[12] —, —, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.
[13] R. Fefferman, *Calderón–Zygmund theory for product domains*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), 840–843.
[14] M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. 34 (1985), 777–799.
[15] —, —, *A discrete transform and decompositions of distribution spaces*, preprint.
[16] L. Hörmander, *Estimates for translation invariant operators in L_p-spaces*, Acta Math. 104 (1960), 93–139.
[17] P. I. Lizorkin, *Limiting cases of theorems on $\mathcal{F}L^p$-multipliers*, Trudy Mat. Inst. Steklov. 173 (1986), 164–180 (in Russian); English transl.: Proc. Steklov Inst. Math. 173 (1987), 177–194.
[18] A. Miyachi, *Notes on Fourier multipliers for H^p, BMO and the Lipschitz spaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 30 (1983), 221–242.
[19] —, *Estimates for pseudo-differential operators with exotic symbols*, ibid. 34 (1987), 81–110.
[20] J. Peetre, *On space of Triebel–Lizorkin type*, Ark. Math. 13 (1975), 123–130.
[21] A. Seeger, *On quasiradial Fourier multipliers and their maximal functions*, J. Reine Angew. Math. 370 (1986), 61–73.
[22] —, *Some inequalities for singular convolution operators in L^p-spaces*, Trans. Amer. Math. Soc. 308 (1988), 259–272.
[23] —, *Estimates near L^1 for Fourier multipliers and maximal functions*, Arch. Math. (Basel) 53 (1989), 188–193.
[24] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.
[25] P. Tomas, *Restriction theorems for the Fourier transform*, in: Proc. Sympos. Pure Math. 35, Amer. Math. Soc., 1979, 111–114.
[26] H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel 1983.

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The convolution equation of Choquet and Deny on semigroups

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Abstract. We characterize the nonnegative solutions f of the convolution equation $f(x) = \int_S f(x+y)\,d\sigma(y)$, $\forall x \in S$, where S is a locally compact, separable, metrizable abelian semigroup with cancellation, and σ is a nonnegative measure. The technique is to identify the extreme rays of the cone of solutions. The case where S is a group was studied by Choquet and Deny.

§ 1. Introduction. Consider the convolution equation

$$\mu = \mu * \sigma \tag{1.1}$$

on a locally compact abelian group G, where σ, μ are regular Borel measures on G, $\sigma \geqslant 0$ is given, and μ is to be determined. Choquet and Deny [4] showed that if σ is a probability measure, and if the regularization of μ is bounded (i.e., $\mu * \varphi$ is bounded for any continuous function φ on G with compact support), then $\mu = f \cdot \omega$, where ω is the Haar measure on G, and f satisfies

$$f(x) = f(x+y), \quad \forall x \in G,\ y \in \operatorname{supp} \sigma,$$

i.e. f is a periodic function with periods $y \in \operatorname{supp} \sigma$. The equation in the form

$$f(x) = \int_G f(x-y)\,d\sigma(y), \quad \forall x \in G, \tag{1.2}$$

was later considered by Doob, Snell and Williamson by a simple martingale argument [7] (see also [15, p. 151]). The result has important applications in renewal processes [8].

The nonnegative measures μ satisfying (1.1) were characterized by Deny [6]: Suppose in addition G is metrizable and separable, and $\operatorname{supp} \sigma$ generates the group G. Then the extreme rays of the cone

$$H = \{\mu \geqslant 0 \colon \mu * \sigma = \mu\}$$

are of the form $\mu = cg \cdot \omega$, where $c > 0$ is a constant, and g is a nonnegative

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exponential function on G (i.e. g is continuous and $g(x+y) = g(x)g(y)$, $\forall x, y \in G$) with $\int_G g(-y)\,d\sigma(y) = 1$. The Choquet theorem implies that $\mu = f\cdot\omega$, where f is a mixture of the cg's.

A similar equation

$$f(x) = \int_0^\infty f(x+y)\,d\sigma(y) \qquad \forall x\in[0, \infty), \tag{1.3}$$

has drawn much attention recently. When σ is a probability measure, the bounded continuous solutions f can easily be characterized (the proof is the same as in the group case, cf. e.g. [6], [15]). In [12], Lau and Rao characterized the nonnegative Lebesgue locally integrable solutions f of (1.3) for $\sigma \geqslant 0$. Various applications and extensions can be found in [5, 12, 17]. The generalization of (1.3) to semigroups was first attempted by Davies and Shanbhag [5]. They pointed out that the solutions are more complicated, due to the lack of structures on semigroups. Their representation of f, however, depending on a martingale limit and a technical probability space they constructed, lacks the clarity of Deny's results.

Our main purpose in this paper is to extend (1.1) and (1.3) to the semigroup setting via Deny's approach.

Let $(S, +, *)$ be a locally compact, separable, metrizable abelian semigroup which satisfies the cancellation law and has a continuous involution. Let $M^+(S)$ denote the set of nonnegative regular Borel measures (possibly unbounded) on S, with the weak topology generated by $C_c(S)$, the space of continuous functions with compact supports. For $\mu, \nu \in M^+(S)$, let $\mu\bullet\nu$ be defined by

$$\mu\bullet\nu(E) = \int_S \mu(E+y^*)\,d\nu(y),$$

where E is any Borel subset in S. The modification of (1.1) is the integral equation

$$\mu\bullet\sigma = \mu, \tag{1.4}$$

where $\mu, \sigma\in M^+(S)$. Note that if S is a group and $x^* = -x$, then (1.4) is just (1.1). If $x^* = x$ and S admits a translation invariant measure, then (1.4) is the extension of (1.3) to semigroups (Theorems 6.1–6.3).

There are two difficulties in studying (1.4) from the analysis point of view:

(i) The translation operator $x\to\mu_x$, where $\mu_x(E) = \mu(x+E)$, is not continuous on S (see §2 for a simple example in $[0, \infty)$).

(ii) The existence of a translation invariant measure on S is not guaranteed.

In order to overcome these, we consider

DEFINITION 1.1. Let S_0 denote the set of elements x in S which have a compact neighborhood U such that for any $y\in S$, $y+U$ is also a neighborhood of $y+x$. We call S_0 the *fundamental ideal* of S.

The notion was introduced by the authors in connection with embedding semigroups into groups. It is clear that if S is a group, then $S_0 = S$. It is also proved that if $S_0 \neq \emptyset$, then S admits a unique translation invariant measure ω [13], and the mapping $(y, \mu)\to\mu_y$ from $S\times M^+(S_0)$ to $M^+(S_0)$ is jointly continuous (Theorem 2.5). This suggests that the appropriate integral equation to be studied is

$$\mu\bullet\sigma = \mu, \tag{1.5}$$

where $\sigma\in M^+(S)$ is given, and $\mu\in M^+(S_0)$ is to be determined.

The paper is arranged as follows: in §2 we recapitulate some properties of the fundamental ideal S_0 from [13], and prove the continuity property of translation of measures stated above. The existence of a translation invariant measure is also discussed.

In §3, we define the $\bullet$ convolution of measures, and derive some basic limit theorems we need from the above continuity property.

In §4, we consider the convolution equation $\mu\bullet\sigma = \mu$ with $\mu\in M^+(S_0)$, and $\sigma\in M^+(S)$. We introduce the cones

$$H_0 = \{\mu\in M^+(S_0):\ \mu\bullet\sigma = \mu\}, \qquad C_0 = \{\mu\in M^+(S_0):\ \mu\bullet\sigma \leqslant \mu\}.$$

An analogous argument to Deny [6] shows that C_0 is a well capped lattice cone, and H_0 is a hereditary subcone of C_0.

In §5, we study the extreme elements of H_0 via the cone C_0, and its relationship to exponential functions (Theorem 5.4, Corollary 5.9). The Choquet theorem enables us to give an integral representation of the solutions to the equation (1.5) in terms of the exponential functions and the translation invariant measure on S (Theorem 5.8, Corollary 5.9). The main restriction of the above results is that $S = S(\sigma)$, the closed subsemigroup spanned by the support of σ. Unlike the group case, this condition is too stringent for semigroups. In Theorems 5.14, we extend the representation theorem to the case of $S(\sigma)$ with the *component generating property* (Definition 5.10), which will cover most of the interesting cases.

We conclude this paper in §6 by reducing our results in §5 to the special case

$$f(x) = \int_S f(x+y)\,d\sigma(y), \qquad \forall x\in S.$$

We also give various examples to illustrate the theorems and corollaries obtained.

