On algebraic generation of $B(X)$
by two subalgebras with square zero

by

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Abstract. We prove several results on algebraic generation of $B(X)$ by two subalgebras with square zero, one of them being finite-dimensional. These results are motivated by a counterexample to one problem posed by W. Żelazko.

Let $X$ be a real or complex Banach space with $\dim X > 1$. We say that the algebra $B(X)$ of all its continuous endomorphisms is $\tau$-generated by its subset $\mathcal{S}$ if it coincides with the smallest $\tau$-closed subalgebra of $B(X)$ containing $\mathcal{S}$. Here $\tau$ denotes some topology on $B(X)$. When $\tau$ is the discrete topology we say that $\mathcal{S}$ algebraically generates $B(X)$. In other words, $\mathcal{S}$ algebraically generates $B(X)$ if each operator $T$ in $B(X)$ is a linear combination of finite products of elements of $\mathcal{S}$. In [3] W. Żelazko raised the question whether $B(X)$ is generated by two of its abelian subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e. whether it coincides with the smallest $\tau$-closed subalgebra containing $\mathcal{A}_1$ and $\mathcal{A}_2$. In the case when $X$ is a separable Hilbert space it was known earlier that $B(X)$ is strongly generated by two operators and hence by two commutative subalgebras. In [1] it is shown that $B(H)$ is strongly generated by two unitary operators, and in [2] that it is strongly generated by two hermitian operators. For an arbitrary subset $\mathcal{S}$ of $B(X)$ we put $\mathcal{S}^2 = \{ T_1 T_2 : T_1, T_2 \in \mathcal{S} \}$; thus a subalgebra $\mathcal{A} \subseteq B(X)$ of square zero is automatically commutative. It is proved in [4] that for any Banach space $X$ with $\dim X > 1$ the algebra $B(X)$ is strongly generated by two subalgebras with square zero.

The situation is completely different if instead of generation in the strong operator topology we consider algebraic generation: there exist Banach spaces $X$ for which $B(X)$ cannot be algebraically generated by any number of subalgebras of square zero. On the other hand, many Banach spaces are “$n$th powers” and for such spaces the algebra $B(X)$ is algebraically generated by two subalgebras of square zero. More precisely, we have the following result.

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Theorem 1. If $X$ can be decomposed into a direct sum of closed linear subspaces 

$$X = X_0 \oplus X_1 \oplus \ldots \oplus X_n, \quad n \geq 1,$$

with the $X_i$ all isomorphic to one another, then the algebra $B(X)$ is algebraically generated by two subalgebras with square zero, one of them being $n$-dimensional. For $n = 1$ the converse is also true.

For the proof of this theorem as well as for other results on algebraic generation of $B(X)$ by two subalgebras with square zero we refer to [3]. In the same paper the problem of the converse to Theorem 1 for $n > 1$ was posed. In order to answer this question we shall first prove a somewhat surprising result which states that if $B(X)$ is algebraically generated by two subalgebras of square zero, then we can also find two such subalgebras with one of them being finite-dimensional. Next, we will generalize Theorem 1. Recall that a closed subspace $Y$ of a Banach space $X$ is called complemented if there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$. It will be shown that $B(X)$ is algebraically generated by two subalgebras of square zero if there is a direct sum decomposition $X = X_0 \oplus X_1 \oplus \ldots \oplus X_n$ into closed linear subspaces such that there exist complemented subspaces $Y_i \subset X_i$ isomorphic to $X_i$, $i = 1, \ldots, n$, and a complemented subspace $Y_0 \subset X_0 \oplus \ldots \oplus X_n$ isomorphic to $X_0$. As a consequence, for an “$n$th power” $X$ ($n > 1$) the algebra $B(X)$ is algebraically generated by two subalgebras with square zero, the dimension of one of them being 1 or 2. It follows that the algebras $B(C^{\omega(d+1)})$ are counterexamples to the following question posed by W. Żelazko in [3]: if $B(X)$ is algebraically generated by two subalgebras with square zero, one of them being $n$-dimensional, is it then true that $X = X_0 \oplus \ldots \oplus X_n$, with the $X_i$ isomorphic to one another?

