Remarques. Soient K un corps commutatif quelconque et E un espace vectoriel sur K, de dimension finie supérieure ou égale à 3. Si A et $B \in L(E)$ avec $A^2 = B^2 = 0$, le raisonnement ci-dessus prouve que si $\operatorname{Lat} A \cap \operatorname{Lat} B = \{0, E\}$, alors $A+B$ est inversible. Mais is K est algébriquement clos, alors $\operatorname{Lat} A \cap \operatorname{Lat} B$ n'est jamais trivial. En effet, si $\operatorname{Lat} A \cap \operatorname{Lat} B = \{0, E\}$, alors $\operatorname{Hlat}(AB+BA) = \{0, E\}$ (treillis des sous-espaces hyperinvariants), d'où $AB+BA = \beta I$ pour un $\beta \in K$, et dans ces conditions pour tout $x \in \operatorname{Ker} A$, $\operatorname{Vect}(x, Bx) \in \operatorname{Lat} A \cap \operatorname{Lat} B$, ce qui contredit l'hypothèse.

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Received November 23, 1989 (2625)
Revised version February 16, 1990

STUDIA MATHEMATICA 97 (2) (1990)

On algebraic generation of $B(X)$ by two subalgebras with square zero*

by

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Abstract. We prove several results on algebraic generation of $B(X)$ by two subalgebras with square zero, one of them being finite-dimensional. These results are motivated by a counterexample to one problem posed by W. Żelazko.

Let X be a real or complex Banach space with $\dim X > 1$. We say that the algebra $B(X)$ of all its continuous endomorphisms is *τ-generated* by its subset $\mathscr{S}$ if it coincides with the smallest τ-closed subalgebra of $B(X)$ containing $\mathscr{S}$. Here τ denotes some topology on $B(X)$. When τ is the discrete topology we say that $\mathscr{S}$ *algebraically generates* $B(X)$. In other words, $\mathscr{S}$ algebraically generates $B(X)$ if each operator T in $B(X)$ is a linear combination of finite products of elements of $\mathscr{S}$. In [3] W. Żelazko raised the question whether $B(X)$ is generated by two of its abelian subalgebras $\mathscr{A}_1$ and $\mathscr{A}_2$, i.e. whether it coincides with the smallest τ-closed subalgebra containing $\mathscr{A}_1$ and $\mathscr{A}_2$. In the case when X is a separable Hilbert space it was known earlier that $B(X)$ is strongly generated by two operators and hence by two commutative subalgebras. In [1] it is shown that $B(H)$ is strongly generated by two unitary operators, and in [2] that it is strongly generated by two hermitian operators. For an arbitrary subset $\mathscr{S}$ of $B(X)$ we put $\mathscr{S}^2 = \{T_1 T_2 \colon T_1, T_2 \in \mathscr{S}\}$; thus a subalgebra $\mathscr{A} \subset B(X)$ of square zero is automatically commutative. It is proved in [4] that for any Banach space X with $\dim X > 1$ the algebra $B(X)$ is strongly generated by two subalgebras with square zero.

The situation is completely different if instead of generation in the strong operator topology we consider algebraic generation: there exist Banach spaces X for which $B(X)$ cannot be algebraically generated by any number of subalgebras of square zero. On the other hand, many Banach spaces are "nth powers" and for such spaces the algebra $B(X)$ is algebraically generated by two subalgebras of square zero. More precisely, we have the following result.

1985 *Mathematics Subject Classification*: Primary 47A05.
* This work was supported by the Research Council of Slovenia.

THEOREM 1. *If X can be decomposed into a direct sum of closed linear subspaces*

$$X = X_0 \oplus X_1 \oplus \ldots \oplus X_n, \quad n \geqslant 1,$$

with the X_i all isomorphic to one another, then the algebra $B(X)$ is algebraically generated by two subalgebras with square zero, one of them being n-dimensional. For $n = 1$ the converse is also true.

