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Characterizations of subnormal operators

by

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Abstract. Some necessary and sufficient conditions for a d-tuple of commuting operators to have commuting normal extensions are formulated in terms of positive definiteness and conditional positive definiteness. Then new characterizations of algebraic normal operators are obtained.

1. Introduction. Among various characterizations of subnormal families of bounded operators, there are ones which can be formulated in terms of positive definiteness. In this paper we are mostly interested in the criteria of Halmos–Bram–Ito (cf. [6], [2], [7]), Embry–Lubin (cf. [4], [9]) and Lambert–Lubin (cf. [8], [9]). In Section 4 we show that all of them remain true if we replace the positive definiteness by a weaker notion of conditional positive definiteness, provided the operators in question are contractions (compare Theorem 4.1 with Theorem 2.1). This fact enables us to distinguish a wide class $\mathscr{H}_1$ of entire functions φ having the following property (see Section 5):

(P_d) if $\mathscr{S} = (S_1, \ldots, S_d)$ is a d-tuple of commuting contractions on a Hilbert space H such that the function $\mathbf{N}^d \ni a \to \varphi(\|\mathscr{S}^a f\|^2) \in \mathbf{C}$ is positive definite over the $*$-semigroup $\mathfrak{N}_d$ for every $f \in H$, then $\mathscr{S}$ is subnormal.

In Section 6 we extend the class $\mathscr{H}_1$ to another one $\mathscr{H}$, whose members have property (P_1) within the set of algebraic contractions. Using the results of that section we prove in [16] that for φ in $\mathscr{H}$, the composition operator $C_A f = f \circ A$, $f \in L^2(\mathbf{R}^n, \varphi(\|x\|^2)dx)$, induced by an $n \times n$ matrix A is subnormal if and only if A is normal in $(\mathbf{R}^n, \|\cdot\|)$.

In Section 3 we show that if $\mathscr{S}$ is a d-tuple of commuting operators on H, X is a linear subspace of H consisting of the vectors f such that the function $\mathbf{N}^d \times \mathbf{N}^d \ni (a, b) \to (\mathscr{S}^a f, \mathscr{S}^b f) \in \mathbf{C}$ is positive definite over $\mathfrak{M}_d$ and $H = \bigvee \{\mathscr{S}^a X\colon a \in \mathbf{N}^d\}$, then $\mathscr{S}$ is subnormal. This is not the case when X consists of the vectors f such that the function $\mathbf{N}^d \ni a \to \|\mathscr{S}^a f\|^2 \in \mathbf{C}$ is positive definite over $\mathfrak{N}_d$.

2. Definitions and background results. Denote by $B(H)$ the C^*-algebra of all bounded linear operators on a Hilbert space H over a field $\mathbf{F}$ (we consider either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers). If $\{H_\omega\colon \omega \in \Omega\}$ is a family of subsets of H, then we denote by $\bigvee \{H_\omega\colon \omega \in \Omega\}$ the

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closed linear span of the set $\bigcup\{H_\omega\colon \omega\in\Omega\}$. Given a nonempty set G, denote by $F(G,H)$ the set of all functions from G to H vanishing off finite subsets of G. $F_0(G,H)$ is the set of all f from $F(G,H)$ such that $\sum_x f(x)=0$.

A kernel $B\colon G\times G\to B(H)$ is said to be *positive definite* (resp. *conditionally positive definite*) if

(2.1) $$B(x,y)=B(y,x)^*,\quad x,y\in G,$$

(2.2) $$\sum_{x,y}(B(x,y)f(y),f(x))\geqslant 0,\quad f\in F(G,H)\ \text{(resp. } f\in F_0(G,H)).$$

For this topic the reader may consult [11] and [12].

Let $\mathfrak{G}=(G,+,*)$ be an additive $*$-semigroup. We say that a function $E\colon G\to B(H)$ is *positive definite over* $\mathfrak{G}$ (resp. *conditionally positive definite over* $\mathfrak{G}$) if the kernel $B\colon G\times G\to B(H)$ defined by $B(x,y)=E(x^*+y)$, $x,y\in G$, is positive definite (resp. conditionally positive definite). It is obvious that each function which is positive definite over $\mathfrak{G}$ is conditionally positive definite over $\mathfrak{G}$. The converse statement is not true in general (see Example 5.4).

In the sequel $\mathbf{N}$ stands for the additive semigroup of all nonnegative integers. The direct product $\mathbf{N}^d$ of d copies of the semigroup $\mathbf{N}$, equipped with the identity involution, becomes a $*$-semigroup. Name it $\mathfrak{N}_d$. The set $\mathbf{N}^d$ with pointwise defined partial order relation $\leqslant$ is an upward directed set. Denote by e the member of $\mathbf{N}^d$ with all coordinates 1.

The product semigroup $\mathbf{N}^d\times\mathbf{N}^d$ with involution $(a,b)^*=(b,a)$, $a,b\in\mathbf{N}^d$, is a $*$-semigroup. Denote it by $\mathfrak{M}_d$. Notice that the $*$-semigroups $\mathfrak{M}_d$ and $\mathfrak{N}_{2d}$ are not $*$-isomorphic.

A sequence $\{w_n\}_{n=0}^\infty$ of real numbers is said to be a *Stieltjes moment sequence* if there is a positive finite Borel measure μ on the nonnegative reals $\mathbf{R}_+$ such that

(2.3) $$w_n=\int t^n\,d\mu(t),\quad n\geqslant 0.$$

It is well known (cf. [1], Theorem 6.2.5) that a sequence $\{w_n\}_{n=0}^\infty$ is a Stieltjes moment sequence if and only if the sequences $\{w_n\}_{n=0}^\infty$ and $\{w_{n+1}\}_{n=0}^\infty$ are positive definite over $\mathfrak{N}_1$. Notice that if $\{w_n\}_{n=0}^\infty$ is a Stieltjes moment sequence such that $\limsup w_n^{1/n}\leqslant 1$, then $\{w_n\}_{n=0}^\infty$ is decreasing. To see this take μ as in (2.3). Since the μ-essential supremum of the identity function on $\mathbf{R}_+$ equals $\lim(\int t^n\,d\mu(t))^{1/n}$ (cf. [13], p. 73), the closed support of μ is included in $[0,1]$. Consequently, by (2.3), the sequence $\{w_n\}_{n=0}^\infty$ is decreasing.