The authors are indebted to Professor C. R. Rao and Dr. D. N. Shanbhag for bringing their attention to the semigroup case, and for some valuable discussions.

§2. Fundamental ideal. Let $(S, +)$ be an abelian semigroup. S is said to satisfy the *cancellation law* if for any $x, y, z\in S$, $x+z = y+z$ implies that $x = y$. A map $*\colon S\to S$ is called an *involution* if for any $x, y\in S$, $(x+y)^* = x^*+y^*$ and $(x^*)^* = x$.

Throughout this paper, we assume that S is a locally compact, separable, metrizable, abelian semigroup, satisfies the cancellation law, and has a continuous involution (see [1] for various examples).

Let S_0 be the fundamental ideal defined in § 1. It is easy to prove (cf. [13]):

(i) S_0 is an open ideal of S, and is closed under involution.

(ii) $S = S_0$ if and only if $0 \in S_0$. In particular, $S = S_0$ if S is a group.

(iii) If S is a subsemigroup of a topological group G, and if $\operatorname{int} S \neq \emptyset$, then $S_0 = \operatorname{int} S$, where $\operatorname{int} S$ is the interior of S in G.

The following embedding theorem is also proved in [13]:

THEOREM 2.1. *S_0 can be embedded into a locally compact, abelian group as an open subsemigroup. Moreover, S_0 is maximal in the sense that if an ideal I is open and is embeddable into a topological group as an open subsemigroup, then $I \subseteq S_0$.*

For any locally compact Hausdorff space X, we let $C_c(X)$ denote the space of continuous functions on X with compact supports, $M(X)$ the set of regular Borel mesures on X, and $M^+(X)$ the subset of nonnegative measures. If $C_c(X)$ is given the inductive topology, then its dual space can be identified with $M(X)$ [3]. The topology on $M(X)$ we use is the weak topology generated by $C_c(X)$. If further X is separable and metrizable, then $C_c(X)$ is separable and $M^+(X)$ is metrizable. It is known that

THEOREM 2.2. *Let X be locally compact, separable and metrizable. Let $\mu_n, \mu \in M^+(X)$. Then the following are equivalent:*

(i) $\mu_n \to \mu$.

(ii) *For any compact subset K, and for any open subset V with compact closure,*

$$\limsup \mu_n(K) \leqslant \mu(K), \qquad \liminf \mu_n(V) \geqslant \mu(V).$$

(iii) *For any Borel subset E with compact closure, and with $\mu(\partial E) = 0$,*

$$\lim \mu_n(E) = \mu(E)$$

(∂A is the boundary of A).

Let $\mu \in M(S)$. We denote the *y-translation* of μ by μ_y, i.e., $\mu_y(E) = \mu(y+E)$ for any Borel subset E of S. The following two examples show that translations of measures on S do not have the desirable convergence properties.

EXAMPLE 2.3. Let $S = \mathbf{R}_+$ with identity involution. Let $\mu_n = \delta_{1-1/n}$, $\mu = \delta_1$ be the point mass measures at $1-1/n$ and 1 respectively. Let $y = 1$. Then $\mu_n \to \mu$, but $(\mu_n)_y = 0 \not\to \mu_y = \delta_0$.

EXAMPLE 2.4. Let S be as above, let $y_n = 1+1/n$, $y = 1$ and $\mu = \delta_1$. Then $y_n \to y$, but $\mu_{y_n} = 0 \not\to \mu_y = \delta_0$.

The fundamental ideal turns out to be significant in this respect.

THEOREM 2.5. *The map $(y, \mu) \to \mu_y$ from $S \times M^+(S_0)$ to $M^+(S_0)$ is jointly continuous.*

We will need the following lemma.

LEMMA 2.6. *Let V be an open subset of S_0. Then:*

(i) *$y+V$ is open for any $y \in S$.*

(ii) *Let $K \subseteq V$ be compact, and let $\{y_n\}$, y be in S with $\lim_{n\to\infty} y_n = y$. Then for sufficiently large n, we have*

$$y_n + K \subseteq y+V, \qquad y+K \subseteq y_n + V.$$

Proof. By Theorem 2.1 we can assume without loss of generality that S_0 is an open subset of a topological group G. Since S_0 is an ideal, for any $y \in S$ and for any fixed $a \in S_0$, $a+y$ is in S_0 ($\subseteq G$). It follows that for any open V in S_0, $(a+y)+V$ is open in G, and $y+V = (a+y+V)-a$ is open in G, hence open in S_0.

For (ii), we let $b_n = (a+y)-(a+y_n)$. Then $b_n \in S_0 - S_0 \subseteq G$. The two inclusions are equivalent to $-b_n + K \subseteq V$, $b_n + K \subseteq V$, for sufficiently large n. These obviously hold for groups.

Proof of Theorem 2.5. In view of Theorem 2.2, we need to show that for any compact subset $K \subseteq S_0$, and for any open subset V of S_0 with compact closure,

$$\limsup \mu_n(y_n + K) \leqslant \mu(y+K), \qquad \liminf \mu_n(y_n + V) \geqslant \mu(y+V).$$

For $\varepsilon > 0$, it follows from the regularity of μ and the metric on S that there exists an open subset $O \subseteq S_0$ such that $K \subseteq O$, $\mu(\partial(y+O)) = 0$ and $\mu(y+O) \leqslant \mu(y+K)+\varepsilon$. We hence have

$$\begin{aligned}\limsup \mu_n(y_n+K) &\leqslant \limsup \mu_n(y+O) && \text{(Lemma 2.6 (ii))}\\ &= \mu(y+O) && \text{(Theorem 2.2 (iii))}\\ &\leqslant \mu(y+K)+\varepsilon.\end{aligned}$$

The proof of the second inequality is similar.

In [13], we proved

THEOREM 2.7. *If $S_0 \neq \emptyset$, then S admits a unique translation invariant measure ω, and $S_0 \subseteq \operatorname{supp} \omega$.*

A measure $0 \neq \tau \in M^+(S_0)$ is called *quasi-invariant* if for each $y \in S$, $\tau_y = \tau$ or $\tau_y = 0$. We use J to denote the set $\{y \in S\colon \tau_y \neq 0\}$, and call it the *translation set* of τ.

Our aim in the rest of the section is to prove:

THEOREM 2.8. *Let $\tau \in M^+(S_0)$ be quasi-invariant, and assume $J \cap S_0 \neq \emptyset$. Then $\tau = c\omega$ on $\operatorname{supp}\tau$ for some $c > 0$.*

We will need the following two propositions.

PROPOSITION 2.9. *Let $\tau \in M^+(S_0)$ be quasi-invariant. Then J is an open and closed subsemigroup of S, and $S \backslash J$ is an open and closed ideal of S.*

Proof. By the continuity of $y \to \tau_y$ (Theorem 2.5), and the alternative expression of J as $\{y \in S: \tau_y = \tau\}$, J is open and closed. For any $x, y \in J$, $\tau_{x+y} = (\tau_x)_y = \tau_y = \tau$. This implies that J is a semigroup. The assertion for $S \backslash J$ follows similarly.

PROPOSITION 2.10. *Let $\tau \in M^+(S_0)$ be quasi-invariant. Then*

(i) $(\operatorname{supp} \tau) + J \subseteq \operatorname{supp} \tau$.
(ii) *If $J \cap S_0 \neq \emptyset$, then $J \cap S_0 = \operatorname{supp} \tau$.*

Proof. (i) Let $x \in \operatorname{supp} \tau$, $y \in J$, and let $U \subseteq S_0$ be an open neighborhood of $x+y$ in S_0. The continuity of addition yields a neighborhood $V \subseteq S_0$ of x such that $y + V \subseteq U$, hence

$$\tau(U) \geqslant \tau(y+V) = \tau_y(V) = \tau(V) > 0.$$

This implies that $x + y \in \operatorname{supp} \tau$.

(ii) Let $x \in J \cap S_0$ and let $y \in \operatorname{supp} \tau$. By (i), $x + y \in \operatorname{supp} \tau$. For any neighborhood V of x, $y + V$ is a neighborhood of $x + y$ (Lemma 2.6(i)), and

$$0 < \tau(y+V) = \tau_y(V) = \tau(V).$$

This implies that $x \in \operatorname{supp} \tau$, and $J \cap S_0 \subseteq \operatorname{supp} \tau$. To prove the reverse inclusion, we let $x \in \operatorname{supp} \tau$ $(\subseteq S_0)$, and fix a $y \in J \cap S_0$. Let V be a neighborhood of x such that $y + V \subseteq x + S_0$, and let W be a neighborhood of y such that $y + V = x + W$. Since $x + y \in \operatorname{supp} \tau$ (by the part just proved), we have

$$0 < \tau(y+V) = \tau(x+W) = \tau_x(W).$$

This implies that $x \in J$ and $\operatorname{supp} \tau \subseteq J \cap S_0$.