For the proof of this theorem as well as for other results on algebraic generation of $B(X)$ by two subalgebras with square zero we refer to [3]. In the same paper the problem of the converse to Theorem 1 for $n > 1$ was posed. In order to answer this question we shall first prove a somewhat surprising result which states that if $B(X)$ is algebraically generated by two subalgebras of square zero, then we can also find two such subalgebras with one of them being finite-dimensional. Next, we will generalize Theorem 1. Recall that a closed subspace Y of a Banach space X is called *complemented* if there exists a closed subspace $Z \subset X$ such that $X = Y \oplus Z$. It will be shown that $B(X)$ is algebraically generated by two subalgebras of square zero if there is a direct sum decomposition $X = X_0 \oplus X_1 \oplus \ldots \oplus X_n$ into closed linear subspaces such that there exist complemented subspaces $Y_i \subset X_0$ isomorphic to X_i, $i = 1, \ldots, n$, and a complemented subspace $Y_0 \subset X_1 \oplus \ldots \oplus X_n$ isomorphic to X_0. As a consequence, for an "nth power" X ($n > 1$) the algebra $B(X)$ is algebraically generated by two subalgebras with square zero, the dimension of one of them being 1 or 2. It follows that the algebras $B(C^{6k \pm 1})$ are counterexamples to the following question posed by W. Żelazko in [3]: if $B(X)$ is algebraically generated by two subalgebras with square zero, one of them being n-dimensional, is it then true that $X = X_0 \oplus \ldots \oplus X_n$, with the X_i isomorphic to one another?

We shall need the following notations. For a nonvoid subset $\mathscr{S} \subset B(X)$ put

$$\operatorname{Ker} \mathscr{S} = \bigcap \{\operatorname{Ker} T\colon\ T \in \mathscr{S}\}, \qquad \operatorname{Im} \mathscr{S} = \operatorname{span}\left(\bigcup \{\operatorname{Im} T\colon\ T \in \mathscr{S}\}\right).$$

Thus $\operatorname{Ker} \mathscr{S}$ is a closed linear subspace of X and $\operatorname{Im} \mathscr{S}$ is a linear, but not necessarily closed subspace of X. For a closed linear subspace $Y \subset X$ we put

$$\mathscr{A}(Y) = \{T \in B(X)\colon \operatorname{Im} T \subset Y \subset \operatorname{Ker} T\}.$$

This is clearly a closed subalgebra of $B(X)$ with square zero.

THEOREM 2. *Suppose that for a real or complex Banach space X the algebra $B(X)$ is algebraically generated by subalgebras $\mathscr{A}_0$ and $\mathscr{A}_1$, both of square zero. Then there exist two subalgebras $\mathscr{B}_0, \mathscr{B}_1 \subset B(X)$ of square zero, one of them being finite-dimensional, such that $B(X)$ is algebraically generated by $\mathscr{B}_0 \cup \mathscr{B}_1$.*

Proof. It was proved by W. Żelazko [3] that under the assumption of our theorem there is a direct sum decomposition $X = X_0 \oplus X_1$ into closed linear subspaces, where $X_0 = \operatorname{Ker} \mathscr{A}_0$ and $X_1 = \operatorname{Ker} \mathscr{A}_1$. Consequently, $\mathscr{A}_0 \subset \mathscr{A}(X_0)$ and $\mathscr{A}_1 \subset \mathscr{A}(X_1)$. It follows that every operator in $B(X)$ is a linear combination of finite products of elements of $\mathscr{A}(X_0) \cup \mathscr{A}(X_1)$. Every such product is of one of the following forms:

S, $S \in \mathscr{A}(X_0)$,
T, $T \in \mathscr{A}(X_1)$,
ST, $S \in \mathscr{A}(X_0)$, $T \in \mathscr{A}(X_1)$,
TS, $S \in \mathscr{A}(X_0)$, $T \in \mathscr{A}(X_1)$.

