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Quadratic functionals and Jordan $*$-derivations

by

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Abstract. Let X be a left A-module where A is a real Banach $*$-algebra with an identity element. In this note the general form of quadratic functionals defined on X is obtained; the notion of Jordan $*$-derivations arises here.

1. Introduction. In this note we shall consider only real Banach algebras. An algebra is called a *$*$-algebra* if there exists an involution, that is, a linear anti-isomorphism of period two. All algebras will have an identity element denoted by 1.

Let A be a Banach $*$-algebra.

DEFINITION 1.1. A function $D: A \to A$ is a *Jordan $*$-derivation* if for all $a, b \in A$

$$D(a+b) = D(a)+D(b), \tag{1}$$

$$D(a^2) = aD(a)+D(a)a^*. \tag{2}$$

Usually, a mapping $D: B \to B$, where B is an algebra, is defined to be a derivation if it is linear and satisfies $D(ab) = aD(b)+D(a)b$. In the present note we will not consider linear mappings. We will be interested in additive mappings which are not homogeneous in general. Therefore a *Jordan derivation* $J: B \to B$ is defined to be an additive function satisfying $J(a^2) = aJ(a)+J(a)a$. Over a commutative algebra with the trivial involution, $a^* = a$, the set of all Jordan $*$-derivations is equal to the set of all Jordan derivations as well as to the set of all (nonlinear) derivations [2].

A function D_x: $A \to A$, $x \in A$, defined by $D_x(a) = ax - xa^*$ will be called an *inner Jordan $*$-derivation*. It is easy to see that the vector space of all inner Jordan $*$-derivations on A is a subspace of the space $\mathscr{D}$ of all Jordan $*$-derivations. In two special cases we will find that the reverse inclusion holds as well. For a similar result concerning linear derivations we refer to [1].

In the present note we will prove some equivalent characterizations and useful properties of Jordan $*$-derivations.

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Let X be a left A-module. We always assume that $1x = x$ for all $x \in X$. A mapping $Q\colon X \to A$ will be called an *A-quadratic functional* if it satisfies the parallelogram law

$$Q(x+y)+Q(x-y) = 2Q(x)+2Q(y), \quad x, y \in X, \tag{3}$$

and the homogeneity

$$Q(ax) = aQ(x)a^*, \quad x \in X,\ a \in A. \tag{4}$$

In the special case $A = \mathbf{R}$, (4) becomes $Q(tx) = t^2 Q(x)$ for all real t and all vectors $x \in X$. S. Kurepa has proved the following theorem [5]:

THEOREM 1.2. *Let X be a real vector space and let Q be a real-valued quadratic functional defined on X. If $\{e_\alpha;\, \alpha \in J\}$ is an algebraic basic set in X (Hamel basis for X over the field of real numbers), with J well ordered by $<$, then*

$$Q\Big(\sum_{\alpha\in J} t_\alpha e_\alpha\Big) = \sum_{\alpha,\beta} b_{\alpha\beta} t_\alpha t_\beta + \sum_{\alpha<\beta} \big(D_{\alpha\beta}(t_\alpha)t_\beta - D_{\alpha\beta}(t_\beta)t_\alpha\big),$$

for all $\sum_{\alpha\in J} t_\alpha e_\alpha \in X$, where $b_{\alpha\beta}$ are real constants and $t \mapsto D_{\alpha\beta}(t)$ is a derivation.

Our main result is a generalization of this theorem. We will determine all quadratic functionals on an arbitrary left A-module X, where A is a real Banach $*$-algebra. The approach in this general setting, specialized to the case $A = \mathbf{R}$, gives a shorter proof than the original one.

