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# Uncomplementability of the spaces of norm continuous functions in some spaces of "weakly" continuous functions

by

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**Abstract.** The paper deals with the complementability problem for the spaces of norm continuous functions (from compact spaces to Banach spaces) in some spaces of weaker-than-norm (e.g., weakly or weak*) continuous functions. The results obtained are fairly general and strongly support the conjecture that complementability can occur only if the spaces in question coincide.

**Introduction and main results.** Throughout, we let $K$ denote an *infinite* compact Hausdorff space, $X$ a Banach space, and $\tau$ a linear Hausdorff topology on $X$ which is *weaker* than the norm topology. Then $C(K;X)$, the Banach space of all (norm) continuous functions from $K$ into $X$, is obviously a closed linear subspace of $C(K;X,\tau)$, the Banach space of all $\tau$-continuous (norm) bounded functions from $K$ to $X$. (Of course, both spaces are endowed with the sup-norms.) This paper is concerned with the following

CONJECTURE. $C(K;X)$ is not complemented in $C(K;X,\tau)$ unless $C(K;X) = C(K;X,\tau)$.

As yet, we have been unable to verify this conjecture in general. Our main result in this direction is the following

THEOREM 1. *If $X$ contains a $\tau$-convergent sequence which is not norm convergent, then $C(K;X)$ is not complemented in $C(K;X,\tau)$.*

This, in particular, covers the two most important cases.

COROLLARY 1. *If $X$ does not have the Schur property, then $C(K;X)$ is uncomplemented in $C(K;X,w)$, the space of all weakly continuous functions from $K$ to $X$.*

COROLLARY 2. *If $X$ is infinite-dimensional, then $C(K;X^*)$ is uncomplemented in $C(K;X^*,w^*)$, the space of all weak* continuous functions from $K$ to $X^*$.*

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(Of course, to deduce Corollary 2 from Theorem 1 one has to appeal to the Josefson–Nissenzweig theorem, see [2; p. 219].)

Let us recall at this point that the spaces $C(K; X^*, w^*)$, $C(K; X^*, w)$ and $C(K; X^*)$ are isometrically isomorphic to the spaces of bounded operators, weakly compact operators, and compact operators from $X$ to $C(K)$, respectively; see [3; Theorem VI.7.1]. (Some additional information about other Banach spaces representable in the form $C(K; X^*, w^*)$ can be found in [1].) It follows that our Corollary 2 is a particular case of Feder's result [4; Cor. 1] that the space $K(X, Y)$ of compact operators is uncomplemented in the space $L(X, Y)$ of bounded operators whenever $\dim X = \infty$ and $Y$ contains an isomorphic copy of $c_0$. In general, the problem considered in this paper somewhat resembles the (as yet open) question of whether $K(X, Y)$ is always uncomplemented in $L(X, Y)$ unless $K(X, Y) = L(X, Y)$; see [4] again for more about this and references to earlier works.

Now, back to our Conjecture, suppose $C(K; X) \neq C(K; X, \tau)$ (so that there is a $\tau$-compact set which is not norm compact) but the assumption made in Theorem 1 is not satisfied, that is, $\tau$- and norm-convergent sequences in $X$ are the same. (We must confess that, at the time of writing this, we do not know of any example of such a situation.) In this case, attempting to extend or modify the method used in the proof of Theorem 1, we encountered a serious difficulty. Roughly speaking, its nature seems to lie in that we were unable to produce functions which are $\tau$- but not norm-continuous, and yet have some sort of "good" expansions with norm-continuous terms. (Comp. Problem 2 at the end of [4].) Nevertheless, though not for the original space $K$, we are able to prove that, in this situation, the assertion of Theorem 1 holds for $K = \beta\mathbf{N}$, the Stone–Čech compactification of $\mathbf{N} = \{1, 2, \ldots\}$.

THEOREM 2. *If $X$ contains a $\tau$-compact set which is not norm compact, then $C(\beta\mathbf{N}; X)$ is not complemented in $C(\beta\mathbf{N}; X, \tau)$.*

Finally, we prove a result which tells us that our conjecture cannot be disproved by means of a "trivial" counterexample.

THEOREM 3. *If $C(K; X) \neq C(K; X, \tau)$, then $C(K; X)$ is of infinite codimension in $C(K; X, \tau)$. In fact, $C(K; X, \tau)$ has a subspace $Z \cong c_0$ such that $Z \cap C(K; X) = \{0\}$.*

*Notation.* Our Banach space terminology and notation are more or less standard, as in [2] for instance. We explain here only some additional notation that will be used in this paper.