Suppose we are given a d-tuple $\mathscr{S}=(S_1,\ldots,S_d)$ of commuting operators on a Hilbert space H over $\mathbf{F}$. We say that $\mathscr{S}$ is *subnormal* if there exists a d-tuple $\mathscr{N}=(N_1,\ldots,N_d)$ of commuting normal operators on some Hilbert space $K\supset H$ such that $S_j\subset N_j$, $j=1,\ldots,d$.

Define functions $E_{\mathscr{S}}\colon \mathbf{N}^d\to B(H)$ and $B_{\mathscr{S}}\colon \mathbf{N}^d\times\mathbf{N}^d\to B(H)$ by

$$E_{\mathscr{S}}(a)=(\mathscr{S}^a)^*\mathscr{S}^a,\quad a\in\mathbf{N}^d,$$
$$B_{\mathscr{S}}(a,b)=(\mathscr{S}^b)^*\mathscr{S}^a,\quad (a,b)\in\mathbf{N}^d\times\mathbf{N}^d,$$

where $\mathscr{S}^a=S_1^{a_1}S_2^{a_2}\ldots S_d^{a_d}$ for $a=(a_1,\ldots,a_d)\in\mathbf{N}^d$. As the following theorem shows the positive definiteness over $\mathfrak{N}_d$ (resp. $\mathfrak{M}_d$) of the function $E_{\mathscr{S}}$ (resp. $B_{\mathscr{S}}$) completely characterizes subnormality.

THEOREM 2.1. *Let $\mathscr{S}$ be a d-tuple of commuting operators on H. Then the following conditions are equivalent*:

(S) *$\mathscr{S}$ is subnormal,*
(HBI) *$B_{\mathscr{S}}$ is positive definite over $\mathfrak{M}_d$,*
(EL) *$E_{\mathscr{S}}$ is positive definite over $\mathfrak{N}_d$,*
(wHBI) *for every $f\in H$, $(B_{\mathscr{S}}(\cdot)f,f)$ is positive definite over $\mathfrak{M}_d$,*
(LL) *for every $f\in H$, $(E_{\mathscr{S}}(\cdot)f,f)$ is positive definite over $\mathfrak{N}_d$.*

The equivalence (S)$\Leftrightarrow$(HBI) was proved by Halmos and Bram in case $d=1$ (cf. [6], [2] and [3]) and by Ito in case $d\geqslant 1$ (cf. [7] and [20]). The present formulation of the Halmos–Bram–Ito condition (HBI) is essentially due to Sz.-Nagy (cf. [18]). The equivalence (S)$\Leftrightarrow$(EL) was proved by Embry in case $d=1$ (cf. [4] and [3]) and by Lubin in case $d\geqslant 1$ (cf. [9] and [20]). Next, the equivalence (EL)$\Leftrightarrow$(LL) was proved by Lambert in case $d=1$ (cf. [8]) and by Lubin in case $d\geqslant 1$ (cf. [9]). Although our formulation of the Lambert–Lubin condition (LL) differs a little from the original one, they are still equivalent (cf. [17], Theorem 11). Finally, the implications (HBI)$\Rightarrow$(wHBI) and (wHBI)$\Rightarrow$(LL) can be easily verified by the reader.

3. Improved Halmos–Bram–Ito criterion. Our goal in this section is to strengthen the Halmos–Bram–Ito criterion for subnormality of d-tuples $\mathscr{S}$ of commuting operatos on a Hilbert space H over $\mathbf{F}$. To begin with notice that for $f\in H$, the scalar function $(B_{\mathscr{S}}(\cdot)f,f)$ is positive definite over $\mathfrak{M}_d$ if and only if

$$\sum_{a,b}\sum_{p,q}(\mathscr{S}^{b+p}f,\mathscr{S}^{a+q}f)z(p,q)\overline{z(a,b)}\geqslant 0,\quad z\in F(\mathbf{N}^d\times\mathbf{N}^d,\mathbf{C}).$$

Therefore the result stated below can be regarded as a generalization of Theorem 2 of [14] as well as of criterion (wHBI) of Theorem 2.1.

THEOREM 3.1. *Let $\mathscr{S}$ be a d-tuple of commuting operators on a Hilbert space H over* $\mathbf{F}$. *Let X be a linear subspace of H such that*

(i) $H=\bigvee\{\mathscr{S}^aX\colon a\in\mathbf{N}^d\}$,
(ii) *for every $f\in X$, $(B_{\mathscr{S}}(\cdot)f,f)$ is positive definite over $\mathfrak{M}_d$.*

Then $\mathscr{S}$ is subnormal.

