for $i = 1, \ldots, n$. The decomposition of X implies $I = P_0 + P_1$ where P_0 and P_1 are the projections of X onto X_0 and $\bigoplus_{i=1}^{n} X_i$ respectively. We have clearly $P_0 = S_0 R_0$ and $P_1 = \sum_{i=1}^{n} R_i S_i$. As in the previous theorem one can now prove that $B(X)$ is algebraically generated by the subalgebras $\mathscr{A}(X_0)$ and $\operatorname{span}(R_0, R_1, \ldots, R_n)$, both of square zero.

COROLLARY 4. *If X is an "n-th power" ($n > 1$), then $B(X)$ is algebraically generated by two subalgebras with square zero, the dimension of one of them being* 1 *or* 2.

Proof. If n is even, we are done. Suppose now that n is odd. Then X can be decomposed into a direct sum of closed linear subspaces

$$X = (X_1 \oplus \ldots \oplus X_m) \oplus (X_{m+1} \oplus \ldots \oplus X_{2m}) \oplus X_{2m+1}$$

with the X_i all isomorphic to one another. Set

$$\hat{X}_0 = X_1 \oplus \ldots \oplus X_m, \quad \hat{X}_1 = X_{m+1} \oplus \ldots \oplus X_{2m}, \quad \hat{X}_2 = X_{2m+1}.$$

We can now find linear homeomorphisms V_0 of $\hat{X}_0$ onto $\hat{X}_1$, $V_1 = V_0^{-1}$ of $\hat{X}_1$ onto $\hat{X}_0$, and V_2 of $\hat{X}_2$ onto X_1. We define R_0, R_1 and R_2 as in Theorem 3 and notice that $R_0 = R_1$. Thus, $B(X)$ is algebraically generated by $\mathscr{A}(\hat{X}_0)$ and $\operatorname{span}(R_0, R_2)$. This completes the proof.

Let us conclude with an open problem which is a modification of a question posed by W. Żelazko: Does the fact that $B(X)$ is algebraically generated by two subalgebras with square zero imply that X is an "nth power" ($n > 1$)? In particular, we do not know whether there exists a Banach space X which is not an "nth power" and satisfies the assumptions of Theorem 3.

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On the positivity of the unit element in a normed lattice ordered algebra

by

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Abstract. An elementary proof of the following result is given: if $T\colon E \to E$ is a Cesàro bounded (or Abel bounded) linear operator on the normed Riesz space E and $T \geqslant I$, then $T = I$. In particular, if T is a contraction and $T \geqslant I$, then $T = I$. As a corollary we obtain that if A is a normed lattice ordered algebra with unit element e and $\|e\| \leqslant 1$, then $e \geqslant 0$.

Recently, E. Scheffold (private communication, unpublished) informed us of the following result: if A is a (real) Banach lattice algebra with multiplicative unit element e and $\|e\| \leqslant 1$ (so $\|e\| = 1$), then $e \geqslant 0$. His proof makes essential use of Kakutani's fixed point theorem to prove that if T is a linear operator on a Banach lattice E such that $T \geqslant I$ and $\|T\| \leqslant 1$ (whence $\|T\| = 1$), then $T = I$ (where I is the identity mapping on E). The result then follows by considering left or right multiplication by $|e|$.

Subsequently B. de Pagter showed us that Scheffold's result could be obtained by a semigroup approach under weaker hypotheses. We give the details of de Pagter's proof.

THEOREM 1. *Let E be a Banach lattice and $T\colon E \to E$ a linear operator on E such that*

(a) $T \geqslant I$,
(b) *T is power bounded* (i.e., $M = \sup_{m \geqslant 1} \|T^m\| < \infty$).

Then $T = I$.