First of all, for $X$ and $\tau$ as above, we denote by $\varkappa(X)$, $\varkappa_0(X)$, $\varkappa(X, \tau)$ and $\varkappa_0(X, \tau)$ the spaces of all sequences $(x_n)$ in $X$ which are, respectively, relatively norm compact; norm null; norm bounded and relatively $\tau$-compact; norm bounded and $\tau$-null. All these spaces are equipped with the sup-norms under which each of them, except possibly the third one, is a Banach space. As concerns the space $\varkappa(X, \tau)$, it is easily seen that if the topology $\tau$ is such that a set in $X$ is relatively $\tau$-compact iff it is relatively sequentially $\tau$-compact, then also $\varkappa(X, \tau)$ is a Banach space. If $(x_n)$ is a sequence in $X$ then, for each $n$, we denote by $\hat{x}_n$ the sequence $(0, \ldots, 0, x_n, 0, \ldots)$, where $x_n$ stands in the $n$th position.



If $M$ is an infinite subset of $\mathbf{N}$, then we denote by $l_\infty(M)$ the subspace of $l_\infty$ consisting of those elements whose supports are contained in $M$; clearly, $l_\infty(M)$ is isometrically isomorphic to $l_\infty$: $l_\infty(M) \cong l_\infty$.

By an *operator* between two Banach spaces we always mean a continuous linear operator.

**Some auxiliary results.** The first result we need is Proposition 4 in [5]:

PROPOSITION 1. *If $T: l_\infty \to l_\infty$ is an operator such that $T|c_0 = 0$, then there exists an infinite subset $M$ of $\mathbf{N}$ for which $T|l_\infty(M) = 0$.*

The next result is basic for all that follows.

PROPOSITION 2. *Every Banach space $X$ satisfies the following condition.*

(C) *Whenever $(x_n)$ is a (bounded) sequence in $X$ for which there exists an operator $T: l_\infty \to \varkappa(X)$ such that $Te_n = \hat{x}_n$ for all $n$, then $(x_n) \in \varkappa(X)$.*

Proof. Let $(x_n)$ and $T$ be as required in (C).

We first show that (C) is satisfied when $X = l_\infty$. For every $n \in \mathbf{N}$, let $P_n$ be the $n$th coordinate projection $(z_j) \to \hat{z}_n$ in $\varkappa(l_\infty)$, and let $R_n$ be the $n$th coordinate projection in $l_\infty$. Since $TR_n e_j = P_n T e_j = \hat{x}_n$ if $j = n$, and $= 0$ otherwise, all the operators

$$TR_n - P_n T: \ l_\infty \to \hat{X}_n = \{(z_k) \in \varkappa(l_\infty): \ z_k = 0 \ \text{for} \ k \neq n\} \cong l_\infty$$

vanish on $c_0$. Now it follows easily from Proposition 1 that there exists an infinite subset $M$ of $\mathbf{N}$ such that $TR_n = P_n T$ on $l_\infty(M)$ for all $n$. Thus if $a = (a_j) \in l_\infty(M)$, then $(Ta)_n$, the $n$th coordinate of $Ta$, equals $a_n x_n$ for $n = 1, 2, \ldots$ In particular, if $a$ is the characteristic function of $M$, then $(Ta)_n = x_n$ for all $n \in M$. Since $Ta \in \varkappa(l_\infty)$, the sequence $(x_n)_{n\in M}$ is relatively norm compact. This argument can be easily modified to show that every subsequence of $(x_n)$ contains a relatively norm compact subsequence. Consequently, $(x_n) \in \varkappa(l_\infty)$.

Now, let $X$ be arbitrary and suppose $(x_n) \notin \varkappa(X)$. Then, by passing to a subsequence of $(x_n)$ (and modifying $T$ suitably), we may assume that $\|x_m - x_n\| > \varepsilon$ for all $m \neq n$ and some $\varepsilon > 0$. For each pair of distinct indices $m$, $n$ choose a norm one functional $x^*_{mn} \in X^*$ so that $|x^*_{mn}(x_m - x_n)| > \varepsilon$. Arrange these $x^*_{mn}$'s in a single sequence $(z^*_n)$ and consider the operator

$$S: X \to l_\infty, \quad x \to (z^*_n(x)).$$

Since

$$\|Sx_m - Sx_n\| \geqslant |x^*_{mn}(x_m - x_n)| > \varepsilon \quad \text{for } m \neq n,$$

$(Sx_n) \notin \varkappa(l_\infty)$.

Consider also the induced operator

$$\tilde{S}: \varkappa(X) \to \varkappa(l_\infty), \qquad (y_n) \to (Sy_n),$$

and the operator

$$R = \tilde{S}T: l_\infty \to \varkappa(l_\infty).$$

Then $Re_n = (0, \ldots, 0, Sx_n, 0, \ldots)$, with $Sx_n$ as the $n$th entry, $n = 1, 2, \ldots$ Since $(Sx_n) \notin \varkappa(l_\infty)$ and since we have already seen above that $l_\infty$ satisfies (C), a contradiction arises. ■

The following result, an easy consequence of Proposition 2, will be our main tool in the proofs of Theorems 1 and 2. (However, modifying our arguments slightly, we could have used Proposition 2 directly as well.)