Proof of Theorem 2.8. By Theorem 2.1, we can assume that S_0 is an open subsemigroup of a locally compact abelian group G. Since J is open in S, $J \cap S_0$ is an open subsemigroup of G also. It follows from the proof of Theorem 1.5 in [13] that τ on $J \cap S_0$ can be extended to a translation invariant measure on G. By the uniqueness of such measure, we conclude that $\tau = c\omega$ on $\operatorname{supp} \tau = J \cap S_0$.

The following example shows that Theorem 2.8 is false if we do not assume $J \cap S_0 \neq \emptyset$.

EXAMPLE 2.11. Let $S = \{(x, y): x > 1, y > 0\}$ be a subsemigroup of $\mathbf{R}^2$. Let μ be any measure on S supported by $\{(x, y) \in S: x = 3/2\}$. Then any translation of μ is a zero measure, it is a quasi-invariant measure with $J = \emptyset$, and the conclusion of Theorem 2.8 does not hold.

§3. Convolution on semigroups. Let μ, ν be in $M^+(S)$, and let $\mu \bullet \nu$ be a set function on the family $\mathscr{K}$ of compact subsets of S defined by

$$(3.1) \qquad \mu \bullet \nu(K) = (\mu \times \nu)\{(x, y): x, y \in S,\ x \in y^* + K\}, \qquad K \in \mathscr{K}.$$

PROPOSITION 3.1. *Let $\mu, \nu \in M^+(S)$. Suppose $\mu \bullet \nu(K) < \infty$ for all $K \in \mathscr{K}$. Then $\mu \bullet \nu$ can be extended to a regular Borel measure on S.*

Proof. It is direct to show that $\mu \bullet \nu$ is a regular content on $\mathscr{K}$, and hence can be extended to a regular Borel measure on S [10, Chapter 10].

DEFINITION 3.2. Let $\mu, \nu \in M^+(S)$ with $\mu \bullet \nu(K) < \infty$ for all $K \in \mathscr{K}$. We call the measure defined above the *generalized convolution* (g-convolution) of μ and ν, and denote it by $\mu \bullet \nu$.

It is easy to show that if ν has compact support, the g-convolution $\mu \bullet \nu$ is well defined for all $\mu \in M^+(S)$. For any Borel subset E in S, we have

$$\mu \bullet \nu(E) = \int_{S \times S} \chi_{y^*+E}(x)\, d(\mu \times \nu)(x, y) = \int_S \int_S \chi_E(x)\, d\mu_{y^*}(x)\, d\nu(y)$$
$$= \int_S \mu(y^* + E)\, d\nu(y).$$

It also follows from a limit argument that

$$(3.2) \qquad \mu \bullet \nu(\varphi) = \int_S \int_S \varphi(x)\, d\mu_{y^*}(x)\, d\nu(y), \qquad \varphi \in C_c(S).$$

If S is a group and the involution is $x^* = -x$, then from (3.2),

$$\mu \bullet \nu(\varphi) = \int_S \int_S \varphi(x - y^*)\, d\mu(x)\, d\nu(y) = \int_S \int_S \varphi(x+y)\, d\mu(x)\, d\nu(y) = \mu * \nu(\varphi)$$

for all $\varphi \in C_c(S)$, and the two convolutions coincide.

The g-convolution on semigroups does not obey the same laws as convolution on groups:

EXAMPLE 3.3. Let $S_1 = (\mathbf{R}_+, +)$, $S_2 = (\mathbf{R}, +)$, and let the involutions on S_1 and S_2 be the identities. Then for any $x, y \geqslant 0$, the g-convolution of δ_x, δ_y on S_1 is

$$\delta_x \bullet \delta_y = \begin{cases} \delta_{x-y} & \text{if } x \geqslant y, \\ 0 & \text{if } x < y, \end{cases}$$

which is not necessarily equal to $\delta_x \bullet \delta_y$ $(= \delta_{x-y})$ on S_2.

EXAMPLE 3.4. Let S be a group with the identity involution. Then for any $x, y, z \in S$,

$$(\delta_x \bullet \delta_y) \bullet \delta_z = \delta_{x-y-z}, \qquad \delta_x \bullet (\delta_y \bullet \delta_z) = \delta_{x-y+z},$$
$$\delta_x \bullet \delta_y = \delta_{x-y}, \qquad \delta_y \bullet \delta_x = \delta_{y-x}.$$

It follows that g-convolution is neither associative nor commutative.

PROPOSITION 3.5. *Let $\mu, \nu, \sigma \in M^+(S)$ be such that all the convolutions involved exist. Then*

$$(\mu \bullet \nu) \bullet \sigma = \mu \bullet (\nu * \sigma) = (\mu \bullet \sigma) \bullet \nu.$$

Proof. For any Borel subset E of S,

$$\big((\mu \bullet \nu) \bullet \sigma\big)(E) = \int_S (\mu \bullet \nu)(z^* + E)\, d\sigma(z) = \int_S \int_S \mu(y^* + z^* + E)\, d\nu(y)\, d\sigma(z)$$

$$= \int_S \int_S \mu\big((y+z)^* + E\big)\, d\nu(y)\, d\sigma(z) = \int_S \mu(x^* + E)\, d(\nu * \sigma)(x)$$

$$= \mu \bullet (\nu * \sigma)(E).$$

The second identity follows from $\mu \bullet (\nu * \sigma) = \mu \bullet (\sigma * \nu) = (\mu \bullet \sigma) \bullet \nu$.

PROPOSITION 3.6. *If in addition S is a group, then for any $\mu, \nu, \sigma \in M^+(S)$ so that the convolutions involved exist, we have*

$$\mu \bullet \nu = (\nu \bullet \mu)^{\sim}, \qquad \mu \bullet (\nu \bullet \sigma) = (\mu * \sigma) \bullet \nu,$$

where $\tilde{\nu}(E) = \nu(-E^*)$.

We remark that if S is not a group, then the right-hand side of the first identity is undefined, and the second identity is false: e.g. let $S = \mathbf{R}_+$, and let $a, b \in S$ with $a > b$; then $\delta_a \bullet (\delta_b \bullet \delta_a) = 0$, $(\delta_a * \delta_b) \bullet \delta_a = \delta_b$.

Proof. For any Borel subset E in S,

$$(\nu \bullet \mu)^{\sim}(E) = \int_S \int_S \chi_{x^* - E^*}(y)\, d\nu(y)\, d\mu(x) = \int_S \int_S \chi_E(-y^* + x)\, d\mu(x)\, d\nu(y)$$

$$= \int_S \int_S \chi_{y^* + E}(x)\, d\mu(x)\, d\mu(y) = (\mu \bullet \nu)(E).$$

Also

$$\mu \bullet (\nu \bullet \sigma) = \big((\nu \bullet \sigma) \bullet \mu\big)^{\sim} = \big(\nu \bullet (\sigma * \mu)\big)^{\sim} = \big(\nu \bullet (\mu * \sigma)\big)^{\sim} = (\mu * \sigma) \bullet \nu.$$

In order to discuss the continuity of the $\bullet$ operation, we will need to restrict our consideration to $M^+(S_0)$ (see Examples 2.3, 2.4 and Theorem 2.5).

PROPOSITION 3.7. *Let $\mu \in M^+(S_0)$, $\nu \in M^+(S)$. Then $\mu \bullet \nu(K)$ is well defined by (3.1) for any compact subset K in S_0. Furthermore, if $\mu \bullet \nu(K) < \infty$ for all such K, then $\mu \bullet \nu$ can be extended to a regular Borel measure on S_0.*

Proof. For any compact set K in S_0, the ideal property of S_0 implies that $y^* + K \subseteq S_0$ and $\mu \bullet \nu(K) = \int_S \mu(y^* + K)\, d\nu(y)$ is defined. The extension of $\mu \bullet \nu$ to a regular Borel measure on S_0 is the same as in Proposition 3.1.

The above proposition allows us to regard $M^+(S_0)$ as an ideal in $M^+(S)$ under the g-convolution. If we interchange the roles of μ and ν, we have

PROPOSITION 3.8. *Let $\mu \in M^+(S)$, $\nu \in M^+(S_0)$, and let μ_0 be such that $\mu_0 = \mu$ on S_0, $\mu_0 = 0$ on $S \backslash S_0$. Then for any compact subset K of S,*

$$\mu \bullet \nu(K) = \mu_0 \bullet \nu(K).$$

If the above expression is finite for all K, then $\mu \bullet \nu$ can be extended to a regular Borel measure on S, and the above equality holds for all Borel sets.