Recall that $B\colon X \times X \to A$ is called an *A-sesquilinear functional* if

$$B(a_1x_1 + a_2x_2, y) = a_1B(x_1, y) + a_2B(x_2, y), \quad x_1, x_2, y \in X,\ a_1, a_2 \in A, \tag{5}$$

$$B(x, b_1y_1 + b_2y_2) = B(x, y_1)b_1^* + B(x, y_2)b_2^*, \quad x, y_1, y_2 \in X,\ b_1, b_2 \in A. \tag{6}$$

Each A-sesquilinear functional gives rise to an A-quadratic functional by $Q(x) = B(x, x)$. It is natural to ask whether for each A-quadratic functional Q there exists a sesquilinear functional B such that $Q(x) = B(x, x)$. This problem was treated by many authors [2], [3], [5]–[15]. Let us mention only two results. The answer is in the negative in the case $A = \mathbf{R}$ [5]. On the other hand, if A is a complex $*$-algebra then each A-quadratic functional can be represented by an A-sesquilinear functional [11]. This last result can be considered as a generalization of the Jordan–von Neumann characterization of inner product spaces [4]. Our main result implies that Jordan $*$-derivations arise as a "measure" of the representability of quadratic functionals by sesquilinear ones. More precisely, it will be proved that each A-quadratic functional can be represented by an A-sesquilinear functional if X is a free A-module and the function $F\colon A \to \mathscr{D}$ defined by $F(x) = D_x$ is onto. If F is also one-to-one, then the assumption that X is a free module can be omitted. We shall also see that the answer to our question is in the negative if there exists a Jordan $*$-derivation on A which is not inner.

2. Jordan $*$-derivations

THEOREM 2.1. *Let A be a real Banach $*$-algebra and $D\colon A \to A$ an additive function. Then the following assertions are equivalent:*

(i) *D is a Jordan $*$-derivation,*

(ii) *for all invertible $a \in A$*

$$D(a) = -aD(a^{-1})a^*, \tag{7}$$

(iii) *for all $a, b \in A$*

$$D(aba) = abD(a) + aD(b)a^* + D(a)b^*a^*. \tag{8}$$

Proof. (ii)$\Rightarrow$(i). (7) yields $D(1) = 0$. If a is invertible and $\|a\| < 1$, then $1+a$, $1-a$, $1-a^2$ are invertible as well, moreover $(a-1)^{-1} - (a^2-1)^{-1} = (a^2-1)^{-1}a$. We will show that for such an a we have

$$D(a^2) = aD(a) + D(a)a^*.$$

Indeed,

$$\begin{aligned} D(a) + a^{-1}D(a)a^{*-1} &= D(a) - D(a^{-1}) = D(a - a^{-1}) = D(a^{-1}(a^2-1)) \\ &= -a^{-1}(a^2-1)D\big((a^2-1)^{-1}a\big)(a^{*2}-1)a^{*-1} \\ &= -a^{-1}(a^2-1)D\big((a-1)^{-1}\big)(a^{*2}-1)a^{*-1} \\ &\quad + a^{-1}(a^2-1)D\big((a^2-1)^{-1}\big)(a^{*2}-1)a^{*-1} \\ &= a^{-1}(a+1)D(a-1)(a^*+1)a^{*-1} - a^{-1}D(a^2-1)a^{*-1} \\ &= (1+a^{-1})D(a)(1+a^{*-1}) - a^{-1}D(a^2)a^{*-1}. \end{aligned}$$

Hence $0 = a^{-1}D(a) + D(a)a^{*-1} - a^{-1}D(a^2)a^{*-1}$, which yields $D(a^2) = aD(a) + D(a)a^*$.

Consider now an invertible element a with norm greater than 1. Then for some positive integer n, $n^{-1}a$ is invertible with norm smaller than 1 and certainly $a = n(n^{-1}a)$. Since D is additive, (2) holds also in this case. Now, let a be arbitrary. Choosing an integer $n > \|a\|$, we have

$$D(a^2) - 2nD(a) = D\big((a-n)^2\big) = (a-n)D(a) + D(a)(a^*-n),$$

so that (2) is valid for this a as well.