We shall need the following notations. For a nonvoid subset $\mathcal{S} \subset B(X)$ put

$$\ker \mathcal{S} = \bigcap \{\ker T: T \in \mathcal{S}\}, \quad \im \mathcal{S} = \operatorname{span}(\bigcup \{\im T: T \in \mathcal{S}\}).$$

Thus $\ker \mathcal{S}$ is a closed linear subspace of $X$ and $\im \mathcal{S}$ is a linear, but not necessarily closed subspace of $X$. For a closed linear subspace $Y \subset X$ we put

$$\mathcal{A}(Y) = \{T \in B(X): \im T \subset Y \subset \ker T\}.$$

This is clearly a closed subalgebra of $B(X)$ with square zero.

Theorem 2. Suppose that for a real or complex Banach space $X$ the algebra $B(X)$ is algebraically generated by subalgebras $\mathcal{A}_0$ and $\mathcal{A}_1$, both of square zero. Then there exist two subalgebras $\mathcal{A}_0, \mathcal{A}_1 \subset B(X)$ of square zero, one of them being finite-dimensional, such that $B(X)$ is algebraically generated by $\mathcal{A}_0 \cup \mathcal{A}_1$.

Proof. It was proved by W. Żelazko [3] that under the assumption of our theorem there is a direct sum decomposition $X = X_0 \oplus X_1$ into closed linear subspaces, where $X_0 = \ker \mathcal{A}_0$ and $X_1 = \ker \mathcal{A}_1$. Consequently, $\mathcal{A}_0 \subset \mathcal{A}(X_0)$ and $\mathcal{A}_1 \subset \mathcal{A}(X_1)$, it follows that every operator in $B(X)$ is a linear combination of finite products of elements of $\mathcal{A}(X_0) \cup \mathcal{A}(X_1)$. Every such product is of one of the following forms:

$$S, \quad T \in \mathcal{A}(X_0), \quad S_i, \quad T_i \in \mathcal{A}(X_1), \quad T_i S_i, \quad T_i S_j, \quad T_i T_j.$$ 

This follows immediately from the obvious fact that the relations $P, Q \in \mathcal{A}(X_0)$ and $Q \in \mathcal{A}(X_1)$ imply $PQ \in \mathcal{A}(X_0)$ for $i = 0, 1$. The decomposition $X = X_0 \oplus X_1$ implies $I = P_0 + P_1$, where $I$ is the identity operator and $P_i$ is the corresponding projection of $X$ onto $X_i$, $i = 0, 1$. Now, since $B(X)$ is algebraically generated by $\mathcal{A}(X_0)$ and $\mathcal{A}(X_1)$ we can write

$$P_0 = \sum_{i=1}^n S_i T_i, \quad P_1 = \sum_{i=n+1}^m T_i S_i,$$

where $S_i \in \mathcal{A}(X_0), T_i \in \mathcal{A}(X_1), i = 1, \ldots, n$. Let $T$ be an arbitrary bounded operator on $X$. Using the obvious relation $T = (P_0 + P_1) T (P_0 + P_1)$ we have

$$T = \sum_{i=1}^n \sum_{j=1}^n S_i T_i T_j S_j + \sum_{i=n+1}^m \sum_{j=1}^m T_i S_i T_j S_j + \sum_{i=1}^n \sum_{j=n+1}^m S_i T_i T_j S_j.$$

Since $S_i, T_j \in \mathcal{A}(X)$ for all $R \in B(X)$ we see that $B(X)$ is algebraically generated by $\mathcal{A}_0 = \mathcal{A}(X_0)$ and $\mathcal{A}_1 = \operatorname{span}(T_1, \ldots, T_m)$.

We shall now extend Theorem 1.

Theorem 3. Let $X$ be a real or complex Banach space and suppose that there is a direct sum decomposition $X = X_0 \oplus X_1 \oplus \ldots \oplus X_n$ into closed linear subspaces such that (i) there exists a complemented subspace $Y_0 \subset X_1 \oplus \ldots \oplus X_n$ isomorphic to $X_n$, and (ii) there exist complemented subspaces $Y_i \subset X_i$ isomorphic to $X_i, i = 1, \ldots, n$. Then $B(X)$ is algebraically generated by two subalgebras of square zero.