This follows immediately from the obvious fact that the relations $P, R \in \mathscr{A}(X_i)$ and $Q \in B(X)$ imply $PQR \in \mathscr{A}(X_i)$ for $i = 0,1$. The decomposition $X = X_0 \oplus X_1$ implies $I = P_0 + P_1$ where I is the identity operator and P_i is the corresponding projection of X onto X_i, $i = 0,1$. Now, since $B(X)$ is algebraically generated by $\mathscr{A}(X_0)$ and $\mathscr{A}(X_1)$ we can write

$$P_0 = \sum_{i=1}^{n} S_i T_i, \qquad P_1 = \sum_{i=n+1}^{m} T_i S_i,$$

where $S_i \in \mathscr{A}(X_0)$, $T_i \in \mathscr{A}(X_1)$, $i = 1, \ldots, m$. Let T be an arbitrary bounded operator on X. Using the obvious relation $T = (P_0 + P_1) T (P_0 + P_1)$ we have

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} S_i T_i T S_j T_j + \sum_{i=n+1}^{m} \sum_{j=1}^{n} T_i S_i T S_j T_j + \sum_{i=1}^{n} \sum_{j=n+1}^{m} S_i T_i T T_j S_j + \sum_{i=n+1}^{m} \sum_{j=n+1}^{m} T_i S_i T T_j S_j.$$

Since $S_i R S_j \in \mathscr{A}(X_0)$ for all $R \in B(X)$ we see that $B(X)$ is algebraically generated by $\mathscr{B}_0 = \mathscr{A}(X_0)$ and $\mathscr{B}_1 = \operatorname{span}(T_1, \ldots, T_m)$.

We shall now extend Theorem 1.

THEOREM 3. *Let X be a real or complex Banach space and suppose that there is a direct sum decomposition $X = X_0 \oplus X_1 \oplus \ldots \oplus X_n$ into closed linear subspaces such that* (i) *there exists a complemented subspace $Y_0 \subset X_1 \oplus \ldots \oplus X_n$ isomorphic to X_0, and* (ii) *there exist complemented subspaces $Y_i \subset X_0$ isomorphic to X_i, $i = 1, \ldots, n$. Then $B(X)$ is algebraically generated by two subalgebras of square zero.*

Proof. We denote by V_i a linear homeomorphism of X_i onto Y_i, $i = 0, 1, \ldots, n$. Choose closed complements Z_i of Y_i in X, $i = 0, 1, \ldots, n$, satisfying

$$X_0 \subset Z_0 \quad \text{and} \quad X_1 \oplus \ldots \oplus X_n \subset Z_i, \quad i = 1, \ldots, n.$$

We put

$$R_0 x = \begin{cases} V_0 x & \text{for } x \in X_0, \\ 0 & \text{for } x \in X_1 \oplus \ldots \oplus X_n, \end{cases} \qquad S_0 x = \begin{cases} V_0^{-1} x & \text{for } x \in Y_0, \\ 0 & \text{for } x \in Z_0, \end{cases}$$

$$R_i x = \begin{cases} V_i^{-1} x & \text{for } x \in Y_i, \\ 0 & \text{for } x \in Z_i, \end{cases} \qquad S_i x = \begin{cases} V_i x & \text{for } x \in X_i, \\ 0 & \text{for } x \in \bigoplus_{j \neq i} X_j, \end{cases}$$

for $i = 1, \ldots, n$. The decomposition of X implies $I = P_0 + P_1$ where P_0 and P_1 are the projections of X onto X_0 and $\bigoplus_{i=1}^{n} X_i$ respectively. We have clearly $P_0 = S_0 R_0$ and $P_1 = \sum_{i=1}^{n} R_i S_i$. As in the previous theorem one can now prove that $B(X)$ is algebraically generated by the subalgebras $\mathscr{A}(X_0)$ and $\operatorname{span}(R_0, R_1, \ldots, R_n)$, both of square zero.

COROLLARY 4. *If X is an "n-th power" $(n > 1)$, then $B(X)$ is algebraically generated by two subalgebras with square zero, the dimension of one of them being* 1 *or* 2.