(i)$\Rightarrow$(iii). In (2), replace a by $a+b$ to get

$$D(ab) + D(ba) = bD(a) + aD(b) + D(a)b^* + D(b)a^* \tag{9}$$

for all $a, b \in A$. Consider now $x = D\big(a(ab+ba) + (ab+ba)a\big)$. Using (9) we see that

$$\begin{aligned} x &= aD(ab+ba) + (ab+ba)D(a) + D(ab+ba)a^* + D(a)(b^*a^* + a^*b^*) \\ &= 2abD(a) + a^2D(b) + aD(a)b^* + 2aD(b)a^* + baD(a) \\ &\quad + bD(a)a^* + 2D(a)b^*a^* + D(b)a^{*2} + D(a)a^*b^*. \end{aligned}$$

On the other hand,

$$x = 2D(aba)+D(a^2b)+D(ba^2) = 2D(aba)+bD(a^2)+a^2D(b)+D(a^2)b^*+D(b)a^{*2}$$
$$= 2D(aba)+baD(a)+bD(a)a^*+a^2D(b)+aD(a)b^*+D(a)a^*b^*+D(b)a^{*2}.$$

Comparing the two expressions for x we arrive at (8).

(iii) $\Rightarrow$ (ii). Take $b = a^{-1}$ in (8).

Remark 2.2. In [15] it has been proved that an additive function f defined on a complex Banach $*$-algebra which satisfies $f(a) = -af(a^{-1})a^*$ for all invertible $a\in A$ is an inner Jordan $*$-derivation. This is a direct consequence of our theorem: just substitute a in (8) by i. So in this special case the function $F\colon A\to\mathscr{D}$ defined by $F(x) = D_x$ is onto. But it is also one-to-one. Indeed, if $x\neq y$, then $D_x(i) = 2ix \neq 2iy = D_y(i)$.

Let us complete our discussion of Jordan $*$-derivations with a lemma which will be needed in the sequel.

Lemma 2.3. *Let A be a real Banach $*$-algebra and $D\colon A\to A$ a Jordan $*$-derivation. Then for all a,b,c and invertible $d\in A$ we have*

(i) $dD(d^{-1}a)d^* = D(ad)-aD(d)-D(d)a^*$,

(ii) $D(cbca) = cD(ba)c^*+cbD(ca)-cbD(a)c^*+D(ca)b^*c^*-cD(a)b^*c^*$.

Proof. (i) Define $e = ad$, which yields $a = ed^{-1}$. Thus,

$$dD(d^{-1}a)d^* = dD(d^{-1}ed^{-1})d^* = eD(d^{-1})d^*+D(e)+dD(d^{-1})e^*$$
$$= adD(d^{-1})d^*+D(ad)+dD(d^{-1})d^*a^*$$

by (8). Using (7) completes the proof.

(ii) Consider first an invertible $a\in A$ and arbitrary $b,c\in A$. We have

$$D(cbca) = D\big(ca(a^{-1}b)ca\big) = cbD(ca)+caD(a^{-1}b)a^*c^*+D(ca)b^*c^*$$

by (8). Now (i) implies $aD(a^{-1}b)a^* = D(ba)-bD(a)-D(a)b^*$. Inserting this in the previous relation gives (ii) in this case.

Next, let a be an arbitrary element of A. For an integer $n>\|a\|$ the element $n-a$ is invertible so that (ii) is valid for $n-a$, c and b. Using (8) and the relation $D(n) = nD(1) = 0$ we get the desired result.

3. Quadratic functionals. In this section we will find an explicit expression for a quadratic functional Q defined on a left A-module X where A is a real Banach $*$-algebra. For this purpose let us solve a certain system of functional equations.

Theorem 3.1. *Let A be a real Banach $*$-algebra. Suppose that $f,g\colon A\to A$ satisfy*

$$(10)\qquad \begin{aligned} 2f(a)+2f(b) &= 4f\big(\tfrac{1}{2}(a+b)\big)+(a-b)g(0)(a^*-b^*),\\ 2g(a)+2g(b) &= 4g\big(\tfrac{1}{2}(a+b)\big)+(a-b)f(0)(a^*-b^*),\\ f(c) &= cg(c^{-1})c^* \end{aligned}$$

for all $a,b\in A$ and all invertible $c\in A$. Then there exist (and are uniquely determined) an element $x\in A$ and a Jordan $*$-derivation D such that $f(a) = ag(0)a^*+ax+xa^*+f(0)+D(a)$ for all $a\in A$.