Proof. Without loss of generality we can assume that all the operators $S_1,\ldots,S_d$ are contractions. Denote by H_0 the closure of X. Notice that $(B_{\mathscr{S}}(\cdot)f,f)$ is positive definite over $\mathfrak{M}_d$ for every $f\in H_0$. Define the function $A\colon \mathbf{N}^d\times\mathbf{N}^d\to B(H_0)$ by

$$A(x)=P_0B_{\mathscr{S}}(x)|_{H_0},\quad x\in\mathbf{N}^d\times\mathbf{N}^d,$$

where P_0 is the orthogonal projection of H onto H_0. Since each S_j is a contraction, we have $\sup\{\|A(x)\|: x\in \mathbf{N}^d\times\mathbf{N}^d\}\leqslant 1$. Moreover, if $f\in H_0$, then (ii) implies

$$\Big(\big(\sum_{x,y} A(x^*+y)\lambda(y)\overline{\lambda(x)}\big)f, f\Big) = \sum_{x,y}\big(B_{\mathscr{S}}(x^*+y)f, f\big)\lambda(y)\overline{\lambda(x)} \geqslant 0.$$

It follows from Proposition 3.4 of [10] that the function A is positive definite over $\mathfrak{M}_d$.

Now we show that $B_{\mathscr{S}}$ is positive definite over $\mathfrak{M}_d$. Take $f\in F(\mathbf{N}^d\times\mathbf{N}^d, Z)$, where Z is the linear span of the set $\bigcup\{\mathscr{S}^a H_0: a\in\mathbf{N}^d\}$. Then for any $(a,b)\in\mathbf{N}^d\times\mathbf{N}^d$ there exists $g_{ab}\in F(\mathbf{N}^d, H_0)$ such that $f(a,b)=\sum_k \mathscr{S}^k g_{ab}(k)$ (if $f(a,b)=0$, then we set $g_{ab}=0$). Define the function $h\in F(\mathbf{N}^d\times\mathbf{N}^d, H_0)$ by

$$h(a,b)=\sum_{u+k=a} g_{ub}(k), \qquad (a,b)\in\mathbf{N}^d\times\mathbf{N}^d.$$

Then, since A is positive definite over $\mathfrak{M}_d$, we get

$$\begin{aligned}\sum_{a,b}\sum_{p,q}\big(B_{\mathscr{S}}((p,q)^*+(a,b))f(a,b), f(p,q)\big) &= \sum_{a,b}\sum_{p,q}\big(\mathscr{S}^{a+q}f(a,b), \mathscr{S}^{b+p}f(p,q)\big)\\ &= \sum_{a,b}\sum_{p,q}\sum_{k,l}\big(\mathscr{S}^{a+q+k}g_{ab}(k), \mathscr{S}^{b+p+l}g_{pq}(l)\big)\\ &= \sum_{a,b,k}\sum_{p,q,l}\big(A((p+l,q)^*+(a+k,b))g_{ab}(k), g_{pq}(l)\big)\\ &= \sum_{a,b}\sum_{p,q}\big(A((p,q)^*+(a,b))h(a,b), h(p,q)\big)\geqslant 0.\end{aligned}$$

This and the denseness of Z in H imply that $B_{\mathscr{S}}$ is positive definite over $\mathfrak{M}_d$. Thus, by (HBI) of Theorem 2.1, $\mathscr{S}$ is subnormal. ∎

As a simple consequence of Theorem 3.1 we get the following generalization of Corollary 1 of [22].

COROLLARY 3.2. *Let $\mathscr{S}=(S_1,\ldots,S_d)$ be a d-tuple of commuting operators on H and let X be a linear subspace of H such that $H=\bigvee\{\mathscr{S}^a X: a\in\mathbf{N}^d\}$. Then the following conditions are equivalent:*

(i) *$\mathscr{S}$ is subnormal,*

(ii) *for every $f\in X$, $\mathscr{S}_f := ((S_1)_f,\ldots,(S_d)_f)$ is subnormal,*

where $(S_j)_f$, $j=1,\ldots,d$, is the restriction of S_j to $\bigvee\{\mathscr{S}^a f: a\in\mathbf{N}^d\}$.

Proof. (i)⇒(ii) is obvious. Assume (ii). Then, applying (wHBI) of Theorem 2.1 to $\mathscr{S}_f$, we find that $(B_{\mathscr{S}}(\cdot)f,f)=(B_{\mathscr{S}_f}(\cdot)f,f)$ is positive definite over $\mathfrak{M}_d$ for every $f\in X$. Therefore the subnormality of $\mathscr{S}$ follows from Theorem 3.1. ∎

Theorem 3.1 can be used to obtain a new version of Lambert's characterization of subnormal weighted shifts with operator weights (cf. [8], Theorem 3.2). For simplicity we restrict our attention to a single operator.

Given a Hilbert space H we denote by $l_2(H)$ the orthogonal direct sum of countably many copies of H and by Π_n, $n\in\mathbf{N}$, the isometric embedding of H onto the nth summand of $l_2(H)$. Let $\{A_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence of operators on H. Let $A_{(0)}=0$ and for $n\geqslant 1$ let $A_{(n)}=A_{n-1}A_{n-2}\ldots A_0$. Define S on $l_2(H)$ by $S(f_0,f_1,\ldots)=(0,A_0f_0,A_1f_1,\ldots)$. The operator S is called a *weighted shift* with *weight sequence* $\{A_n\}_{n=0}^{\infty}$. Notice that S always satisfies the equalities $S\Pi_n=\Pi_{n+1}A_n$, $n\in\mathbf{N}$.

COROLLARY 3.3. *Let S be a weighted shift with weight sequence $\{A_n\}_{n=0}^{\infty}$ such that $\ker A_n^*=\{0\}$ for all $n\in\mathbf{N}$. Then S is subnormal if and only if the sequence $\{\|A_{(n)}f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $f\in H$.*

Proof. If S is subnormal, then, by (LL) of Theorem 2.1, the sequences $\{\|S^n\Pi_0 f\|^2\}_{n=0}^{\infty}$ and $\{\|S^{n+1}\Pi_0 f\|^2\}_{n=0}^{\infty}$ are positive definite over $\mathfrak{N}_1$ for all $f\in H$. Since $\|S^n\Pi_0 f\|^2=\|\Pi_n A_{(n)}f\|^2=\|A_{(n)}f\|^2$, the sequence $\{\|A_{(n)}f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for all $f\in H$.