Proof. We make use of the ideal property of S_0: for any $y \in S_0$, and for any compact subset K of S with $y^* + K \subseteq S_0$,

$$(\mu \bullet \nu)(K) = \int_{S_0} \mu(y^* + K)\, d\nu(y) = \int_{S_0} \mu_0(y^* + K)\, d\nu(y) = (\mu_0 \bullet \nu)(K).$$

Some convergence properties of $\bullet$ can be derived from Theorem 2.5.

THEOREM 3.9. *Let $\{\mu_n\}, \mu \in M^+(S_0)$ with $\mu_n \xrightarrow{w} \mu$, and let $\sigma \in M^+(S)$. If either*

(i) *$\mu_n \uparrow$, and $\mu \bullet \sigma$ exists, or*

(ii) *there exists $\nu \in M^+(S_0)$ such that $\mu_n \leqslant \nu$ for all n, and $\nu \bullet \sigma$ exists,*

then $\nu_n \bullet \sigma \xrightarrow{w} \mu \bullet \sigma$.

Proof. For any $\varphi \in C_c(S_0)$, with $\varphi \geqslant 0$,

$$(\mu_n \bullet \sigma)(\varphi) = \int_S \Big(\int_{S_0} \varphi(x)\, d(\mu_n)_{y^*}(x)\Big)\, d\sigma(y), \qquad (\mu \bullet \sigma)(\varphi) = \int_S \Big(\int_{S_0} \varphi(x)\, d\mu_{y^*}(x)\Big)\, d\sigma(y).$$

The assertion follows from Theorem 2.5, the monotone convergence theorem (case (i)), and the bounded convergence theorem (case (ii)).

THEOREM 3.10. *Let $\{\mu_n\}$, $\{\nu_n\}$, μ, ν be in $M^+(S_0)$ with $\mu_n \xrightarrow{w} \mu$, $\nu_n \xrightarrow{w} \nu$. Let $\sigma \in M^+(S)$, and $\mu_n \bullet \sigma \leqslant \nu_n$ for all n. Then $\mu \bullet \sigma$ exists, and $\mu \bullet \sigma \leqslant \nu$.*

Proof. Let $\varphi \in C_c(S_0)$ with $\varphi \geqslant 0$. Then

$$(\mu \bullet \sigma)(\varphi) = \int_S \Big(\int_{S_0} \varphi(x)\, d\mu_{y^*}(x)\Big)\, d\sigma(y) \leqslant \liminf \int_S \Big(\int_{S_0} \varphi(x)\, d(\mu_n)_{y^*}(x)\Big)\, d\sigma(y)$$

$$= \liminf \int_{S_0} \varphi(x)\, d(\mu_n \bullet \sigma)(x) \leqslant \int_{S_0} \varphi(x)\, d\nu(x).$$

(The first inequality follows from Theorem 2.5 and Fatou's Lemma.)

THEOREM 3.11. *Let* $\mu\in M^+(S_0)$, *and let* $\{\sigma_n\}$, σ *be in* $M^+(S)$ *with* $\operatorname{supp}\sigma_n$, $\operatorname{supp}\sigma\subseteq K$ *for some compact* K *in* S, *and* $\sigma_n \xrightarrow{w} \sigma$. *Then* $\mu\bullet\sigma_n \xrightarrow{w} \mu\bullet\sigma$.

Proof. It is clear from the expression $\mu\bullet\nu(E)=\int_S \mu(y^*+E)\,d\nu(y)$ that $\mu\bullet\sigma_n$, $\mu\bullet\sigma$ are finite on compact sets, and hence they are in $M^+(S_0)$ (Proposition 3.7). Let $\varphi\in C_c(S_0)$. Then $\int_{S_0}\varphi(x)\,d\mu_{y^*}(x)$ is a continuous function of y (Theorem 2.5), and

$$\lim(\mu\bullet\sigma_n)(\varphi)=\lim\int_K\Big(\int_{S_0}\varphi(x)\,d\mu_{y^*}(x)\Big)\,d\sigma_n(y)=\int_K\Big(\int_{S_0}\varphi(x)\,d\mu_{y^*}(x)\Big)\,d\sigma(y)=\mu\bullet\sigma(\varphi).$$

§4. The cones. Throughout the rest of the paper, we assume that $0\neq\sigma\in M^+(S)$. All unexplained terms involving convex cones in this section can be found in [3] or [16].

DEFINITION 4.1. Let

$$C_0=\{\mu\in M^+(S_0):\ \mu\bullet\sigma\leqslant\mu\},\qquad H_0=\{\mu\in M^+(S_0):\ \mu\bullet\sigma=\mu\}.$$

We call $\mu\in C_0$ a *σ-superharmonic measure*, and $\mu\in H_0$ a *σ-harmonic measure*.

It is easy to show that both C_0 and H_0 are convex cones. They are closed under g-convolution in the following sense.

PROPOSITION 4.2. *Let* $\mu\in C_0$ *(or* H_0*). Then for any* $\nu\in M^+(S)$ *such that* $\mu\bullet\nu$ *exists,* $\mu\bullet\nu\in C_0$ *(*H_0*, respectively).*

Proof. The result follows from $(\mu\bullet\nu)\bullet\sigma=(\mu\bullet\sigma)\bullet\nu\leqslant\mu\bullet\nu$ (Proposition 3.4).

We call a cone C in a locally convex space *well capped* if $C=\bigcup K$, where K is compact and K, $C\backslash K$ are convex [16].

PROPOSITION 4.3. C_0 *is metrizable, weakly closed in* $M^+(S_0)$, *and well capped.*

Proof. Let $\{\mu_n\}$ be in C_0 with $\mu_n\xrightarrow{w}\mu\in M^+(S_0)$. Since $\mu_n\bullet\sigma\leqslant\mu_n$, Theorem 3.10 implies that $\mu\bullet\sigma\leqslant\mu$, which means $\mu\in C_0$, and C_0 is weakly closed. That C_0 is metrizable is inherited from $M^+(S_0)$, and it is well capped by [16, Theorem 11.5].

By using the same proof as in Deny [6], we can show that C_0 has the following decomposition property (Riesz decomposition property). For any $\mu\in C_0$,

$$\mu=\tau+\eta$$

where $\eta=\lim_{n\to\infty}\mu\bullet\sigma^n$ is σ-harmonic, $\tau=\lim_{n\to\infty}\zeta\bullet\sum_{k=1}^{n}\sigma^k$, with $\zeta=\mu-\mu\bullet\sigma$, and $\mu\bullet\sigma^n$ is defined by $(\mu\bullet\sigma^{n-1})\bullet\sigma$. Furthermore, if $\mu=\tau'+\eta'$ is another decomposition of μ with $\tau'\in C_0$, $\eta'\in H_0$, then $\eta'\leqslant\eta$. From this it follows that

PROPOSITION 4.4. *The cone* H_0 *is a hereditary subcone of* C_0 *(i.e., for any* $\mu\in H_0$, $\nu\in C_0$ *if* $\mu-\nu\in C_0$, *then* $\nu\in H_0$*).*

Let $S(\sigma)$ denote the closed subsemigroup generated by $\operatorname{supp}\sigma$ in S.

LEMMA 4.5. *Let* S *be a Hausdorff topological semigroup, and let* σ *be a probability measure on* S. *Let* f *be a bounded continuous function on* S *satisfying*

$$f(x)=\int_S f(x+y^*)\,d\sigma(y),\qquad \forall x\in S.$$

Then $f(x+y^*)=f(x)$, $\forall x\in S$, $y^*\in S(\sigma)$.

Proof. Let $\tilde f(x)=f(x^*)$. The above equation can be rewritten as

$$\tilde f(x)=\int_S \tilde f(x+y)\,d\sigma(y),\qquad \forall x\in S.$$

The same martingale argument as in [15, p. 151] will prove the assertion.

