad (n = 1, 2, \ldots)$$

([1], Chapter 7) and (4.7) in [5] we get the following rule for Itô integration and differentiation:

$$I\Big(\sum_{k=0}^{\infty} b_k J_k(t,\omega)\Big) = 2b_0 J_1(t,\omega) + 2\sum_{k=2}^{\infty}\Big(\sum_{j=1}^{k-1} b_j\Big) J_k(t,\omega),$$

$$D_I\Big(\sum_{k=0}^{\infty} b_k J_k(t,\omega)\Big) = \tfrac{1}{2}b_1 J_0(t,\omega) + (\tfrac{1}{2}b_2 - b_0) J_1(t,\omega) + \tfrac{1}{2}\sum_{k=2}^{\infty}(b_{k+1} - b_k) J_k(t,\omega).$$

In particular, $I(J_0(t,\omega)) = 2J_1(t,\omega)$ and

$$I(J_n(t,\omega)) = 2\sum_{k=n+1}^{\infty} J_k(t,\omega) \qquad (n = 1, 2, \ldots).$$

By Corollary 1.1 and formula (3.1) the translates of Bessel analytic processes have a representation

$$(T_c J_n)(t,\omega) = (2\pi t)^{-1/2}\int_{-\infty}^{\infty} J_n(iu + c + W(t,\omega))\exp(-u^2/2t)\,du.$$

As a consequence of Theorem 4.1 in [5], formula (3.1) and Theorem 2.2 we get the following statement.

THEOREM 3.2. *Suppose that a sequence $c_1, c_2, \ldots$ of distinct nonzero complex numbers fulfils the condition*

$$\sum_{k=1}^{\infty} |c_k|^{-p} = \infty$$

for some $p > 1$. Then for every $r = 0, 1, \ldots$ and $T \in (0, \infty]$ the linear span of translates $(T_{c_k} J_r)(t,\omega)$ $(k = 1, 2, \ldots)$ is dense in $\mathscr{A}_T$. Moreover, the condition $p > 1$ is essential.

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