Suppose now that $\{\|A_{(n)}f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $f\in H$. Fixing $f\in H$ we can find a positive finite Borel measure μ on $\mathbf{R}_+$ such that $\|A_{(n)}f\|^2=\int t^n\,d\mu(t)$ for $n\in\mathbf{N}$. Thus

$$\begin{aligned}(B_S(m,n)\Pi_0 f,\Pi_0 f) &= (S^m\Pi_0 f, S^n\Pi_0 f)=\delta_{m,n}\|A_{(n)}f\|^2\\ &= (2\pi)^{-1}\int_0^{2\pi}\int_{\mathbf{R}_+}(e^{i\vartheta}t^{1/2})^m\overline{(e^{i\vartheta}t^{1/2})^n}\,d\mu(t)\,d\vartheta, \qquad m,n\in\mathbf{N}.\end{aligned}$$

Using this formula one can show that $(B_S(\cdot)h,h)$ is positive definite over $\mathfrak{M}_1$ for all $h\in\Pi_0 H$. Since $\ker A_n^*=\{0\}$ for any $n\in\mathbf{N}$, we get $l_2(H)=\bigvee\{S^n\Pi_0 H: n\in\mathbf{N}\}$. Thus the subnormality of S follows from Theorem 3.1 applied to $X=\Pi_0 H$. ∎

It is a simple observation that Theorem 3.1 remains true if we replace the linearity of X by the linearity of $\bar{X}$. However, we cannot drop the assumption about the linearity of X (or $\bar{X}$) in general.

EXAMPLE 3.4. Let $H=\mathbf{C}^2$, $f_1=(1,\sqrt{2})$ and $f_2=(1,-\sqrt{2})$. Denote by A the operator on H such that $Af_1=\sqrt{2}f_1$ and $Af_2=-\sqrt{2}f_2$. Let S be the weighted shift with weight sequence $A_n=A$, $n\in\mathbf{N}$. Then $l_2(H)=\bigvee\{S^n X: n\in\mathbf{N}\}$, where $X=\{\Pi_0 f_1,\Pi_0 f_2\}$. Arguing as in the proof of Corollary 3.3 one can show that the function $(B_S(\cdot)h,h)$ is positive definite over $\mathfrak{M}_1$ for every $h\in X$. However, S is not subnormal, because A is not normal (use Corollary 3.3 and then (LL) of Theorem 2.1). ∎

Notice also that B_S cannot be replaced by E_S in Theorem 3.1.

EXAMPLE 3.5. Let $H=\mathbf{C}^2$ and $f_0=(0,1)$. Define $S\in B(H)$ by $S(\alpha,\beta)=(\alpha+\beta,0)$ for $\alpha,\beta\in\mathbf{C}$. Put $X=\mathbf{C}f_0$. Then $H=\bigvee\{S^n X: n\in\mathbf{N}\}$ and $\{\|S^n h\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for $h\in X$ (which implies that $(E_S(\cdot)h,h)$ is positive definite over $\mathfrak{N}_1$ for $h\in X$). However, S is not subnormal. ∎

We close this section with the observation that Theorem 3.1 can also be formulated for d-tuples $\mathscr{S}$ composed of invertible operators on H such that $H=\bigvee\{\mathscr{S}^a X: a\in\mathbf{Z}^d\}$.

4. Conditional positive definiteness and subnormality. Recently M. Thill has proved (cf. [21], Prop. 7) that if the sequence $\varphi\colon \mathbf{N}\to\mathbf{C}$ is conditionally positive definite over $\mathfrak{N}_1$ and φ satisfies the inequality $|\varphi(n)|\leqslant v\alpha^n$, $n\in\mathbf{N}$, for some nonnegative reals v and $\alpha<1$, then φ is positive definite over $\mathfrak{N}_1$. Though this is not true for bounded sequences, it is true, as the following theorem shows, for contractive operator-valued functions $E_{\mathscr{S}}$ and $B_{\mathscr{S}}$. As a consequence we obtain new criteria for subnormality of contractions.

THEOREM 4.1. *If $\mathscr{S}=(S_1,\ldots,S_d)$ is a d-tuple of commuting contractions on H, then the following conditions are equivalent*:

(i) *$\mathscr{S}$ is subnormal,*

(ii) *$B_{\mathscr{S}}$ is conditionally positive definite over $\mathfrak{M}_d$,*

(iii) *$E_{\mathscr{S}}$ is conditionally positive definite over $\mathfrak{N}_d$,*

(iv) *for every $f\in H$, $(E_{\mathscr{S}}(\cdot)f,f)$ is conditionally positive definite over $\mathfrak{N}_d$.*

The following lemma plays a crucial role not only in the proof of Theorem 4.1.

LEMMA 4.2. *If $\mathscr{S}=(S_1,\ldots,S_d)$ is a d-tuple of commuting contractions on H, then*

(i) *all the nets $\{E_{\mathscr{S}}(a)\colon a\in\mathbf{N}^d\}$, $\{E_{\mathscr{S}}(2me)\colon m\in\mathbf{N}\}$ and $\{E_{\mathscr{S}}(b+me)\colon m\in\mathbf{N}\}$, $b\in\mathbf{N}^d$, are convergent in the strong operator topology to the same positive operator $D_{\mathscr{S}}\in B(H)$,*

(ii) *if $E_{\mathscr{S}}$ is conditionally positive definite over $\mathfrak{N}_d$, then*

$$\sum_{a,b}(E_{\mathscr{S}}(a+b)f(b),f(a))\geqslant\|D_{\mathscr{S}}^{1/2}(\sum_a f(a))\|^2,\quad f\in F(\mathbf{N}^d,H),\tag{4.1}$$