A measure $\mu\in M^+(S_0)$ is said to be *σ-shift bounded* if the set $\{\mu_y:\ y\in S(\sigma)\}$ is weakly bounded in $M^+(S_0)$. For any $\varphi\in C_c(S_0)$, let $\mu\bullet\varphi$ be the function defined by

$$(\mu\bullet\varphi)(y)=\int_{S_0}\varphi(x)\,d\mu_{y^*}(x)=\mu_{y^*}(\varphi),\qquad y\in S(\sigma).$$

It follows from Theorem 2.5 that $\mu\bullet\varphi$ is a continuous function on $S(\sigma)$. Also, μ is σ-shift bounded if and only if $\mu\bullet\varphi$ is a bounded continuous function on $S(\sigma)$ for any $\varphi\in C_c(S_0)$.

We will use the following theorem in the next section.

THEOREM 4.6. *Let* σ *be a probability measure on* S, *and let* $\mu\in H_0$ *be σ-shift bounded. Then* $\mu_{x^*}=\mu$ *for every* $x\in S(\sigma)$.

Proof. Let $\varphi\in C_c(S_0)$. The above remark implies that $\mu\bullet\varphi$ is a bounded continuous function on $S(\sigma)$, and for any $x\in S(\sigma)$,

$$\int_S \mu\bullet\varphi(x+y)\,d\sigma(y)=\int_S\Big(\int_{S_0}\varphi(z)\,d\mu_{(x+y)^*}(z)\Big)\,d\sigma(y)=(\mu_{x^*}\bullet\sigma)(\varphi)=\mu_{x^*}(\varphi)$$
$$=\mu\bullet\varphi(x).$$

It follows from Lemma 4.5 that $\mu\bullet\varphi$ is a periodic function with periods $x\in S(\sigma)$, and hence $\mu_{x^*}=\mu$ for $x\in S(\sigma)$.

§5. Extreme rays and integral representations. A nonzero function $g\colon S\to\mathbf{R}$ is called an *exponential function* if g is continuous and

$$g(x+y)=g(x)g(y),\qquad \forall x,y\in S.$$

A measure $\mu\in M^+(S_0)$ is called an *exponential measure* if $\mu_y=g(y)\mu$, for some $g\geqslant 0$ on S. Let

$$T=\{y\in S:\ \mu_y\neq 0\}.$$

It follows from Theorem 2.5 and the definition that the associated function g for an exponential measure is actually an exponential function, and $y \in T$ if and only if $g(y) > 0$.

LEMMA 5.1. *Let $\mu \in M^+(S_0)$ be an exponential measure. Then T is both open and closed.*

Proof. That T is open follows from Theorem 2.5. To show that T is closed, let g be the associated exponential function for μ, and let $\{y_n\}$ be in T with $\lim_{n\to\infty} y_n = y \in S$. Pick an $x \in \operatorname{supp}\mu$, and a neighborhood U of x such that $\mu(U) < \infty$. Since $y+U$ is a neighborhood of $y+x$ (Lemma 2.6), there exists a neighborhood V of x such that $y_n + V \subseteq y+U$ for large n. It follows that

$$0 < g(y_n)\mu(V) = \mu(y_n+V) \leqslant \mu(y+U) = g(y)\mu(U) < \infty.$$

This implies that $\mu_y \neq 0$ and T is closed.

THEOREM 5.2. *Let $0 \neq \mu \in M^+(S_0)$. Then μ is an exponential measure with $\mu_y \neq 0$ for some $y \in S_0$ if and only if $\mu = cg\cdot\omega$ on S_0 for some $c \geqslant 0$, and some nonnegative exponential function g on S.*

Proof. The hypothesis implies that $S_0 \neq \emptyset$, and hence the unique translation invariant measure ω on S exists (Theorem 2.8). The sufficiency of the theorem is clear.

To prove the necessity, let g be such that $\mu_y = g(y)\mu$, and define h on S by

$$h(x) = \begin{cases} 1/g(x) & \text{if } g(x) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Then h is continuous (Lemma 5.1), and is also exponential. Let $\tau = h\cdot\mu$. We claim that τ is quasi-invariant: for any $\varphi \in C_c(S_0)$, and $y \in S$,

$$\tau_y(\varphi) = \int_{S_0} \varphi(x)h(x+y)\,d\mu_y(x) = \int_{S_0} \varphi(x)h(x)h(y)g(y)\,d\mu(x)$$

$$= h(y)g(y)\int_{S_0} \varphi(x)h(x)\,d\mu(x) = h(y)g(y)\tau(\varphi).$$

The claim follows by observing that $h(y)g(y)$ equals 1 if and only if $y \in T$, and equals 0 otherwise. By Theorem 2.8 we have $\tau = c\omega$ on T, and μ is of the required form.

We will use $E(\sigma)$ to denote the set of exponential measures in H_0, and let

$$E_0(\sigma) = \{\mu \in E(\sigma)\colon\ \mu_y \neq 0 \text{ for some } y \in S_0\}.$$

Our aim is to characterize the extreme rays of H_0 as the exponential measures. Recall that H_0 is a hereditary subcone of C_0 (Proposition 4.4), hence the extreme rays of H_0 is the same as the extreme rays of C_0 that are contained in H_0. We denote the set by ∂H_0, and will use this fact without specifying which cone we are referring to.

PROPOSITION 5.3. *Let $\mu \in M^+(S_0)$ be an exponential measure, and let g be its associated exponential function. Then $\mu \in E(\sigma)$ if and only if $\int_S g(y^*)\,d\sigma(y) = 1$.*

Proof. Let $\varphi \in C_c(S_0)$. Then

$$\mu\bullet\sigma(\varphi) = \int_S\int_{S_0} \varphi(x)\,d\mu_{y^*}(x)\,d\sigma(y) = \int_S\int_{S_0} \varphi(x)g(y^*)\,d\mu(x)\,d\sigma(y) = \mu(\varphi)\int_S g(y^*)\,d\sigma(y).$$

The assertion follows from this.

THEOREM 5.4. *Let $S = S(\sigma)$, and let $H_0 \neq \{0\}$. Then $E_0(\sigma) \subseteq \partial H_0 \subseteq E(\sigma)$.*

Proof. Let $0 \neq \mu \in H_0$ be extreme. For any $x \in \operatorname{supp}\sigma$, and for any neighborhood V of x with compact closure, let σ_V be the restriction of σ to V. Then $\mu\bullet\sigma_V \in C_0$ (Proposition 4.2). If we write

$$\mu = \tfrac{1}{2}(\mu+\mu\bullet\sigma_V)+\tfrac{1}{2}(\mu-\mu\bullet\sigma_V),$$

then $\mu\pm(\mu\bullet\sigma_V) \in C_0$, and μ being extreme in C_0 implies that $\mu\bullet\sigma_V = \alpha_V\mu$ for some $\alpha_V \geqslant 0$. Let

$$\tau_V = \sigma_V/\sigma(V), \qquad \beta_V = \alpha_V/\sigma(V).$$

Then $\mu\bullet\tau_V = \beta_V\mu$. Let $\{V_n\}$ be a decreasing sequence of neighborhoods of x with compact closures such that $\lim_{n\to\infty} V_n = \{x\}$. Then $\lim \tau_{V_n} = \delta_x$ in $M^+(S)$, and hence Theorem 3.11 implies that

$$\mu_{x^*} = \mu\bullet\delta_x = \lim_{n\to\infty} \mu\bullet\tau_{V_n} = \big(\lim_{n\to\infty}\beta_{V_n}\big)\mu.$$

Denote $\lim_{n\to\infty}\beta_{V_n}$ by $g(x^*)$. The above equality can be rewritten as $\mu_x = g(x)\mu$, for all $x \in \operatorname{supp}\sigma$. Since $S(\sigma) = S$, $\mu \in E(\sigma)$.