Proof. Put $S = T - I$. Then $S \geqslant 0$, so $e^{tS} \geqslant 0$ for all $t \geqslant 0$. But

$$\|e^{tT}\| = \left\| \sum_{n=0}^{\infty} \frac{(tT)^n}{n!} \right\| \leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!} \|T^n\| \leqslant M e^t$$

for all $t \geqslant 0$ implies $\|e^{tS}\| \leqslant M$ for all $t \geqslant 0$. Observe now that

$$e^{tS} = I + tS + t^2 S^2/2! + \ldots \geqslant tS \geqslant 0$$

for all $t \geqslant 0$ and hence $0 \leqslant S \leqslant e^{tS}/t$ for all $t > 0$. Consequently, $\|S\| \leqslant M/t$ for all $t > 0$, showing that $S = 0$ and $T = I$.

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In the present note we give an elementary proof of Scheffold's result which even holds in noncomplete normed lattice ordered algebras.

THEOREM 2. *Let A be a normed lattice ordered algebra with unit element e. Then the following statements are equivalent.*

(i) $e \geqslant 0$.
(ii) $|e|$ *is power bounded (i.e.,* $K = \sup_{m\geqslant 1} \| \, |e|^m \| < \infty$).
(iii) e^+ *is power bounded (i.e.,* $M = \sup_{m\geqslant 1} \|(e^+)^m\| < \infty$).

In particular, if $\|e\| \leqslant 1$, *then* $e \geqslant 0$.

Proof. *Assume* (i). If $e \geqslant 0$, then $e = |e|$, so $K = \sup_{m\geqslant 1} \|e^m\| = \|e\| < \infty$, and (ii) follows.

Assume (ii). It follows from $(e^+)^m \leqslant |e|^m$ $(m = 1, 2, \ldots)$ that $M \leqslant K$, which shows (iii).

Assume (iii). Put $u = e^+$ and $v = e^-$. It follows from $u \geqslant e$ that $u^k \geqslant e$ $(k = 1, 2, \ldots)$ and hence

$$u^{k+1} - u^k = u^k(u-e) \geqslant u - e = v \geqslant 0$$

$(k = 1, 2, \ldots)$. Thus

$$u^{n+1} = \sum_{k=1}^{n} (u^{k+1} - u^k) + u \geqslant nv \geqslant 0$$

and consequently $0 \leqslant n\|v\| \leqslant \|u^{n+1}\| = \|(e^+)^{n+1}\| \leqslant M$ $(n = 1, 2, \ldots)$. Therefore, $v = 0$, in other words $e = u \geqslant 0$ and (i) is proved.

If $\|e\| \leqslant 1$, then

$$\|(e^+)^m\| \leqslant \| \, |e|^m \| \leqslant \| \, |e| \, \|^m = \|e\|^m \leqslant 1$$

$(m = 1, 2, \ldots)$, so $M \leqslant K \leqslant 1 < \infty$ and thus $e \geqslant 0$.

Remark 3. It is shown in the last part of the above proof that $\|e\| \leqslant 1$ implies that e^+ is power bounded. The converse of this observation does not hold. By way of example, take $A = \mathbf{R}^2$ with the pointwise vector space operations, partial ordering and multiplication, so

$$a \cdot b = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_2\beta_2 \end{pmatrix}$$

for all $a, b \in A$. Providing A with the $\|\cdot\|_1$-norm, A becomes a Banach lattice algebra with positive unit element $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It follows from

$$M = \sup_{m\geqslant 1} \|(e^+)^m\| = \sup_{m\geqslant 1} \|e^m\| = \|e\| < \infty$$

that e^+ is power bounded. However, $\|e\|_1 = 2 > 1$.

The argument in the proof of Theorem 2 modifies easily to show that if T is a power bounded linear operator (say, $M = \sup_{m\geqslant 1} \|T^m\| < \infty$) on a normed Riesz space E which satisfies $T \geqslant I$, then necessarily $T = I$. Particularly, if T is a contraction and $T \geqslant I$, then $T = I$.