Proof. Set $x = \frac{1}{2}\big(f(1)-f(0)-g(0)\big) = \frac{1}{2}\big(g(1)-f(0)-g(0)\big)$. Define $D,E\colon A\to A$ by

$$(11)\qquad \begin{aligned} f(a) &= ag(0)a^*+ax+xa^*+f(0)+D(a),\\ g(a) &= af(0)a^*+ax+xa^*+g(0)+E(a). \end{aligned}$$

The relation $f(c) = cg(c^{-1})c^*$ and (11) yield $D(a) = aE(a^{-1})a^*$ for all invertible $a\in A$. Putting $b=0$ in (10) we get $2f(a)+2f(0) = 4f(\frac{1}{2}a)+ag(0)a^*$. Together with (11) this implies $D(a) = 2D(\frac{1}{2}a)$. On the other hand, using the first equation of (11) in (10) we get $2D\big(\frac{1}{2}(a+b)\big) = D(a)+D(b)$. The last two relations yield the additivity of D.

Consider now an invertible element $a\in A$ such that $\|a\|<1$. Then $1+a$ is invertible and $(1+a)^{-1} = 1-(1+a)^{-1}a$. From the definition of x and (11) it follows that $D(1) = E(1) = 0$. Using additivity of D and the relation $D(a) = aE(a^{-1})a^*$ we get

$$D(a) = D(1+a) = (1+a)E\big((1+a)^{-1}\big)(1+a^*) = -(1+a)E\big((1+a)^{-1}a\big)(1+a^*)$$
$$= -aD\big(a^{-1}(1+a)\big)a^* = -aD(a^{-1}+1)a^* = -aD(a^{-1})a^*.$$

Since D is additive, $D(a) = -aD(a^{-1})a^*$ for all invertible $a<A$. The proof is now complete.

Suppose now that X is a left A-module and $Q\colon X\to A$ an A-quadratic functional. Let a pair of elements $\{x,y\}$ generate X. We define $f(a) = Q(x+ay)$ and $g(a) = Q(ax+y)$. For an invertible $a\in A$ we have $f(a) = Q\big(a(a^{-1}x+y)\big)$. Using the homogeneity of Q we obtain $f(a) = ag(a^{-1})a^*$. The parallelogram law implies

$$2f(a)+2f(b) = 2Q(x+ay)+2Q(x+by) = Q\big(2\big(x+\tfrac{1}{2}(a+b)y\big)\big)+Q\big((a-b)y\big)$$
$$= 4Q\big(x+\tfrac{1}{2}(a+b)y\big)+(a-b)Q(y)(a^*-b^*) = 4f\big(\tfrac{1}{2}(a+b)\big)+(a-b)g(0)(a^*-b^*).$$

Hence according to Theorem 3.1 we have $Q(x+ay) = aQ(y)a^*+ac+ca^*+Q(x)+D(a)$, where $c = \frac{1}{2}\big(Q(x+y)-Q(x)-Q(y)\big)$ and D is a Jordan $*$-derivation. For $a,b\in A$, where a is invertible, we get

$$Q(ax+by) = Q\big(a(x+a^{-1}by)\big) = aQ(x+a^{-1}by)a^*$$
$$= bQ(y)b^*+bca^*+acb^*+aQ(x)a^*+aD(a^{-1}b)a^*.$$