Let $\mu \in E_0(\sigma)$ with $\mu_y = g(y)\mu$, $y \in S$. By Theorem 5.2, $\mu = cg\cdot\omega$. Let $T = \{y \in S\colon\ g(y) > 0\}$. The definition of $E_0(\sigma)$ implies that $T\cap S_0 \neq \emptyset$. Define $\hat{g}(x) = g(x^*)$, and

$$h(x) = \begin{cases} 1/g(x) & \text{if } g(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\nu \in H_0$ is such that $\mu-\nu \in H_0$. Then $\nu \leqslant \mu$, and hence $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$. Observing that $\operatorname{supp}\mu = T\cap S_0$ (Proposition 2.10(ii)), we have for any $\varphi \in C_c(S_0)$,

$$(5.1)\qquad \int_{S_0} \varphi(x)\,d\nu_{y^*}(x) = 0, \qquad \forall y \notin T^*.$$

Let $\hat{\nu} = h\cdot\nu$ and $\hat{\sigma} = \hat{g}\cdot\sigma$. For any $\varphi \in C_c(S_0)$,

$$(\hat{\nu}\bullet\hat{\sigma})(\varphi) = \int_S\Big(\int_{S_0} \varphi(x)\,d\hat{\nu}_{y^*}(x)\Big)\,d\hat{\sigma}(y) = \int_S\Big(\int_{S_0} \varphi(x)\,h(x+y^*)\,d\nu_{y^*}(x)\Big)\,d\hat{\sigma}(y)$$

$$= \int_S\Big(\int_{S_0} \varphi(x)\,h(x)\,d\nu_{y^*}(x)\Big)\,h(y^*)\,g(y^*)\,d\sigma(y)$$

$$= \int_{T^*}\Big(\int_{S_0} \varphi(x)\,h(x)\,d\nu_{y^*}(x)\Big)\,d\sigma(y)$$

$$= \int_S\Big(\int_{S_0} \varphi(x)\,h(x)\,d\nu_{y^*}(x)\Big)\,d\sigma(y) \qquad \text{(by (5.1))}$$

$$= \int_{S_0} \varphi(x)\,h(x)\,d(\nu\bullet\sigma)(x) = \int_{S_0} \varphi(x)\,d\hat{\nu}(x) = \hat{\nu}(\varphi),$$

i.e. $\hat{\nu}$ is $\hat{\sigma}$-harmonic. Moreover, Proposition 5.3 implies that

$$\hat{\sigma}(S) = \int_S g(y^*)\,d\sigma(y) = 1, \tag{5.2}$$

and hence $\hat{\sigma}$ is a probability measure with $S(\hat{\sigma}) = T^*$. Note that $\hat{\nu} = h\cdot\nu \leqslant h\cdot\mu$, and $h\cdot\mu$ equals a scalar multiple of the translation invariant measure ω restricted to T (Theorem 2.8), $\hat{\nu}$ is thus $\hat{\sigma}$-shift bounded. It follows from Theorem 4.6 that

$$\hat{\nu}_{x^*} = \hat{\nu}, \qquad \forall x\in S(\hat{\sigma}), \tag{5.3}$$

i.e. $\hat{\nu}_x = \hat{\nu}$ for all $x\in T$. By Theorem 2.8, $\hat{\nu} = c'\cdot\omega$ on T, and $\nu = g\cdot\hat{\nu} = c' g\cdot\omega = c_1\mu$. This shows that $\mu\in E_0(\sigma)$ is extreme, i.e., $\mu\in\partial H_0$.

We remark that the above inclusions may be proper in view of the following semigroups modified from Davies and Shanbhag [5]:

EXAMPLE 5.5. Let $S = S_1\cup S_2$, where $S_1 = \{0\}\times\mathbf{R}_+$, $S_2 = [1,\infty)\times\mathbf{R}_+$, and S has the usual addition and identity involution. Let $\sigma > 0$ be such that $\operatorname{supp}\sigma = S$, and $\sigma|_{S_1}$ is a probability measure. Let μ be the 1-dimensional Lebesgue measure on $\{(x, y)\in S\colon x = 3/2\}$, and $\mu = 0$ elsewhere. The fundamental ideal S_0 equals $(1,\infty)\times\mathbf{R}_+$ and $\mu\in\partial H_0$. However, the translation set of μ, $\{z\in S\colon \mu_z\neq 0\}$, equals S_1, which does not intersect S_0, hence $\mu\notin E_0(\sigma)$.

EXAMPLE 5.6. Let $S = \{(0, 0)\}\cup S_2$, where S_2 is as above, and let $\sigma\geqslant 0$ be such that $\operatorname{supp}\sigma = S$ and $\sigma\{(0,0)\} = 1$. Then any μ which has support contained in $\{(x, y)\in S\colon 1 < x < 2\}$ satisfies $\mu_z = 0$ if $z\neq(0,0)$, and hence $\mu\bullet\sigma = \mu$. Thus $\mu\in E(\sigma)$, but μ is not an extreme element of H_0 in general.

Let $\mathscr{E}(\sigma)$ be the set of nonnegative exponential functions g on S with

$$\int_S g(y^*)\,d\sigma(y) = 1,$$

and $g\not\equiv 0$ on S_0. $\mathscr{E}(\sigma)$ is endowed with the topology of uniform convergence on compact sets.

LEMMA 5.7. *The map $i\colon \mathbf{R}_+\times\mathscr{E}(\sigma)\to E_0(\sigma)$, $i(\alpha, g) = \alpha g\cdot\omega$, is a surjective homeomorphism.*

Proof. It follows from Theorem 5.2 that the map is a bijection. We need only show that the topologies on the two sets coincide.

Let $g_n\in\mathscr{E}(\sigma)$ and $\lim_{n\to\infty} g_n = g\in\mathscr{E}(\sigma)$. For any $\varphi\in C_c(S_0)$, the sequence $\{\varphi g_n\}$ and φg vanish outside $\operatorname{supp}\varphi$, and $\{\varphi g_n\}$ converges to φg uniformly. The bounded convergence theorem implies that

$$\lim_{n\to\infty} g_n\cdot\omega(\varphi) = g\cdot\omega(\varphi), \qquad \forall\varphi\in C_c(S_0).$$

Hence $g_n\cdot\omega \xrightarrow{w} g\cdot\omega$, and that i is continuous follows easily.

To show that i^{-1} is continuous, let $\alpha_n g_n\cdot\omega \xrightarrow{w} \alpha g\cdot\omega$. Then

$$\lim_{n\to\infty}(\alpha_n g_n\cdot\omega)_{y_n} = (\alpha g\cdot\omega)_y,$$

for any $\{y_n\}$, y in S with $y_n\to y$ (Theorem 2.5). It follows from

$$(g_n\cdot\omega)_{y_n} = g_n(y_n)\,g_n\cdot\omega \quad\text{and}\quad (g\cdot\omega)_y = g(y)\,g\cdot\omega$$

that $\lim_{n\to\infty} g_n(y_n) = g(y)$ for any sequence $\{y_n\}$ in S with $\lim_{n\to\infty} y_n = y$, and hence $\{g_n\}$ converges to g uniformly on compact sets. This implies that $g_n\cdot\omega\xrightarrow{w} g\cdot\omega$, and we have $\lim_{n\to\infty}\alpha_n = \alpha$.

THEOREM 5.8. *Suppose $S = S(\sigma)$. Then for $0\neq\mu\in M^+(S_0)$, $\mu\in H_0$ if and only if for each $y\in S_0$, $\mu_y = f_y\cdot\omega$ on S_0, where*

$$f_y = \int_{\mathscr{E}(\sigma)} g(y)\,g\,dP(g)$$

for some positive measure P on $\mathscr{E}(\sigma)$.

Proof. Let $\mu = H_0$. Since C_0 is metrizable and well capped (Proposition 4.3), the Choquet theorem implies that

$$\mu = \int_{\partial C_0} \nu\,dQ(\nu),$$

where ∂C_0 denotes the set of extreme rays of C_0, and Q a probability measure on ∂C_0. Since $\mu\in H_0$, Q actually has support contained in ∂H_0, and hence

$$\mu = \int_{\partial H_0} \nu\,dQ(\nu).$$

Note that $\partial H_0\subseteq E(\sigma)$ (Theorem 5.4), and for $\nu\in\partial H_0$ and $y\in S_0$, $\nu_y = 0$ if $\nu\notin E_0(\sigma)$. Hence Lemma 5.7 implies that

$$\mu_y = \int_{E_0(\sigma)} \nu_y \, dQ(\nu) = \Big(\int_{\mathbf{R}_+ \times \mathscr{E}(\sigma)} \alpha g(y)\, g \, dQ(\alpha g)\Big)\,\omega$$

$$= \int_{\mathscr{E}(\sigma)} g(y)\, g \Big(\int_{\mathbf{R}_+} \alpha\, dQ(\alpha g)\Big)\,\omega.$$

Let $dP(g) = \int_{\mathbf{R}_+} \alpha\, dQ(\alpha g)$. Then μ has the representation as claimed.

The converse of the statement is direct.

COROLLARY 5.9. *Suppose $S = S(\sigma)$, and S_0 is dense in S. Then*

$$E_0(\sigma) = \partial H_0 = E(\sigma),$$

and for $\mu \in M^+(S_0)$, $\mu \in H_0$ if and only if $\mu = f \cdot \omega$ on S_0, where

$$f = \int_{\mathscr{E}(\sigma)} g \, dP(g)$$

for some positive measure P on $\mathscr{E}(\sigma)$.