Note in this connection that a positive power bounded operator T on a normed Riesz space E (or even on a Banach lattice E) need not be contractive. By way of example, take $E = (\mathbf{R}^2, \|\cdot\|_1)$ and $\mathrm{matrix}(T) = \begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}$. Then $\|T\|_1 = 3/2$, so T is not contractive. However,

$$\mathrm{matrix}(T^n) = \begin{pmatrix} 1 & 1+1/2+\ldots+1/2^{n-1} \\ 0 & 1/2^n \end{pmatrix} \leqslant \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

so T is power bounded.

THEOREM 4. *If* $T\colon E \to E$ *is a power bounded linear operator on the normed Riesz space* E *and* $T \geqslant I$, *then* $T = I$.

Proof. For all $x \in E^+$ we have $Tx - x \geqslant 0$, so $T^k \geqslant I$ yields $T^{k+1}x - T^k x \geqslant Tx - x \geqslant 0$ $(k = 1, 2, \ldots)$. It follows from

$$T^{n+1}x = \sum_{k=1}^{n} (T^{k+1}x - T^k x) + Tx \geqslant n(Tx - x) \geqslant 0$$

that $0 \leqslant n\|Tx - x\| \leqslant \|T^{n+1}x\| \leqslant M\|x\|$ $(n = 1, 2, \ldots)$. Hence, $Tx = x$ for all $x \in E^+$, so $T = I$.

Notice that Theorem 2 is a special case of Theorem 4. Indeed, let A be a normed lattice ordered algebra with unit element e such that e^+ is power bounded, say $M = \sup_{m\geqslant 1} \|(e^+)^m\| < \infty$. Denote by L_{e^+} the left multiplication by e^+, so $L_{e^+}(x) = e^+x$ for all $x \in A$ (the same argument works for the right multiplication R_{e^+} with e^+ and also for the left or right multiplication with $|e|$). Since $L_{e^+}^m$ is the left multiplication with $(e^+)^m$, it follows from

$$\|L_{e^+}^m x\| = \|(e^+)^m x\| \leqslant \|(e^+)^m\| \cdot \|x\| \leqslant M\|x\|$$

for all $x \in A$ $(m = 1, 2, \ldots)$ that L_{e^+} is power bounded. On the other hand, $L_{e^+}(x) = e^+x \geqslant ex = x$ for all $x \in A^+$ gives $L_{e^+} \geqslant I$ as well. By Theorem 4, $L_{e^+} = I$, so in particular $L_{e^+}(e) = e^+e = e^+ = e$, showing that $e \geqslant 0$.

Remark 5. In connection with Theorem 4 it is natural to ask whether it is also true that $T = I$, $r(T) \leqslant 1$ (so $r(T) = 1$) implies $T = I$.

This conjecture is violated by the next example. Take $E = (C([0, 1]), \|\cdot\|_\infty)$ and define $T\colon E \to E$ by

$$(Tf)(x) = f(x) + \int_0^x f(t)\,dt$$

for all $x \in [0, 1]$. Hence, $T = I + V$ with V the Volterra integral operator defined by

$$(Vf)(x) = \int_0^x f(t)\,dt$$

for all $x \in [0, 1]$. Then T is a linear operator and $V \geqslant 0$ yields $T \geqslant I$. It is well known that $r(V) = 0$, so $r(T) = 1$. Obviously, $T \neq I$.

Another example is provided by taking $E = \mathbf{R}^2$ and $\mathrm{matrix}(T) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

It should be observed that the power bounded case in Theorem 4 can easily be reduced to the contractive case, by defining an equivalent Riesz norm in E with respect to which T is contractive. Indeed, if $M = \sup_{n \geqslant 0} \|T^n\| < \infty$ and we define

$$|||x||| = \sup_{n \geqslant 0} \|T^n |x|\,\|$$

for all $x \in E$, then $|||\cdot|||$ is a Riesz norm in E, $\|x\| \leqslant |||x||| \leqslant M\|x\|$ for all $x \in E$ and $|||T||| \leqslant 1$.

In the next example we present a Banach lattice algebra A with unit element e for which e^+ is not power bounded and e is not positive.