PROPOSITION 3. (a) *Assume that $X$ contains a $\tau$-convergent sequence which is not norm convergent. Then there exists no operator $T: \varkappa_0(X, \tau) \to \varkappa(X)$ such that*

$$(*) \qquad T|\varkappa_0(X) = \mathrm{id}_{\varkappa_0(X)}.$$

*In particular, $\varkappa_0(X)$ is not complemented in $\varkappa_0(X, \tau)$, and $\varkappa(X)$ is not complemented in $\varkappa(X, \tau)$.*

(b) *Assume that $X$ contains a $\tau$-compact set which is not norm compact. Then there exists no operator $T: \varkappa(X, \tau) \to \varkappa(X)$ satisfying $(*)$.*

*In particular, $\varkappa(X)$ is not complemented in $\varkappa(X, \tau)$.*

Proof. (a) Suppose such an operator $T$ exists. From the assumption it follows that we can find a sequence $(x_n)$ in $X$ which is normalized and $\tau$-null. Using it we define an operator $S: l_\infty \to \varkappa_0(X, \tau)$ by $S(a_n) = (a_n x_n)$. Then $TS: l_\infty \to \varkappa(X)$ and $(TS)e_n = \hat{x}_n$ for all $n$. Hence, by Proposition 2, $(x_n) \in \varkappa(X)$. Since $x_n \to 0$ $(\tau)$, and the norm topology is stronger than $\tau$, $\|x_n\| \to 0$; a contradiction.

(b) In view of (a) we may additionally assume that every $\tau$-convergent sequence is norm convergent. Then all $\tau$-bounded (in particular, $\tau$-compact) sets are norm bounded. Now, from the assumption in (b) it follows that there exists a sequence $(x_n) \in \varkappa(X, \tau) \setminus \varkappa(X)$. We may then define an operator $S$ from $l_\infty$ to $\varkappa(X, \tau)$ by the same formula as above, and conclude the proof as above. ■

Remark 1. Suppose the topology $\tau$ is such that $(X, \tau)$ admits a nonzero sequentially continuous linear functional. Then $\varkappa_0(X, \tau)$ is not complemented in $\varkappa(X, \tau)$.

Indeed, by assumption we can find a norm one element $x$ in $X$ and a sequentially $\tau$-continuous linear functional $\xi$ on $X$ such that $\xi(x) = 1$. Then the operator $R: l_\infty \to \varkappa(X, \tau)$, $(a_n) \to (a_n x)$, is an isometric embedding, and the operator $P: \varkappa_0(X, \tau) \to \varkappa_0(X, \tau)$, $(x_n) \to (\xi(x_n)x)$, is a projection onto the subspace $R(c_0) \cong c_0$. Hence if $Q: \varkappa(X, \tau) \to \varkappa_0(X, \tau)$ were a projection, then $S = PQR: l_\infty \to R(c_0)$ would map isomorphically $c_0$ onto $R(c_0)$ so that $(S|c_0)^{-1} \circ S$ would be a projection from $l_\infty$ onto $c_0$, which is well known to be impossible.



**Proofs of the main results**

Proof of Theorem 1. Since $K$ is infinite, we can find a sequence $(\varphi_n)$ of continuous functions $\varphi_n: K \to [0, 1]$ with pairwise disjoint supports and such that $\varphi_n(t_n) = 1$ for some point $t_n$ in $K$. Then we can define an operator $Q: \varkappa_0(X, \tau) \to C(K; X, \tau)$ by the equality

$$Q((x_n))(t) = \sum_{n=1}^{\infty} \varphi_n(t) x_n.$$

(Note that the series is $\tau$-uniformly convergent on $K$, hence its sum is indeed a norm bounded $\tau$-continuous function.) Moreover, consider the operator

$$R: C(K; X) \to \varkappa(X), \qquad f \to (f(t_n)).$$

Now, if an onto projection $P: C(K; X, \tau) \to C(K; X)$ existed, then the operator $RPQ: \varkappa_0(X, \tau) \to \varkappa(X)$ would be the identity when restricted to $\varkappa_0(X)$, thus contradicting Proposition 3(a). ■

Remark 2. Let $X$ be as in Theorem 1. If $S: l_\infty \to \varkappa_0(X, \tau)$ is the operator defined in the proof of Proposition 3(a) and $Q$ is the operator defined above, then $J = QS$ is an isometric embedding of $l_\infty$ into $C(K; X, \tau)$ and $J(l_\infty) \cap C(K; X) = J(c_0)$. Note also that the above proof actually shows that $C(K; X)$ is uncomplemented in the subspace $C_s(K; X, \tau)$ of $C(K; X, \tau)$ consisting of functions with norm separable ranges.