Proof. We denote by $V_i$ a linear homeomorphism of $X_i$ onto $Y_i, i = 0, 1, \ldots, n$. Choose closed complements $Z_i$ of $Y_i$ in $X_i, i = 0, 1, \ldots, n$, satisfying

$$X_0 = Z_0 \quad \text{and} \quad X_i \oplus \ldots \oplus X_n \subset Z_i, \quad i = 1, \ldots, n.$$

We put

$$R_0 x = \begin{cases} V_0 x & \text{for } x \in X_0, \\ 0 & \text{for } x \in X_1 \oplus \ldots \oplus X_n, \end{cases} \quad S_0 x = \begin{cases} V_0^{-1} x & \text{for } x \in Y_0, \\ 0 & \text{for } x \in Z_0, \end{cases}$$

$$R_i x = \begin{cases} V_i^{-1} x & \text{for } x \in Y_i, \\ 0 & \text{for } x \in Z_i, \end{cases} \quad S_i x = \begin{cases} V_i x & \text{for } x \in X_i, \\ 0 & \text{for } x \in \oplus \mathcal{A}(X_i), \end{cases}$$

and $\mathcal{A}_0 \subset \mathcal{A}(X_0)$, it follows that every operator in $B(X)$ is a linear combination of finite products of elements of $\mathcal{A}(X_0) \cup \mathcal{A}(X_1)$. Every such product is of one of the following forms:

$$S, \quad T \in \mathcal{A}(X_0), \quad S_i, \quad T_i \in \mathcal{A}(X_1), \quad T_i S_i, \quad T_i S_j, \quad T_i T_j.$$ 

This follows immediately from the obvious fact that the relations $P, Q \in \mathcal{A}(X_0)$ and $Q \in \mathcal{A}(X_1)$ imply $PQ \in \mathcal{A}(X_0)$ for $i = 0, 1$. The decomposition $X = X_0 \oplus X_1$ implies $I = P_0 + P_1$, where $I$ is the identity operator and $P_i$ is the corresponding projection of $X$ onto $X_i, i = 0, 1$. Now, since $B(X)$ is algebraically generated by $\mathcal{A}(X_0)$ and $\mathcal{A}(X_1)$ we can write

$$P_0 = \sum_{i=1}^n S_i T_i, \quad P_1 = \sum_{i=n+1}^m T_i S_i,$$

where $S_i \in \mathcal{A}(X_0), T_i \in \mathcal{A}(X_1), i = 1, \ldots, n$. Let $T$ be an arbitrary bounded operator on $X$. Using the obvious relation $T = (P_0 + P_1) T (P_0 + P_1)$ we have

$$T = \sum_{i=1}^n \sum_{j=1}^n S_i T_i T_j S_j + \sum_{i=n+1}^m \sum_{j=1}^m T_i S_i T_j S_j + \sum_{i=1}^n \sum_{j=n+1}^m S_i T_i T_j S_j.$$
for \( i = 1, \ldots, n \). The decomposition of \( X \) implies \( I = P_0 + P_1 \) where \( P_0 \) and \( P_1 \) are the projections of \( X \) onto \( X_0 \) and \( \bigoplus_{i=1}^n X_i \) respectively. We have clearly \( P_0 = S_0 R_0 \) and \( P_1 = \sum_{i=1}^n R_i S_i \). As in the previous theorem one can now prove that \( B(X) \) is algebraically generated by the subalgebras \( \mathcal{A}(X_0) \) and \( \text{span}(R_0, R_1, \ldots, R_n) \) both of square zero.

**Corollary 4.** If \( X \) is an \("n-th power" \( (n > 1) \), then \( B(X) \) is algebraically generated by two subalgebras with square zero, the dimension of one of them being 1 or 2.