EXAMPLE 6. Consider $A = \mathbf{R}^2$ with coordinatewise addition, scalar multiplication and partial ordering, so A is an Archimedean Riesz space. Define for $p, q, r \in \mathbf{R}$ the following multiplication in A:

$$a \cdot b = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} (\alpha_1 + p\alpha_2)(\beta_1 + p\beta_2) + q\alpha_2\beta_2 \\ r\alpha_2\beta_2 \end{pmatrix}$$

for all $a, b \in A$. This multiplication is automatically commutative and distributive. It is associative if and only if $q = -pr$. Hence, if we define

$$a \cdot b = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} (\alpha_1 + p\alpha_2)(\beta_1 + p\beta_2) - pr\alpha_2\beta_2 \\ r\alpha_2\beta_2 \end{pmatrix}$$

for all $a, b \in A$, then A is an algebra with respect to this multiplication. Positive elements have positive products if $p = 0$, $r \geqslant 0$ or if $p > 0$, $0 \leqslant r \leqslant p$, so in these cases A is an Archimedean lattice ordered algebra in which

$$a \cdot b \cdot c = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \cdot \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} (\alpha_1 + p\alpha_2)(\beta_1 + p\beta_2)(\gamma_1 + p\gamma_2) - pr^2\alpha_2\beta_2\gamma_2 \\ r^2\alpha_2\beta_2\gamma_2 \end{pmatrix}$$

for all $a, b, c \in A$. If $p = 0$, $r > 0$ or if $p > 0$, $0 < r \leqslant p$, then A is an Archimedean lattice ordered algebra with unit element e given by

$$e = \begin{pmatrix} 1 - p/r \\ 1/r \end{pmatrix}.$$

If $0 < r < p$, then e is not positive and

$$e^+ = \begin{pmatrix} 0 \\ 1/r \end{pmatrix}.$$

An easy induction argument shows that

$$(e^+)^n = \begin{pmatrix} p^n/r^n - p/r \\ 1/r \end{pmatrix}$$

$(n = 1, 2, \ldots)$.

If $0 < r < p \leqslant 1$, then A is a Banach lattice algebra with respect to the $\|\cdot\|_1$-norm (observe that $0 < p^2 - pr + r \leqslant 1$ in this case). However, $\|e\|_1 = p/r - 1 + 1/r > 1$ and e^+ is not power bounded.

Added in proof (June 1990). A. R. Schep has pointed out to us that our methods can be combined with an operator transformation used in ergodic theory to obtain a stronger result. Specifically, let us recall that a linear operator T on the Banach lattice E (actually on any normed linear space) is *Cesàro*, or *mean*, *bounded* if the sequence $(\|\sum_{k=0}^{n} T^k\|/(n+1))$ is bounded. Clearly a power bounded operator is Cesàro bounded. In [2, Section 2] Y. Derrienic and M. Lin present an example of a positive Cesàro bounded linear operator on a Banach lattice that is not power bounded. Note in this connection that such an example cannot be given in the finite-dimensional setting, as every positive Cesàro bounded matrix is power bounded.

Schep's observation was that our Theorem 4 remains valid for the wider class of Cesàro bounded operators. He showed this by using the formal transformation of a positive linear operator T on a Banach lattice E to the so-called barycenter $A(T)$ of T defined by $A(T) = \sum_{n=0}^{\infty} \alpha_n T^n$, where the α_n are the coefficients of z^n in the power series expansion of $(1 - \sqrt{1-z})/z$. It was shown by A. Brunel and R. Émilion in [1, Theorem 2.1] that $A(T)$ is power bounded whenever T is Cesàro bounded. It is then easy to verify that $T \geqslant I$ implies $A(T) \geqslant I$. Hence, by Theorem 4, $A(T) = I$; from which $T = I$.

It turns out that his result can be obtained very simply without the use of $A(T)$, and can be generalized a good deal more. Suppose we have a summability method (P) for series (à la G. H. Hardy [4]). We will say that the operator T on the Banach lattice E is *P-bounded* if the P-method applied to the series with nth term $T^n - T^{n-1}$ produces sequences of partial sums which are uniformly norm bounded.