(iii) *if $(E_{\mathscr{S}}(\cdot)f,f)^j$ is conditionally positive definite over $\mathfrak{N}_d$ for some fixed integer $j\geqslant 1$ and $f\in H$, then*

$$\sum_{a,b}(E_{\mathscr{S}}(a+b)f,f)^j z(b)\overline{z(a)}\geqslant\|D_{\mathscr{S}}^{1/2}f\|^{2j}|\sum_a z(a)|^2,\quad z\in F(\mathbf{N}^d,\mathbf{C}).\tag{4.2}$$

Proof. (i) Since $0\leqslant E_{\mathscr{S}}(a_2)\leqslant E_{\mathscr{S}}(a_1)$ for $a_1\leqslant a_2$, the net $\{E_{\mathscr{S}}(a)\colon a\in\mathbf{N}^d\}$ is convergent in the strong operator topology to some positive operator $D_{\mathscr{S}}\in B(H)$ (cf. [19], Prop. II.3.1, for $d=1$). Since the sets $\{2me\colon m\in\mathbf{N}\}$ and $\{b+me\colon m\in\mathbf{N}\}$ are cofinal in $\mathbf{N}^d$, the sequences $\{E_{\mathscr{S}}(2me)\colon m\in\mathbf{N}\}$ and $\{E_{\mathscr{S}}(b+me)\colon m\in\mathbf{N}\}$ are convergent in the strong operator topology to $D_{\mathscr{S}}$, for every $b\in\mathbf{N}^d$.

(ii) Take $f\in F(\mathbf{N}^d,H)$. Then there exists $k\in\mathbf{N}$ such that f vanishes off the set $\{a\in\mathbf{N}^d\colon a\leqslant ke\}$. Let $m>k$. Define the net $f_m\in F(\mathbf{N}^d,H)$ by

$$f_m(a)=\begin{cases} f(a) & \text{for } a\leqslant ke,\\ -\sum_b f(b) & \text{for } a=me,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\sum_{a,b}(E_{\mathscr{S}}(a+b)f_m(b),f_m(a))=\sum_{a,b}(E_{\mathscr{S}}(a+b)f(b),f(a))$$
$$-2\operatorname{Re}\sum_{a,b}(E_{\mathscr{S}}(a+me)f(b),f(a))+\big(E_{\mathscr{S}}(2me)(\sum_a f(a)),\sum_a f(a)\big).$$

Since $\sum_a f_m(a)=0$ for every $m>k$, one can use (i) to obtain

$$\sum_{a,b}(E_{\mathscr{S}}(a+b)f(b),f(a))-\big(D_{\mathscr{S}}(\sum_a f(a)),\sum_a f(a)\big)=\lim_{m\to\infty}\sum_{a,b}(E_{\mathscr{S}}(a+b)f_m(b),f_m(a))\geqslant 0.$$

Thus we have proved (4.1). The proof of (4.2) is similar. This completes the proof of Lemma 4.2. ∎

Proof of Theorem 4.1. (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(iv) are obvious. (iv)$\Rightarrow$(i) follows from (4.2) and (LL) of Theorem 2.1. ∎

Arguing similarly to the proof of (4.1) one can show that if $\mathscr{S}=(S_1,\ldots,S_d)$ is a d-tuple of commuting contractions on H which satisfies

$$\text{(HBI)}'\qquad \sum_{a,b\geqslant 0}(\mathscr{S}^b f(a),\mathscr{S}^a f(b))\geqslant 0,\quad f\in F_0(\mathbf{N}^d,H),$$

then $\mathscr{S}$ is subnormal, provided at least one of the contractions $S_1,\ldots,S_d$ is of class $C_{0\cdot}$ (a contraction S is of class $C_{0\cdot}$ if $S^n\to 0$ in the strong operator topology).

5. Positive definiteness and subnormality. In this section we introduce a class of entire functions φ having property (P_d). Denote by $\mathscr{H}_j$, $j=1,2,\ldots$, the family of all entire functions φ such that

$$\frac{d^n\varphi}{dz^n}(0)\geqslant 0,\quad\text{for every } n\geqslant j+1 \text{ and } n=0,$$

$$\frac{d^j\varphi}{dz^j}(0)>0,$$

$$\frac{d^n\varphi}{dz^n}(0)=0,\quad\text{for every } n=1,\ldots,j-1.$$

Put $\mathscr{H}=\bigcup_{j=1}^{\infty}\mathscr{H}_j$.

The following theorem states that the class $\mathscr{H}_1$ consists of the functions φ that have the property (P_d).

THEOREM 5.1. *Let $\varphi\in\mathscr{H}_1$. Then a d-tuple $\mathscr{S}=(S_1,\ldots,S_d)$ of commuting contractions on H is subnormal if and only if $\varphi\big((E_{\mathscr{S}}(\cdot)f,f)\big)$ is positive definite over $\mathfrak{N}_d$ for every $f\in H$.*

To prove Theorem 5.1 we need the following

LEMMA 5.2. *If $\varphi\in\mathscr{H}_j$ for some fixed $j\geqslant 1$ and $b\colon G\times G\to\mathbf{C}$ is a kernel such that $\varphi\circ(tb)$ is positive definite for every $t>0$, then the kernel b^j is conditionally positive definite.*

Proof. It follows from the definition of $\mathscr{H}_j$ that φ has the power series representation

$$\varphi(z) = a_0 + \sum_{n=j}^{\infty} a_n z^n, \quad z \in \mathbf{C},$$

where $a_j > 0$ and $a_n \geqslant 0$ for every $n \geqslant 0$. Thus for every $z \in F_0(G, \mathbf{C})$ we have

$$\sum_{x,y} b(x,y)^j z(x)\overline{z(y)} = \lim_{t \to 0^+} a_j^{-1} \sum_{x,y} t^{-j}\big(\varphi(tb(x,y)) - \varphi(0)\big) z(x)\overline{z(y)}$$

$$= a_j^{-1} \lim_{t \to 0^+} t^{-j} \sum_{x,y} \varphi(tb(x,y)) z(x)\overline{z(y)} \geqslant 0. \;\blacksquare$$