Using Lemma 2.3 (i) we find that

$$(12)\qquad Q(ax+by) = aQ(x)a^*+acb^*+bca^*+bQ(y)b^*+D(ba)-bD(a)-D(a)b^*.$$

Finally, let a and b be arbitrary elements of A. According to (3) we have for every integer n

$$2Q(ax+by)+2Q(nx) = Q\big((n+a)x+by\big)+Q\big((a-n)x+by\big).$$

If n is large enough then $n+a$ and $n-a$ are invertible, so that we get using (4) and (12)

$$Q(ax+by) = -n^2Q(x)+\tfrac{1}{2}\big((a+n)Q(x)(a^*+n)+(a+n)cb^*+bc(a^*+n)$$
$$+bQ(y)b^*+nD(b)+D(ba)-bD(a)-D(a)b^*+(a-n)Q(x)(a^*-n)$$
$$+(a-n)cb^*+bc(a^*-n)+bQ(y)b^*+D(ba)-nD(b)-bD(a)-D(a)b^*\big).$$

A straightforward computation shows that (12) holds in this general case as well.

On the other hand, if for $c,d,e\in A$ and a Jordan $*$-derivation D, the functional $Q\colon X\to A$ given by (12) is well defined, then Q is a quadratic functional. This is an easy consequence of Lemma 2.3(ii) and the fact that D is additive.

Generalizing this result we shall obtain an extension of Theorem 1.2.

THEOREM 3.2. *Let A be a real Banach $*$-algebra and Q an A-quadratic functional defined on a left A-module X. If a subset $\{x_\alpha;\ \alpha\in J\}\subset X$ generates X, where J is well ordered by $<$, then*

$$(13)\qquad Q\Big(\sum_{\alpha\in J} a_\alpha x_\alpha\Big) = \sum_{\alpha,\beta\in J} a_\alpha c_{\alpha\beta}a_\beta^* + \sum_{\alpha<\beta}\big(D_{\alpha\beta}(a_\beta a_\alpha)-a_\beta D_{\alpha\beta}(a_\alpha)-D_{\alpha\beta}(a_\alpha)a_\beta^*\big)$$

for all $\sum_{\alpha\in J}a_\alpha x_\alpha$, where $c_{\alpha\beta}\in A$ are constants, $c_{\alpha\beta}=c_{\beta\alpha}$, and $D_{\alpha\beta}$ are Jordan $*$-derivations.

Remarks 3.3. In the sum $\sum_{\alpha\in J}a_\alpha x_\alpha$ as well as in other sums of this paper only a finite number of terms may be nonzero. It should be mentioned that the expression $x=\sum_{\alpha\in J}a_\alpha x_\alpha$ is not unique in general and that also constants $c_{\alpha\beta}$ as well as Jordan $*$-derivations $D_{\alpha\beta}$ are not uniquely determined. One can easily prove that if $Q\colon X\to A$ given by (13) is well defined, then Q is a quadratic functional.

Proof. First consider the case that X is generated by a finite set of elements $\{x_1,\ldots,x_n\}$. Choose constants $c_{ij}\in A$, $i<j$, and Jordan $*$-derivations D_{ij}, $i<j$, such that

$$Q(ax_i+bx_j) = aQ(x_i)a^*+ac_{ij}b^*+bc_{ij}a^*+bQ(x_j)b^*$$
$$+D_{ij}(ba)-bD_{ij}(a)-D_{ij}(a)b^*$$

for all $a,b\in A$ and all integers $1\leqslant i<j\leqslant n$. Denote $Q(x_i)$ by c_{ii} and set $c_{ij}=c_{ji}$ for $i>j$. Suppose that for all subsets $K\subset\{1,\ldots,n\}$ of no more than k elements, $2\leqslant k<n$, the restriction $Q|_Y$ is of the form

$$(14)\qquad Q\Big(\sum_{i\in K} a_ix_i\Big) = \sum_{i,j\in K} a_ic_{ij}a_j^* + \sum_{i<j,\,i,j\in K}\big(D_{ij}(a_ja_i)-a_jD_{ij}(a_i)-D_{ij}(a_i)a_j^*\big),$$

where Y is the submodule generated by $\{x_i;\ i\in K\}$. We will show that the restriction $Q|_Z$ to a submodule Z generated by $k+1$ elements is of the same

type. Assume that Z is generated by $\{x_1,\ldots,x_k,x_{k+1}\}$. Define

$$b_1 = Q\Big(\sum_{i=1}^{k+1} a_ix_i\Big),\qquad b_2 = Q\Big(\sum_{i=1}^{k} a_ix_i-a_{k+1}x_{k+1}\Big),$$
$$b_3 = Q\Big(\sum_{i=1}^{k-1} a_ix_i-a_kx_k-a_{k+1}x_{k+1}\Big).$$