Proof. Theorem 5.4 implies that $E_0(\sigma) \subseteq \partial H_0 \subseteq E(\sigma)$. To prove the reverse inclusion, we let $\mu \in E(\sigma)$ with the associated exponential function g. Then

$$1 = \int_S g(y^*)\, d\sigma(y) = \int_{\bar{S}_0} g(y^*)\, d\sigma(y).$$

The continuity of g implies that $g(y) > 0$ for some $y \in S_0$, hence $\mu \in E_0(\sigma)$ by definition.

The second part follows from the same argument as in Theorem 5.8.

DEFINITION 5.10. Let R be a subsemigroup of S. R is said to have the *component generating property* if for each open and closed subsemigroup T of R, there exists a dense subset $D \subseteq S$ such that for any $x \in D$, there is $y \in T$ with $x + y \in R$.

We remark that if S is a subsemigroup of an abelian topological group G, the above property is equivalent to $(R - T) \cap S$ being dense in S.

EXAMPLE 5.11. Let $S = [0, \infty)$ and let R be a subsemigroup which does not contain 0, and is not contained in $\{ak\colon k = 0, 1, 2, \ldots\}$, for any $a > 0$. Then R has the component generating property.

EXAMPLE 5.12 [5]. Let $S = \mathbf{R}_+ \times \mathbf{R}_+$, and let $R = R_1 \cup R_2$, where $R_1 = \{0\} \times \mathbf{R}_+$, $R_2 = [1, \infty) \times \mathbf{R}_+$. Then $R - R_1$ does not contain a subset dense in S, and hence R does not have the component generating property.

Let R be a subsemigroup of S. A measure $\mu \in M^+(S_0)$ is called an *exponential measure with respect to R* if $\mu_y = g(y)\mu$ for some $g \geqslant 0$, and for all $y \in R$.

LEMMA 5.13. *Let R be a subsemigroup of S with the component generating property. Then every $\mu \in M^+(S_0)$ exponential with respect to R is exponential with respect to S.*

Proof. Let $\mu \in M^+(S_0)$ be an exponential measure with respect to R, and let $T = \{y \in R\colon \mu_y \neq 0\}$. The same proof as for Theorem 5.1 implies that T is a both open and closed subsemigroup of R. The component generating property yields a dense subset $D \subseteq S$ such that for any $x \in D$, there exists $y \in T$ with $x + y \in R$. Hence

$$g(x+y)\,\mu = \mu_{x+y} = g(y)\,\mu_x,$$

with $g(y) \neq 0$. Note that $g(x+y)/g(y)$ is independent of $y \in T$, define it to be $g(x)$. By the continuity of $x \to \mu_x$ (Theorem 2.5), g can be extended to S, and the conclusion follows.

THEOREM 5.14. *Suppose $S(\sigma) \subseteq \bar{S}_0$, $S(\sigma)$ is closed under involution, and has the component generating property in S. Then*

$$E_0(\sigma) = \partial H_0 = E(\sigma),$$

and for $\mu \in M^+(S_0)$, $\mu \in H_0$ if and only if $\mu = f \cdot \omega$ on S_0, where

$$f = \int_{\mathscr{E}(\sigma)} g \, dP(g)$$

for some positive measure P on $\mathscr{E}(\sigma)$.

Proof. Let $\mu \in E_0(\sigma)$. We can use the same proof as in Theorem 5.4 to show that $\mu \in \partial H_0$. The only change is in (5.2) and (5.3): $\hat{\sigma}$ will be a probability measure supported by $S(\sigma) \cap T^*$, and hence

$$\hat{\nu}_x = \nu, \qquad \forall x \in S(\sigma)^* \cap T = S(\sigma) \cap T.$$

The component generating property implies that $\hat{\nu}_x = \hat{\nu}$ for all $x \in T$.

Let $\mu \in \partial H_0$. Then Theorem 5.4 implies that μ is a σ-harmonic exponential measure with respect to $S(\sigma)$, and hence with respect to S by Lemma 5.13.

To show that $E_0(\sigma) = E(\sigma)$, we let $\mu \in E(\sigma)$. If $\mu_y = 0$ for all $y \in S_0$, then since $S(\sigma) \subseteq \bar{S}_0$, $\mu_y = 0$ for $[\sigma]$-almost all $y = S$. This implies that

$$\mu(E) = \mu \bullet \sigma(E) = \int_S \int_{S_0} \chi_E(x)\, d\mu_{y^*}(x)\, d\sigma(y) = 0,$$

which is a contradiction. Hence $\mu_y \neq 0$ for some $y \in S_0$, i.e. $\mu \in E_0(\sigma)$, and $E_0(\sigma) \supseteq E(\sigma)$. The integral representation follows readily.

To conclude this section, we will consider a special case for

$$\mu \bullet \sigma = \mu, \tag{5.4}$$

where $\mu \in M^+(S)$ instead of $M^+(S_0)$ as in the previous theorems.

THEOREM 5.15. *Suppose* $S = S(\sigma)$, *and* $\sigma(S \setminus S_0) = 0$. *Then the solution* $\mu \in M^+(S)$ *of* (5.4) *has the representation* $\mu = f \cdot \omega$ *on* S, *where* $f = \int_{\mathscr{E}(\sigma)} g\, dP(g)$ *for some positive measure* P *on* $\mathscr{E}(\sigma)$.

Proof. Let μ_1 be the restriction of μ to S_0. Then (5.4) will be reduced to $\mu_1 \bullet \sigma = \mu_1$ on S_0. By taking $S = S_0$ in Corollary 5.9, we see $\mu_1 = \mu|_{S_0}$ is of the required form on S_0. For $E \subseteq S \setminus S_0$,

$$\mu(E) = \int_S \mu(y^* + E)\, d\sigma(y) = \Big[\int_{\mathscr{E}(\sigma)} \Big(\int_{S_0} g(y^*)\, d\sigma(y) \Big) g\, dP(g) \Big]\, \omega(y^* + E)$$

$$= \Big(\int_{\mathscr{E}(\sigma)} g\, dP(g) \Big) \cdot \omega(E)$$

(the third equality follows from S_0 being an ideal), and hence the result follows.

§ 6. The functional equation and examples. In this section, we will consider the functional equation

$$f(x) = \int_S f(x+y)\, d\sigma(y), \qquad \forall x \in S, \tag{6.1}$$

where $\sigma \in M^+(S)$ is given, and $f \geqslant 0$ is to be determined.

THEOREM 6.1. *Suppose* $S_0 \neq \emptyset$, *and* $S(\sigma) = S$. *If* f *is a nonnegative* $[\omega]$*-locally integrable solution of* (6.1), *then for any* $y \in S_0$,

$$f(\cdot + y) = \int_{\mathscr{E}(\sigma)} g(y)\, g\, dP(g), \qquad [\omega]\text{-a.e. on } S_0,$$

where P *is a positive measure on* $\mathscr{E}(\sigma)$.

Proof. Let f_0 be the restriction of f to S_0, and let $\mu = f_0 \cdot \omega$. Then $\mu \in M^+(S_0)$ and $\mu \bullet \sigma = \mu$. It follows from Theorem 5.8 that f has the desired form.

As consequences of Theorems 5.14 and 5.15, and Corollary 5.9, we have

THEOREM 6.2. *Suppose* $S_0 \neq \emptyset$, $S(\sigma) = S$, *and* $f \geqslant 0$ *is a solution of* (6.1). *If, further, either* (i) S_0 *is dense in* S, *and* f *is continuous, or* (ii) $\sigma(S \setminus S_0) = 0$, *and* f *is* $[\omega]$*-locally integrable, then* $f = \int_{\mathscr{E}(\sigma)} g\, dP(g)$, $[\omega]$*-a.e. on* S, *where* P *is a positive measure on* $\mathscr{E}(\sigma)$.

THEOREM 6.3. *Suppose* $S_0 \neq \emptyset$, $S(\sigma) \subseteq \overline{S}_0$, $S(\sigma)$ *is closed under involution, and has the component generating property in* S. *If* $f \geqslant 0$ *is a continuous solution of* (6.1), *then* $f = \int_{\mathscr{E}(\sigma)} g\, dP(g)$ *on* S, *where* P *is a positive measure on* $\mathscr{E}(\sigma)$.

The following proposition is also useful in reducing equation (6.1) to a simpler one.

PROPOSITION 6.4. *Let* R *be a subsemigroup of* S *which is closed under involution and* $S(\sigma) \subseteq R$. *Suppose there exists* $D \subseteq S$ *such that* $\{s + R \colon s \in D\}$ *is a disjoint family of sets whose union is* S. *Then the nonnegative continuous solution* f *of equation* (6.1) *is of the form*

$$f(s+x) = g_s(x), \qquad s \in D,\ x \in R,$$

where $g_s \geqslant 0$ *is continuous and satisfies* (6.1) *on* R.