Proof of Theorem 5.1. Assume that $\mathscr{S}$ is subnormal and take $f \in H$. Then, by (LL) of Theorem 2.1, $(E_{\mathscr{S}}(\cdot)f, f)$ is positive definite over $\mathfrak{N}_d$. It follows from the Schur theorem (cf. [1], Theorem 3.1.12) that all its powers $(E_{\mathscr{S}}(\cdot)f, f)^j$, $j = 1, 2, \ldots$, are positive definite over $\mathfrak{N}_d$. Since $\varphi \in \mathscr{H}_1$, φ has the power series representation

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbf{C},$$

with all a_n being nonnegative. Thus $\varphi\big((E_{\mathscr{S}}(\cdot)f, f)\big)$ is positive definite over $\mathfrak{N}_d$ for every $f \in H$.

Assume now that $\varphi\big((E_{\mathscr{S}}(\cdot)f, f)\big)$ is positive definite over $\mathfrak{N}_d$ for every $f \in H$. It follows from Lemma 5.2 (put $b(x,y) = (E_{\mathscr{S}}(x+y)f, f)$) that $(E_{\mathscr{S}}(\cdot)f, f)$ is conditionally positive definite over $\mathfrak{N}_d$ for every $f \in H$. Thus, in virtue of Theorem 4.1, $\mathscr{S}$ is subnormal. $\blacksquare$

Consider now a single contraction S on a complex Hilbert space H and the function $\varphi = \exp \in \mathscr{H}_1$. Let $\mathscr{F}(H) = \bigoplus \sum_{n=0}^{\infty} \mathscr{F}_n(H)$, where $\mathscr{F}_n(H)$ is the n-fold Hilbert space symmetric tensor product of H with itself and $\mathscr{F}_0(H) = \mathbf{C}$. Denote by $\mathscr{F}(S)$ the unique operator on $\mathscr{F}(H)$ such that $\mathscr{F}(S)(\mathrm{EXP}(f)) = \mathrm{EXP}(Sf)$ for every $f \in H$, where $\mathrm{EXP}(f) = \bigoplus \sum_{n=0}^{\infty} (n!)^{-1/2} f^{\otimes n}$ (cf. [5] and [15]). Under these circumstances, Theorem 5.1 can be reformulated in terms of quantum field theory as follows.

Corollary 5.3. *If S is a contraction on H, then $\mathscr{F}(S)$ is subnormal if and only if the sequence $\{\|\mathscr{F}(S)^n \mathrm{EXP}(f)\|^2\}_{n=0}^{\infty}$ is positive definite over $\mathfrak{N}_1$ for every $f \in H$.*

Notice that there are nonsubnormal operators A for which the set $\{h\colon \{\|A^n h\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence$\}$ is total in the underlying Hilbert space (see Example 3.5).

We end this section with an example which shows that Theorems 4.1 and 5.1 are not true if we replace contractions by arbitrary operators.

Example 5.4. Let S be a weighted shift operator subordinate to an orthonormal basis $\{f_n\}_{n=0}^{\infty}$ of H with weights $\big((n+2)(n+1)^{-1}\big)^{1/2}$, $n \geqslant 0$, i.e.

$$Sf_n = \big((n+2)(n+1)^{-1}\big)^{1/2} f_{n+1}, \quad n \geqslant 0.$$

Then an easy calculation gives us

$$\exp(\|S^n f\|^2) = \exp(\|f\|^2) \exp(c(f))^n, \quad n \in \mathbf{N},$$

where

$$c(f) = \sum_{k=0}^{\infty} (k+1)^{-1} |(f, f_k)|^2.$$

This means that $\{\exp(\|S^n f\|^2)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence with representing measure $\exp(\|f\|^2)\delta_{\exp(c(f))}$ (δ_z stands for the point mass probability measure concentrated at $z \in \mathbf{C}$). In other words, $\varphi\big((E_S(\cdot)f, f)\big)$ is positive definite over $\mathfrak{N}_1$ for every $f \in H$, where $\varphi = \exp$. However, S is not subnormal, because the sequence of weights of S is not increasing (cf. [3], Prop. III.8.6). Moreover, one can show that $\|S\| = 2^{1/2}$ and the spectral radius of S is less than or equal to 1.

Notice also that S satisfies condition (iv) of Theorem 4.1. Indeed, since $\varphi \in \mathscr{H}_1$ and $\varphi\big((E_S(\cdot)f, f)\big)$ is positive definite over $\mathfrak{N}_1$ for every $f \in H$, Lemma 5.2 implies that $(E_S(\cdot)f, f)$ is conditionally positive definite over $\mathfrak{N}_1$ for all $f \in H$. $\blacksquare$

6. Positive definiteness and normality. In this section we restrict our attention to a single operator. We show that Theoem 4.1 remains true for functions φ from the class $\mathscr{H}$, provided the operator in question is assumed to be algebraic.

Recall that an operator $S \in B(H)$ is said to be *algebraic* if there exists a nonzero polynomial p with coefficients in $\mathbf{F}$ such that $p(S) = 0$. The following proposition will be useful in the second part of this section.