The parallelogram law (3) gives us

$$b_1+b_2 = 2Q\Big(\sum_{i=1}^{k} a_ix_i\Big)+2Q(a_{k+1}x_{k+1}),$$
$$b_2+b_3 = 2Q\Big(\sum_{i=1}^{k-1} a_ix_i-a_{k+1}x_{k+1}\Big)+2Q(a_kx_k),$$
$$b_1+b_3 = 2Q\Big(\sum_{i=1}^{k-1} a_ix_i\Big)+2Q(a_kx_k+a_{k+1}x_{k+1}).$$

Solving this system of equations we obtain

$$b_1 = Q\Big(\sum_{i=1}^{k} a_ix_i\Big)+Q(a_{k+1}x_{k+1})+Q(a_kx_k+a_{k+1}x_{k+1})$$
$$+Q\Big(\sum_{i=1}^{k-1} a_ix_i\Big)-Q\Big(\sum_{i=1}^{k-1} a_ix_i-a_{k+1}x_{k+1}\Big)-Q(a_kx_k).$$

Using (14) we complete the first part of our proof. The theorem is proved in full generality by a simple use of Zorn's lemma.

An A-module X is *free* if there exists a set $\mathscr{B}\subset X$ having the property that every $x\in X$ is uniquely expressible in the form $x=\sum_{i=1}^n a_ix_i$, where n is an integer and $a_i\in A$, $x_i\in\mathscr{B}$. Such a set $\mathscr{B}$ is called a *basic set*.

COROLLARY 3.4. *Let A be a real Banach $*$-algebra. All Jordan $*$-derivations defined on A are inner Jordan $*$-derivations if and only if for each free A-module X and each A-quadratic functional Q defined on X there is an A-sesquilinear functional $B\colon X\times X\to A$ such that $Q(x)=B(x,x)$ for all $x\in X$.*

Proof. Suppose first that for each Jordan $*$-derivation D on A there is $x\in A$ with $D(a)=ax-xa^*$. Let X be a free A-module and $Q\colon X\to A$ an A-quadratic functional. If $\{x_\alpha;\ \alpha\in J\}$ is a basic set in X well ordered by $<$, then there are constants $c_{\alpha\beta}=c_{\beta\alpha}\in A$, $\alpha,\beta\in J$, and Jordan $*$-derivations $D_{\alpha\beta}$, $\alpha<\beta$, such that Q is of the form (13). For each pair $\alpha,\beta\in J$, $\alpha<\beta$, we can find $d_{\alpha\beta}\in A$ such that $D_{\alpha\beta}(a)=ad_{\alpha\beta}-d_{\alpha\beta}a^*$. Set

$$e_{\alpha\beta} = \begin{cases} c_{\alpha\alpha}, & \alpha=\beta,\\ c_{\alpha\beta}-d_{\alpha\beta}, & \alpha<\beta,\\ c_{\alpha\beta}+d_{\beta\alpha}, & \alpha>\beta.\end{cases}$$

Then Q is of the form $Q(\sum_{\alpha\in J} a_\alpha x_\alpha) = \sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} a_\beta^*$. From the definition of a free module it follows that the functional $B\colon X\times X\to A$,

$$B(\sum_{\alpha\in J} a_\alpha x_\alpha, \sum_{\beta\in J} b_\beta x_\beta) = \sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} b_\beta^*,$$

is well defined. Moreover, it is A-sesquilinear and $Q(x) = B(x, x)$ for all $x\in X$.