Proof of Theorem 2. As in the proof of Proposition 3(b) we may additionally assume that the topology $\tau$ and the norm topology have the same convergent sequences. (For, otherwise, Theorem 1 applies.) Then, if we consider the norm on $X$ whose closed unit ball equals the $\tau$-closure of the original unit ball, it is easily seen that the new norm and the original norm are equivalent. We may therefore assume that our original norm has the property that its closed balls are $\tau$-closed. From this it follows that if $A$ is a relatively $\tau$-compact set, then

$$\sup\{\|x\|: x \in A\} = \sup\{\|x\|: x \in \bar{A}^\tau\}.$$

Now it should be clear that there exists an isometric isomorphism between $\varkappa(X, \tau)$ and $C(\beta\mathrm{N}; X, \tau)$ which maps $\varkappa(X)$ onto $C(\beta\mathrm{N}; X)$. Hence, under the additional assumptions made in the course of this proof, Theorem 2 is simply a reformulation of the final statement in Proposition 3(b). ■

Remark 3. If the $\tau$- and norm-convergent sequences in $X$ coincide, then the subspace $C_s(K; X, \tau)$ mentioned in Remark 2 is equal to $C(K; X)$. This follows immediately from the fact that if $Y$ is a norm separable subspace of $X$, then there exists a metrizable linear topology on $Y$ which is weaker than the topology induced on $Y$ by $\tau$ (see [6]) so that $\tau$-compact subsets of $Y$ are norm compact.

Proof of Theorem 3. Let $f \in C(K;X,\tau) \setminus C(K;X)$ and let

$$D(f) = \{t \in K: f \text{ is not norm continuous at } t\}.$$

First consider the case when $D(f)$ has an isolated point $t_0$ so that we can find a neighborhood $U$ of $t_0$ with $U \cap D(f) = \{t_0\}$. Choose a continuous function $\varphi$: $K \to [0,1]$ such that $\varphi(t_0) = 1$ and $\varphi = 0$ on $K \setminus U$. Then the function $\varphi f$ is bounded, $\tau$-continuous and $D(\varphi f) = \{t_0\}$. Thus, replacing $f$ by $\varphi f$ if necessary, we may assume that $D(f) = \{t_0\}$. Since $f$ is not norm continuous, $f(K)$ is not norm compact. Hence there exists a sequence $(t_n)$ in $K$ and an $\varepsilon > 0$ such that $\|f(t_m) - f(t_n)\| > \varepsilon$ for $m \neq n$. We claim that $t_n \to t_0$. If not, then there is a neighborhood $V$ of $t_0$ such that $t_n \notin V$ for $n$ in some infinite subset $M$ of N. But $f|(K \setminus V)$ is norm continuous, hence $\{f(t_n): n \in M\}$ is relatively norm compact — a contradiction. Now, as $t_n \to t_0$, the sequence $(f(t_n))$ is $\tau$-convergent but, by our choice of $t_n$'s, not norm convergent. Hence, by Remark 2, there exists an isometric embedding $J$ of $l_\infty$ into $C(K;X,\tau)$ such that $J(l_\infty) \cap C(K;X) = J(c_0)$. Let $(M_n)$ be a partition of N into an infinite sequence of infinite subsets, and consider the subspace $L$ of $l_\infty$ consisting of those elements which are constant on each of the sets $M_n$. Then $L \cong l_\infty$ and $J(L) \cap C(K;X) = \{0\}$. (Thus we have proved even more than asserted.)

Now consider the opposite case, i.e., when the (nonempty) set $D(f)$ has no isolated points. Then, in particular, $D(f)$ is infinite and we can find a sequence $(G_n)$ of pairwise disjoint nonempty open subsets of $K$ so that each $G_n$ contains a point $t_n$ from $D(f)$. For each $n$ choose a continuous function $\varphi_n$: $K \to [0,1]$ such that $\varphi_n = 0$ on $K \setminus G_n$ and $\varphi_n(t_n) = 1$, and let $x$ be any norm one element in $X$. Let us assume, as we may, that $\|f\| \leqslant 1$. Then the formula

$$S((a_n))(t) = \sum_{n=1}^{\infty} a_n [\varphi_{2n-1}(t) f(t) + \varphi_{2n}(t) x]$$

defines an isometric isomorphism from $c_0$ into $C(K;X,\tau)$ and it is clear that $S(c_0) \cap C(K;X) = \{0\}$. ■

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