Proposition 6.1. *An algebraic operator N on a Hilbert space H over $\mathbf{F}$ is normal if and only if the following condition holds true:*

(i) *if $f \in H$, $t > 0$ and $\limsup_{n\to\infty} \|(tN)^n f\|^{1/n} \leqslant 1$, then the sequence $\{\|(tN)^n f\|\}_{n=0}^{\infty}$ is decreasing.*

Proof. Suppose that N is normal and $\limsup_{n\to\infty} \|(tN)^n f\|^{1/n} \leqslant 1$, where $t > 0$. Since tN is normal, the sequences $\{\|(tN)^n f\|^2\}_{n=0}^{\infty}$ and $\{\|(tN)^n (tNf)\|^2\}_{n=0}^{\infty}$ are positive definite over $\mathfrak{N}_1$ (use (LL) of Theorem 2.1). Thus $\{\|(tN)^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence. This implies that $\{\|(tN)^n f\|\}_{n=0}^{\infty}$ is decreasing (see Section 2).

Assume that N is algebraic and satisfies condition (i). First we consider the case $\mathbf{F} = \mathbf{C}$. Let p be the minimal polynomial for N. Then we can split p into factors as follows:

$$p(z) = (z - z_1)^{n_1} \ldots (z - z_m)^{n_m}, \quad z \in \mathbf{C},$$

where z_j are the distinct complex roots of p of multiplicity n_j, $j = 1, \ldots, m$. Denote by H_j the kernel of the operator $(N-z_j)^{n_j}$ and by N_j the restriction of N to its invariant subspace H_j, $j = 1, \ldots, m$. Then N is the direct sum of $N_1, \ldots, N_m$. Moreover, N_j satisfies condition (i) and $(N_j-z_j)^{n_j} = 0$ for every $j = 1, \ldots, m$. The proof of the normality of N will be split into two steps.

Step 1. *If N satisfies* (i) *and $(N-z)^k = 0$ for some $k \geq 1$, then $N-z = 0$.*

We use induction on k. Assume that $k = 2$. If $z = 0$, then $N^2 f = 0$ for every $f \in H$. It is easy to deduce from (i) that in fact Nf must be 0 for every $f \in H$. If $z \neq 0$, then $(N-z)^2 = 0$ implies

$$N^n f = ((N-z)+z)^n f = z^n f + nz^{n-1}(N-z)f, \quad n \geq 1, \ f \in H.$$

Thus $\limsup_{n\to\infty} \|(z^{-1}N)^n f\|^{1/n} \leq 1$. In virtue of (i), the sequence $\{\|(z^{-1}N)^n f\|^2\}_{n=0}^{\infty}$ is convergent. This fact, combined with the equality

$$\|(z^{-1}N)^n f\|^2 = \|f\|^2 + 2n\mathrm{Re}\left(z^{-1}((N-z)f, f)\right) + n^2|z|^{-2}\|(N-z)f\|^2, \quad n \geq 1,$$

implies that $\|(N-z)f\| = 0$ for every $f \in H$. Consequently $N-z = 0$.

Assume now that $k > 0$. Denote by $\tilde{H}$ the closure of the range of $N-z$. If $\tilde{H} = \{0\}$, then $N-z = 0$. Suppose that $\tilde{H} \neq \{0\}$. Denote by $\tilde{N}$ the restriction of N to its invariant subspace $\tilde{H}$. Then $(\tilde{N}-z)^{k-1} = 0$ and $\tilde{N}$ satisfies (i). It follows from the induction assumption that $\tilde{N}-z = 0$, or equivalently $(N-z)^2 = 0$. This in turn implies that $N-z = 0$.

Step 2. *The spaces H_j and H_k are orthogonal for $j \neq k$.*

Without loss of generality we may assume that $j = 1$, $k = 2$ and $|z_1| \leq |z_2|$. In virtue of Step 1, $N_j h_j = z_j h_j$, $h_j \in H_j$, $j = 1, 2$. This implies that

$$\limsup_{n\to\infty} \|(z_2^{-1}N)^n(h_1+h_2)\|^{1/n} \leq 1, \quad h_j \in H_j, \ j = 1, 2. \tag{6.1}$$

First we consider the case $0 \leq |z_1| < |z_2|$. It follows from (6.1) and (i) that

$$\|h_2\|^2 = \lim_{n\to\infty} \|(z_2^{-1}N)^n(h_1+h_2)\|^2 \leq \|h_1+h_2\|^2, \quad h_j \in H_j, \ j = 1, 2.$$

Consequently

$$\|h_2\|^2 \leq \|h_1+h_2\|^2, \quad h_j \in H_j, \ j = 1, 2. \tag{6.2}$$

One can deduce from (6.2) that H_1 is orthogonal to H_2.

Suppose now that $0 \neq |z_1| = |z_2|$. Using again (6.1) and (i) we conclude that $\lim_{n\to\infty} \|(z_2^{-1}N)^n(h_1+h_2)\|^2$ exists for all $h_1 \in H_1$ and $h_2 \in H_2$. Since

$$\|(z_2^{-1}N)^n(h_1+h_2)\|^2 = \|h_1\|^2 + \|h_2\|^2 + 2\mathrm{Re}\,[(z_1 z_2^{-1})^n(h_1, h_2)],$$

for $h_1 \in H_1$ and $h_2 \in H_2$, we find that $\lim_{n\to\infty} (z_1 z_2^{-1})^n |(h_1, h_2)|$ exists for all $h_1 \in H_1$ and $h_2 \in H_2$. Since $z_1 z_2^{-1} \neq 1$, we have necessarily $(h_1, h_2) = 0$ for all $h_1 \in H_1$ and $h_2 \in H_2$. This completes the proof of Step 2.

Steps 1 and 2 together show that N is the orthogonal sum of normal operators $N_1, \ldots, N_m$. Thus N is normal too.