Suppose now that there is a Jordan $*$-derivation D on A which is not inner. Define $Q\colon A\times A\to A$ by $Q((a,b)) = D(ba) - bD(a) - D(a)b^*$. If there is a sesquilinear functional B which generates Q, then B is of the form $B((a,b),(c,d)) = aed^* + bfc^*$ for some $e, f\in A$. The relation $Q((a,b)) = B((a,b),(a,b))$ with $b = 1$ gives us $D(a) = -ae - fa^*$. Since $D(1) = 0$, we have $e = -f$, so that D is an inner Jordan $*$-derivation. This contradiction completes the proof.

COROLLARY 3.5. *Let A be a real Banach $*$-algebra. Suppose that the mapping $F\colon A\to \mathscr{D}$, $F(x) = D_x$, is one-to-one and onto. Then for each A-quadratic functional Q defined on an arbitrary A-module X there is an A-sesquilinear functional $B\colon X\times X\to A$ such that $Q(x) = B(x, x)$ for all $x\in X$.*

Proof. Let X be generated by a subset $\{x_\alpha;\ \alpha\in J\}$. As before one can prove that Q is of the form $Q(\sum_{\alpha\in J} a_\alpha x_\alpha) = \sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} a_\beta^*$, where $e_{\alpha\beta}\in A$. The relation $\sum_{\alpha\in J} a_\alpha x_\alpha = 0$ yields for every $\sum_{\alpha\in J} b_\alpha x_\alpha \in X$

$$\sum_{\alpha,\beta\in J} b_\alpha e_{\alpha\beta} b_\beta^* + \sum_{\alpha,\beta\in J} b_\alpha e_{\alpha\beta} a_\beta^* + \sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} b_\beta^* + \sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} a_\beta^*$$
$$= Q(\sum_{\alpha\in J} (b_\alpha + a_\alpha) x_\alpha) = Q(\sum_{\alpha\in J} b_\alpha x_\alpha) = \sum_{\alpha,\beta\in J} b_\alpha e_{\alpha\beta} b_\beta^*.$$

Since $\sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} a_\beta^* = 0$, we get

$$\sum_{\alpha,\beta\in J} b_\alpha e_{\alpha\beta} a_\beta^* + \sum_{\alpha,\beta\in J} a_\alpha e_{\alpha\beta} b_\beta^* = 0. \tag{15}$$

Fix $\gamma\in J$ and choose $b_\gamma = 1$ and $b_\alpha = 0$ for $\alpha\neq\gamma$. We get $f + g = 0$, where

$$f = \sum_{\beta\in J} e_{\gamma\beta} a_\beta^*, \quad g = \sum_{\alpha\in J} a_\alpha e_{\alpha\gamma}.$$

On the other hand, if we set in (15) $b_\gamma = c$, $c\in A$, and $b_\alpha = 0$, $\alpha\neq\gamma$, we obtain $cf + gc^* = 0$. Together with $cf + cg = 0$ this implies $cg - gc^* = 0$ for all $c\in A$. Since the mapping F is one-to-one, this yields $g = f = 0$, or (with the same definition of B as in the previous proof)

$$B(\sum_{\alpha\in J} a_\alpha x_\alpha, x_\gamma) = 0 = B(x_\gamma, \sum_{\alpha\in J} a_\alpha x_\alpha)$$

for all $\gamma\in J$. Thus, B is well defined. This completes the proof.

EXAMPLES. Two special cases have been extensively studied. The first one is that A is a complex Banach $*$-algebra. In [6, 10–15] it has been proved that in this case each A-quadratic functional is generated by an A-sesquilinear functional. This is an immediate consequence of our results. According to Remark 2.2 the mapping $F\colon A\to\mathscr{D}$ is one-to-one and onto in this special case, so that the result follows directly from Corollary 3.5. The same result was obtained in the case that A is the field of quaternions [6, 10]. This shows that all Jordan $*$-derivations defined on the field of quaternions are inner.

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