Proof. If n is even, we are done. Suppose now that n is odd. Then X can be decomposed into a direct sum of closed linear subspaces

$$X = (X_1 \oplus \ldots \oplus X_m) \oplus (X_{m+1} \oplus \ldots \oplus X_{2m}) \oplus X_{2m+1}$$

with the X_i all isomorphic to one another. Set

$$\hat{X}_0 = X_1 \oplus \ldots \oplus X_m, \quad \hat{X}_1 = X_{m+1} \oplus \ldots \oplus X_{2m}, \quad \hat{X}_2 = X_{2m+1}.$$

We can now find linear homeomorphisms V_0 of $\hat{X}_0$ onto $\hat{X}_1$, $V_1 = V_0^{-1}$ of $\hat{X}_1$ onto $\hat{X}_0$, and V_2 of $\hat{X}_2$ onto X_1. We define R_0, R_1 and R_2 as in Theorem 3 and notice that $R_0 = R_1$. Thus, $B(X)$ is algebraically generated by $\mathscr{A}(\hat{X}_0)$ and $\operatorname{span}(R_0, R_2)$. This completes the proof.

Let us conclude with an open problem which is a modification of a question posed by W. Żelazko: Does the fact that $B(X)$ is algebraically generated by two subalgebras with square zero imply that X is an "nth power" $(n > 1)$? In particular, we do not know whether there exists a Banach space X which is not an "nth power" and satisfies the assumptions of Theorem 3.

Acknowledgment. I would like to thank Professor W. Żelazko for bringing his paper [3] to my attention.

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Received December 1, 1989
Revised version February 13, 1990 (2626)

STUDIA MATHEMATICA 97 (2) (1990)

On the positivity of the unit element in a normed lattice ordered algebra

by

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Abstract. An elementary proof of the following result is given: if T: $E \to E$ is a Cesàro bounded (or Abel bounded) linear operator on the normed Riesz space E and $T \geqslant I$, then $T = I$. In particular, if T is a contraction and $T \geqslant I$, then $T = I$. As a corollary we obtain that if A is a normed lattice ordered algebra with unit element e and $\|e\| \leqslant 1$, then $e \geqslant 0$.

Recently, E. Scheffold (private communication, unpublished) informed us of the following result: if A is a (real) Banach lattice algebra with multiplicative unit element e and $\|e\| \leqslant 1$ (so $\|e\| = 1$), then $e \geqslant 0$. His proof makes essential use of Kakutani's fixed point theorem to prove that if T is a linear operator on a Banach lattice E such that $T \geqslant I$ and $\|T\| \leqslant 1$ (whence $\|T\| = 1$), then $T = I$ (where I is the identity mapping on E). The result then follows by considering left or right multiplication by $|e|$.

Subsequently B. de Pagter showed us that Scheffold's result could be obtained by a semigroup approach under weaker hypotheses. We give the details of de Pagter's proof.

THEOREM 1. *Let E be a Banach lattice and T: $E \to E$ a linear operator on E such that*

(a) $T \geqslant I$,

(b) *T is power bounded* (*i.e.*, $M = \sup_{m \geqslant 1} \|T^m\| < \infty$).

Then $T = I$.

Proof. Put $S = T - I$. Then $S \geqslant 0$, so $e^{tS} \geqslant 0$ for all $t \geqslant 0$. But

$$\|e^{tT}\| = \left\| \sum_{n=0}^{\infty} \frac{(tT)^n}{n!} \right\| \leqslant \sum_{n=0}^{\infty} \frac{t^n}{n!} \|T^n\| \leqslant M e^t$$

for all $t \geqslant 0$ implies $\|e^{tS}\| \leqslant M$ for all $t \geqslant 0$. Observe now that

$$e^{tS} = I + tS + t^2 S^2/2! + \ldots \geqslant tS \geqslant 0$$

for all $t \geqslant 0$ and hence $0 \leqslant S \leqslant e^{tS}/t$ for all $t > 0$. Consequently, $\|S\| \leqslant M/t$ for all $t > 0$, showing that $S = 0$ and $T = I$.

1985 *Mathematics Subject Classification*: 47B55, 46B30, 06F25.