Suppose now that $\mathbf{F} = \mathbf{R}$, N is algebraic and N satisfies condition (i). Denote by $\hat{H}$ (resp. $\hat{N}$) the complexification of H (resp. N). Notice that $\hat{N}$ is algebraic and satisfies (i). Since Proposition 6.1 was proved for $\mathbf{F} = \mathbf{C}$, $\hat{N}$ is normal. Consequently N is normal too. This completes the proof of Proposition 6.1. ∎

The following proposition extends the Lambert characterization of subnormal operators within the class of algebraic operators.

Proposition 6.2. *Let $j \geq 1$ be a fixed integer. Then an algebraic operator N on a Hilbert space H over $\mathbf{F}$ is normal if and only if the following condition holds true:*

(i) *for every $f \in H$, the sequence $\{\|N^n f\|^{2j}\}_{n=0}^{\infty}$ is positive definite over $\mathfrak{N}_1$.*

Proof. If N is normal, then the sequence $\{\|N^n f\|^2\}_{n=0}^{\infty}$ is positive definite over $\mathfrak{N}_1$ (use (LL) of Theorem 2.1). Thus, in virtue of the Schur theorem (cf. [1], Theorem 3.1.12), so is its jth power.

Suppose that N is algebraic and satisfies condition (i). To prove that N is normal it is enough to show that N satisfies condition (i) of Proposition 6.1. Let $t > 0$ and $f \in H$ be such that $\limsup_{n\to\infty} \|(tN)^n f\|^{1/n} \leq 1$. It follows from (i) that the sequences $\{\|(tN)^n f\|^{2j}\}_{n=0}^{\infty}$ and $\{\|(tN)^{n+1} f\|^{2j}\}_{n=0}^{\infty}$ are positive definite over $\mathfrak{N}_1$. Thus $\{\|(tN)^n f\|^{2j}\}_{n=0}^{\infty}$ is a Stieltjes moment sequence such that $\limsup_{n\to\infty} (\|(tN)^n f\|^{2j})^{1/n} \leq 1$. This implies that $\{\|(tN)^n f\|\}_{n=0}^{\infty}$ is decreasing (see Section 2). This completes the proof of Proposition 6.2. ∎

Now we are in a position to prove the main result of this section.

Theorem 6.3. *Assume that N is an algebraic contraction on a Hilbert space H over $\mathbf{F}$ and $\varphi \in \mathscr{H}$. Then N is normal if and only if $\varphi((E_N(\cdot)f, f))$ is positive definite over $\mathfrak{N}_1$ for every $f \in H$.*

Proof. Assume that $\varphi((E_N(\cdot)f, f))$ is positive definite over $\mathfrak{N}_1$ for every $f \in H$. Since $\varphi \in \mathscr{H}$, there exists $j \geq 1$ such that $\varphi \in \mathscr{H}_j$. It follows from Lemma 5.2 that $(E_N(\cdot)f, f)^j$ is conditionally positive definite over $\mathfrak{N}_1$ for every $f \in H$. Since N is a contraction, we can deduce from inequality (4.2) of Lemma 4.2 that the function $(E_N(\cdot)f, f)^j$ is positive definite over $\mathfrak{N}_1$ for every $f \in H$. This and Proposition 6.2 imply the normality of N.

The converse statement can be proved similarly to the "only if" part of Theorem 5.1. ∎

We conclude the paper with the following open question.

Let $j \geq 2$ be a fixed integer. Is it true that an operator S on a complex Hilbert space H is subnormal if and only if for every $f \in H$, the sequence $\{\|S^n f\|^{2j}\}_{n=0}^{\infty}$ is positive definite over $\mathfrak{N}_1$?

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Weighted norm inequalities for Riesz potentials and fractional maximal functions in mixed norm Lebesgue spaces

by

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Abstract. We prove a norm inequality between the Riesz potential $R_\alpha f$ and the fractional maximal function $M_\alpha f$ in $\mathbf{R}^d$, $0<\alpha<d$. The norm is a weighted mixed Lebesgue norm $L^{p,q}_{w_0,w_1}(\mathbf{R}^m\times\mathbf{R}^n)$, where $0<p,q<\infty$ and $d=m+n$, with weights in A_∞. Our proof makes extensive use of the concept of independence of weights in A_p. It is shown that many of the well known properties of Muckenhoupt weights are true in this more general form, among them the P. W. Jones Factorization Theorem for A_p-weights.

0. Introduction. Let $\mathbf{R}^d$ be the d-dimensional Euclidean space. The Riesz potential of order α, $0<\alpha<d$, of a function f is defined by

$$R_\alpha f(\xi)=\int|\xi-\eta|^{\alpha-d}f(\eta)\,d\eta.$$

For $0\leqslant\alpha<d$ we also define the fractional maximal operator $M_\alpha f(\xi)$ by

$$M_\alpha f(\xi)=\sup|Q|^{\alpha/d-1}\int_Q|f(\eta)|\,d\eta,$$

where the supremum is over all cubes Q with sides parallel to the axes and containing ξ. When $\alpha=0$ we get the usual Hardy–Littlewood maximal operator.

Muckenhoupt and Wheeden [MW, Theorem 1] proved that if $0<p<\infty$ and $0<\alpha<d$ then

$$\int|R_\alpha f(\xi)|^p w(\xi)\,d\xi\leqslant C\int M_\alpha f(\xi)^p w(\xi)\,d\xi, \tag{0.1}$$

where w is a weight in the Muckenhoupt class A_∞ and the constant C is independent of f. The purpose of this paper is to extend (0.1) to certain weighted Lebesgue spaces $L^{p,q}_{w_0,w_1}(\mathbf{R}^d)$ with mixed norm (see Definition 1.2). More precisely, we prove that

$$\|R_\alpha f\|_{p,q,w_0,w_1}\leqslant C\|M_\alpha f\|_{p,q,w_0,w_1}, \tag{0.2}$$

where $0<p,q<\infty$, $0<\alpha<d$ and w_0, w_1 are weights in the Muckenhoupt

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