We will conclude this section by demonstrating the above results with some special cases.

EXAMPLE 6.5 [12]. Let $S = \mathbf{R}_+$ with the identity involution. Suppose $f \geqslant 0$ is a continuous solution of (6.1), where σ is a nonnegative Borel measure. Then $f(x) = p(x) e^{-\alpha x}$, where $\alpha \in \mathbf{R}$ satisfies $\int_0^\infty e^{-\alpha x}\, d\sigma(x) = 1$, and p is a periodic function with periods $\varrho \in \operatorname{supp} \sigma$. In fact, let $R = [0, \infty) \cap T$ where T is the group generated by $S(\sigma)$. Then R satisfies the conditions in Proposition 6.4 (take $D = \{0\}$ if $T = \mathbf{R}$, and $D = [0, d)$ if $T = d\mathbf{Z}$ for some $d > 0$), and the result follows from Theorem 6.3 by taking R as the semigroup.

EXAMPLE 6.6. Let $S = \mathbf{R}_+ \times \mathbf{R}_+$ with the identity involution. If $\sigma(S \setminus S_0) = 0$, and there exist two nonparallel line segments L_1, L_2 in S such that $S(\sigma) \cap L_i$, $i = 1, 2$, are nonlattice subsets, then Theorem 6.3 implies that the nonnegative solution f has the integral representation

$$f(x, y) = \int_A e^{-(\alpha_1 x + \alpha_2 y)}\, d\tau(\alpha_1, \alpha_2), \qquad [\omega]\text{-a.e. on } S,$$

where A is the set of (α_1, α_2) in $\mathbf{R} \times \mathbf{R}$ such that

$$\int_0^\infty \int_0^\infty e^{-(\alpha_1 s + \alpha_2 t)}\, d\sigma(s, t) = 1,$$

and τ is a positive measure on A.

EXAMPLE 6.7. Let S and the equation be as in Example 6.6, and assume that $S(\sigma) \subseteq \{(x, x) \colon x \geqslant 0\}$. We will let $R = S(\sigma)$, and $D = \{(x, 0) \colon x \geqslant 0\} \cup \{(0, y) \colon y > 0\}$. By Proposition 6.4, the nonnegative continuous solution f is of the form

$$f(x, y) = \begin{cases} p_1(x-y)\, e^{-\alpha y}, & x \geqslant y \geqslant 0, \\ p_2(y-x)\, e^{-\alpha x}, & y \geqslant x \geqslant 0, \end{cases}$$

where $\int_0^\infty e^{-\alpha t}\, d\sigma(t) = 1$, and p_1, p_2 are continuous functions on $\mathbf{R}_+$ with $p_1(0) = p_2(0)$.

This form has been used extensively in generalizing exponential functions to higher dimensions in connection with reliability theory (see, e.g., [9] and [14]).

EXAMPLE 6.8. Let $S = S_1 \cup S_2$, where $S_1 = \{0\} \times \mathbf{R}_+$, $S_2 = [1, \infty) \times \mathbf{R}_+$, and let σ be a regular Borel measure in S such that $\operatorname{supp} \sigma = S_1 \cup S_2$. In this case, the fundamental ideal S_0 equals $(1, \infty) \times \mathbf{R}_+$. The exponential functions g are of the form

$$g(x, y) = e^{-(\alpha x + \beta y)}, \quad \forall (x, y) \in S,$$

or

$$g(x, y) = \begin{cases} e^{-\beta y}, & (x, y) \in S_1, \\ 0, & (x, y) \in S_2. \end{cases}$$

Hence, $\mathscr{E}(\sigma)$ is the set of g such that

$$g(x, y) = e^{-(\alpha x + \beta y)}, \quad \forall (x, y) \in S,$$

and satisfies

$$\int_{S_1} e^{-\beta y} \, d\sigma(0, y) + \int_{S_2} e^{-(\alpha x + \beta y)} \, d\sigma(x, y) = 1. \tag{6.2}$$

By Theorem 6.1, the locally integrable solutions f are of the form

$$f(x, y) = \int_A e^{-(\alpha x + \beta y)} \, d\tau(\alpha, \beta), \quad [\omega]\text{-a.e. on } [2, \infty] \times \mathbf{R}_+,$$

for some nonnegative measure τ on A, where A is the set of (α, β) in $\mathbf{R} \times \mathbf{R}$ such that (6.2) holds. A further observation shows that the above expression can be extended to $(x, y) \in [1, \infty) \times \mathbf{R}_+$. It can hence be shown that the continuous solutions f can be represented as

$$f(x, y) = p(x) e^{-\gamma y} + \int_A e^{-(\alpha x + \beta y)} \, d\tau(\alpha, \beta), \quad \forall (x, y) \in S, \tag{6.3}$$

where

$$p(x) = \begin{cases} c, & x = 0, \\ 0, & x \neq 0, \end{cases} \quad \int_0^\infty e^{-\gamma t} \, d\sigma(0, t) = 1.$$

EXAMPLE 6.9. Let $S = \mathbf{R}_+ \times \mathbf{R}_+$, and let $\sigma \geqslant 0$ be such that $\operatorname{supp} \sigma = S_1 \cup S_2$, where $S_1 = \{0\} \times \mathbf{R}_+$, $S_2 = [1, \infty) \times \mathbf{R}_+$ as in Example 6.8. By restricting the integral equation (6.1) to $\operatorname{supp} \sigma$, we see that for a continuous solution f, $f|_{S_1 \cup S_2}$ has the representation (6.3) on $S_1 \cup S_2$. By a simple argument, we can extend the domain of the solution and obtain

$$f(x, y) = p(x) e^{-\gamma y} + \int_A e^{-(\alpha x + \beta y)} \, d\tau(\alpha, \beta), \quad \forall (x, y) \in S,$$

where p is a continuous function with $p(x) = 0$ for $x > 1$, $\int_0^\infty e^{-\gamma t} \, d\sigma(0, t) = 1$, and A, τ are defined as in Example 6.8.

References

[1] C. Berg, J. P. R. Christensen and P. Ressel, *Harmonic Analysis on Semigroups*, Springer, New York 1984.
[2] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.
[3] G. Choquet, *Lectures on Analysis*, Vol. I, II, Benjamin, New York 1969.
[4] G. Choquet et J. Deny, *Sur l'équation de convolution $\mu = \mu * \sigma$*, C. R. Acad. Sci. Paris 250 (1960), 799–801.
[5] P. L. Davies and D. N. Shanbhag, *A generalization of a theorem of Deny with application in characterization theory*, Quart. J. Math. Oxford (2) 38 (1987), 13–34.
[6] J. Deny, *Sur l'équation de convolution $\mu * \sigma = \mu$*, Sémin. Théor. Potent. M. Brelat, Fac. Sci. Paris, 4 ann., 1960.
[7] J. L. Doob, J. L. Snell and R. E. Williamson, *Applications of boundary theory to the sum of independent random variables*, Contributions to Probability and Statistics, Stanford, California, 1960.
[8] W. Feller, *An Introduction to Probability Theory and its Applications*, Wiley, New York 1966.
[9] S. G. Ghurye and A. W. Marshall, *Shock processes with after-effects and multivariate lack of memory*, J. Appl. Probab. 21 (1984), 786–801.
[10] P. Halmos, *Measure Theory*, van Nostrand, Princeton 1950.
[11] A. M. Kagan, Y. Linnik and C. R. Rao, *Characterization Problems in Mathematical Statistics*, Wiley, New York 1965.
[12] K. S. Lau and C. R. Rao, *Integrated Cauchy functional equation and characterization of the exponential law*, Sankhyā Ser. A 44 (1982), 72–90.
[13] K. S. Lau and W. B. Zeng, *Embedding of locally compact semigroups*, submitted.
[14] A. W. Marshall and I. Olkin, *A multivariate exponential distribution*, Ann. Probab. 3 (1964), 352–354.
[15] P. A. Meyer, *Probability and Potentials*, Blaisdell, London 1966.
[16] R. R. Phelps, *Lectures on Choquet's Theorem*, van Nostrand, Princeton 1966.
[17] C. R. Rao and D. N. Shanbhag, *Recent results on characterization of probability distributions: a unified approach through extension of Deny's theorem*, Adv. Appl. Probab. 18 (1979), 660–678.

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