**Proof.** If \( n \) is even, we are done. Suppose now that \( n \) is odd. Then \( X \) can be decomposed into a direct sum of closed linear subspaces

\[
X = (X_1 \oplus \ldots \oplus X_n) \oplus (X_{m+1} \oplus \ldots \oplus X_{2m}) \oplus X_{2m+1}
\]

with the \( X_i \) all isomorphic to one another. Set

\[
\tilde{X}_0 = X_1 \oplus \ldots \oplus X_n, \quad \tilde{X}_1 = X_{m+1} \oplus \ldots \oplus X_{2m}, \quad \tilde{X}_2 = X_{2m+1}.
\]

We can now find linear homeomorphisms \( V_0 \) of \( \tilde{X}_0 \) onto \( X_1 \), \( V_1 = V_0^{-1} \) of \( \tilde{X}_1 \) onto \( \tilde{X}_0 \), and \( V_2 \) of \( \tilde{X}_2 \) onto \( X_1 \). We define \( R_0, R_1, \) and \( R_2 \) as in Theorem 3 and notice that \( R_0 = R_2 \). Thus, \( B(X) \) is algebraically generated by \( \mathcal{A}(\tilde{X}_0) \) and \( \text{span}(R_0, R_1, R_2) \). This completes the proof.

Let us conclude with an open problem which is a modification of a question posed by W. Żelazko: Does the fact that \( B(X) \) is algebraically generated by two subalgebras with square zero imply that \( X \) is an \("n-th power" \( (n > 1) \)? In particular, we do not know whether there exists a Banach space \( X \) which is not an \("n-th power" \) and satisfies the condition of Theorem 3.

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**References**


**On the positivity of the unit element in a normed lattice ordered algebra**

by

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**Abstract.** An elementary proof of the following result is given: if \( T \colon E \to E \) is a Ceder bounded (or Abel bounded) linear operator on the normed Riesz space \( E \) and \( T \geq I \), then \( T = I \). In particular, if \( T \) is a contraction and \( T \geq I \), then \( T = I \). As a corollary we obtain that if \( A \) is a normed lattice ordered algebra with unit element \( e \) and \( \|e\| \leq 1 \), then \( e = 0 \).

Recently, E. Scheffold (private communication, unpublished) informed us of the following result: if \( A \) is a (real) Banach lattice algebra with multiplicative unit element \( e \) and \( \|e\| \leq 1 \) (so \( \|e\| = 1 \)), then \( e = 0 \). His proof makes essential use of Kakutani's fixed point theorem to prove that if \( T \) is a linear operator on a Banach lattice \( E \) such that \( T \geq I \) and \( \|T\| < 1 \) (whence \( \|T\| = 1 \), then \( T = I \) (where \( I \) is the identity mapping on \( E \)). The result then follows by considering left or right multiplication by \( |e| \).

Subsequently B. de Pagter showed us that Scheffold's result could be obtained by a semigroup approach under weaker hypotheses. We give the details of de Pagter's proof.

**Theorem 1.** Let \( E \) be a Banach lattice space and \( T : E \to E \) a linear operator on \( E \) such that

\[(a) \quad T \geq I,
\]

\[(b) \quad T \text{ is power bounded} \quad \text{(i.e.,} \quad M = \sup_{n \geq 1} \|T^n\| < \infty).\]

Then \( T = I \).

**Proof.** Put \( S = T - I \). Then \( S \geq 0 \), so \( e^{\|S\|} \geq 0 \) for all \( t \geq 0 \). But

\[
\|e^t\| = \left\| \sum_{n=0}^{\infty} \frac{(tT)^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \|T^n\| \leq Me^t
\]

for all \( t \geq 0 \) implies \( \|e^S\| \leq M \) for all \( t \geq 0 \). Observe now that

\[
e^S = I + tS + t^2S^2/2! + \ldots = tS \geq 0
\]

for all \( t \geq 0 \) and hence \( 0 \leq S \leq e^{\|S\|}t \) for all \( t > 0 \). Consequently, \( \|S\| \leq M/t \) for all \( t > 0 \), showing that \( S = 0 \) and \( T = I \).

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