EXAMPLE 7. (a) If P is the $(C, 1)$-method, i.e., Cesàro summability, the series $\sum(T^n - T^{n-1})$ has partial sums T^n (if we interpret $T^{-1} = 0$). Thus T is P-bounded means precisely that T is Cesàro bounded.

(b) If P is the A-method, i.e., Abel summability, we go formally from $\sum(T^n - T^{n-1})$ to $\sum \Theta^n (T^n - T^{n-1}) = (1 - \Theta)\sum \Theta^n T^n$ and T is Abel bounded if the series $(1 - \Theta)\sum \Theta^n T^n$ have uniformly norm bounded partial sums for $0 < \Theta < 1$.

Following Hardy we say that a summability method P is *totally regular* if it sums all convergent series to convergent series and all divergent series of positive terms to $+\infty$. Theorem 4 above now generalizes to the following.

THEOREM 8. *If P is a totally regular summability method, $T\colon E\to E$ is a P-bounded linear operator on the normed Riesz space E, and $T\geqslant I$, then $T=I$.*

Proof (outline). Let $x\in E^+$. The inequality $0\leqslant Tx-x\leqslant T^{k+1}x-T^kx$, obtained in the proof of Theorem 4, leads to the inequality

$$P(1)_n(Tx-x)\leqslant P(T^{k+1}x-T^kx)_n$$

for each n, when we contemplate the sequences of partial sums obtained by applying the summability method P to the series with kth terms 1, and $T^{k+1}x-T^kx$ respectively. Taking norms we have

$$P(1)_n\|Tx-x\|\leqslant\|P(T^{k+1}-T^k)_n\|\cdot\|x\|.$$

The right hand side is bounded and $P(1)_n\to\infty$, so $Tx=x$ and we are done.

In the Cesàro bounded case the argument can be written

$$\frac{n}{2}\|Tx-x\|=\frac{1}{n+1}\Big\|\sum_{k=1}^{n}k(Tx-x)\Big\|\leqslant\frac{1}{n+1}\Big\|\sum_{k=1}^{n}T^{k+1}x\Big\|$$
$$\leqslant\frac{1}{n+1}\Big\|\sum_{k=1}^{n}T^{k+1}\Big\|\cdot\|x\|\leqslant M\|x\|,$$

where M is the bound provided by the Cesàro boundedness assumption.

In the Abel bounded case we assume there exists M such that

$$(1-\Theta)\Big\|\sum_{k=1}^{n}\Theta^kT^k\Big\|\leqslant M$$

for all n and all Θ such that $0<\Theta<1$. From $0\leqslant k(Tx-x)\leqslant T^{k+1}x$ for all k, we obtain for $0<\Theta<1$, and all positive integers n,

$$\Big(\sum_{k=1}^{n}\Theta^kk\Big)\|Tx-x\|\leqslant\Big\|\sum_{k=1}^{n}\Theta^kT^{k+1}\Big\|\cdot\|x\|\leqslant\frac{M}{\Theta(1-\Theta)}\|x\|.$$

Letting $n\to\infty$ we obtain

$$\Big(\sum_{k=1}^{\infty}\Theta^kk\Big)\|Tx-x\|=\frac{\Theta}{(1-\Theta)^2}\|Tx-x\|\leqslant\frac{M}{\Theta(1-\Theta)}\|x\|,$$

whence $\Theta^2\|Tx-x\|\leqslant M(1-\Theta)\|x\|$ for $0<\Theta<1$. Letting Θ tend to 1 from below gives $Tx=x$.

W. A. J. Luxemburg pointed out to us that a positive linear operator T on a normed Riesz space E is Cesàro bounded if and only if T is Abel bounded. This was shown by R. Émilion in [3, 1.5 and 1.7] in case E is a Banach lattice.

For the unexplained notions of Riesz space, normed Riesz space, Banach lattice and Banach lattice algebra we refer the reader to [5].

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