Step 4: The polynomial p "vanishes" in (3). Since p and g have no common zero in K the second step implies the existence of $k,\ v\in A,\ v\in A^{-1}$ such that $p+kg = v$. Putting this in (3) yields

$$f+Hg = su\frac{1}{v} \tag{4}$$

with $H\in A$ chosen appropriately.

Step 5: The polynomial s in (4) "vanishes". By this equation s and g can have no common zero in K. (Such a zero would be a common zero of f and g, contradicting $\alpha f+\beta g = 1$.) Now the second step implies the existence of $h,\ w\in A,\ w\in A^{-1}$ such that $s+lg = w$. Together with (4) this yields

$$f+\left(H+lu\frac{1}{v}\right)g = u\frac{1}{v}w.$$

Since the right-hand side is invertible, we are done. ∎

Note that there exist compact sets K with $R(\partial K) = C(\partial K)$, but $A(K)\neq R(K)$; see [12], p. 72, Example 9.8.

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Remarks on singular convolution operators

by

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Abstract. We prove some endpoint estimates for singular convolution operators. For example, let m be a bounded function such that for some suitable $C_0^\infty(\mathbf{R}^n\setminus\{0\})$-function φ, $\varphi m(t\cdot)$ is in the Besov space $B^q_{\alpha 1}$, uniformly in $t>0$. Then m is a Fourier multiplier on $L^p(\mathbf{R}^n)$ if $\alpha = n(1/p-1/2) > n/q$, $1<p\leqslant 2$, and on H^1 if $\alpha = n/2$, $2<q\leqslant\infty$. If m is radial we may replace $B^q_{n/2,1}$ by $B^2_{n/2,1}$.

1. Introduction. The purpose of this paper is to prove endpoint estimates for some classes of multiplier transformations on $L^p(\mathbf{R}^n)$ and other function spaces. We consider a convolution operator T, defined by $Tf = \mathscr{F}^{-1}[m\mathscr{F}f]$, where $\mathscr{F}f$ (or $\hat{f}$) denotes the Fourier transform of f. The M^p-multiplier norm of m is defined as the norm of T as a bounded operator on $L^p(\mathbf{R}^n)$.

To formulate a theorem let us introduce some notation. Let $\varphi\in C_0^\infty(\mathbf{R}^n)$ be supported in $\{\xi;\ 1/4<|\xi|<4\}$ and positive in $\{\xi;\ 1/2<|\xi|<2\}$. Let $\psi\in C_0^\infty(\mathbf{R}^n)$ be supported in $\{x;\ |x|\leqslant 1\}$ and be equal to 1 in $\{x;\ |x|\leqslant 1/2\}$. Define $\Psi_l(x) = \psi(2^{-l}x)-\psi(2^{-l+1}x)$. Then Ψ_l is supported in $\{x;\ 2^{l-2}\leqslant |x|\leqslant 2^l\}$. Ψ_l will be used to decompose the convolution kernel $\mathscr{F}^{-1}[m]$.

THEOREM 1.1. *Suppose that* $1<p<r\leqslant 2$ *and*

$$\|m\|_\infty + \sup_{t>0}\sum_{l>0} 2^{ln(1/p-1/r)}\,\|\varphi m(t\cdot)*\hat{\Psi}_l\|_{M^r}\leqslant A. \tag{1.1}$$

Then m is a multiplier in M^p, and $\|m\|_{M^p}\leqslant cA$. If $p=1<r\leqslant 2$ we have the conclusion that T is a bounded operator on the Hardy space H^1.

This result can be considered as an endpoint version of Hörmander's multiplier theorem [16] (compare (1.2) below). It extends a result by Baernstein and Sawyer [1] who proved that the condition

$$\|m\|_\infty^p + \sup_{t>0}\sum_{l>0} 2^{ln(1-p)}\,\|\varphi m(t\cdot)*\hat{\Psi}_l\|_1^p<\infty$$

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implies that T is a bounded operator on H^p, provided $0 < p < 1$. Theorem 1.1 is also closely related to recent results of A. Carbery [4] and the author [22] on L^p-variants of Calderón–Zygmund theory (see also [6], [7]). While those results can be applied to operators which are bounded on L^p in some open range (p_0, p_0') but not in $(1, p_0]$, our present result can be considered as a variant which is more suitable for endpoint estimates. Specifically, let $B^q_{\alpha 1}(\mathbf{R}^n)$, $\alpha > 0$, be the Besov space with norm

$$\|g\|_{B^q_{\alpha 1}} = \|g\|_q + \sum_{l>0} 2^{l\alpha} \|g * \hat{\Psi}_l\|_q .$$

Then we have the following multiplier criterion:

COROLLARY 1.2. *Suppose* $1 < p \leqslant 2$, $\alpha = n(1/p-1/2) > n/q$. *Then*

$$\|m\|_{M^p} \leqslant c \sup_{t>0} \|\varphi m(t\cdot)\|_{B^q_{\alpha 1}}. \tag{1.2}$$

For $2 < q \leqslant \infty$ *one has*

$$\|m\|_{M(H^1)} \leqslant c \sup_{t>0} \|\varphi m(t\cdot)\|_{B^q_{n/2,1}} \tag{1.3}$$

(*where, of course,* $M(H^1)$ *denotes the multiplier norm on* H^1).

Corollary 1.2 covers the sharp estimates for multipliers like $e^{i|\xi|^a}(1+|\xi|^2)^{-b/2}$ (see Fefferman and Stein [12], Miyachi [16] and Baernstein and Sawyer [1]). Apparently Corollary 1.2 (and Corollary 1.3 below) cannot be obtained by interpolation with known results. In [1] it is shown that (1.3) does not hold with $B^q_{n/2,1}$ replaced by $B^2_{n/2,1}$. A substitute involving $B^2_{n/2,1}$ is contained in [23]. Somewhat different results have been obtained by Besov [2], Carbery [3] and Lizorkin [17]. However, for radial multipliers $m = h(|\cdot|)$ a stronger result is true. If φ_0 is defined similarly to φ above, but now as a function on $\mathbf{R}$, we have

COROLLARY 1.3. *Suppose* $n \geqslant 2$, $1 < p < 2(n+1)/(n+3)$, $\alpha = n(1/p-1/2)$. *Then*

$$\|h(|\cdot|)\|_{M^p} \leqslant c \sup_{t>0} \|\varphi_0 h(t\cdot)\|_{B^2_{\alpha 1}(\mathbf{R})}, \tag{1.4}$$

$$\|h(|\cdot|)\|_{M(H^1)} \leqslant c \sup_{t>0} \|\varphi_0 h(t\cdot)\|_{B^2_{n/2,1}(\mathbf{R})}. \tag{1.5}$$

Corollary 1.3 improves some multiplier estimates for radial multipliers in higher dimensions due to M. Christ [9] and the author [21] (see also Carbery, Gasper and Trebels [5]). A counterexample of A. Miyachi [18] for a radial multiplier shows that neither (1.3) nor (1.5) imply a weak-type (1-1)-estimate. M. Frazier has asked whether the results in [1] have analogues in Triebel–

Lizorkin spaces. The answer is yes and we prove Theorem 1.1 and another generalization of a result in [1] in this slightly more general context.

Section 2 contains some preliminary facts about Triebel–Lizorkin spaces, the statement of multiplier theorems in these spaces and a technical lemma which is not needed for the proof of Theorem 1.1. In Sections 3 and 4 we prove these theorems and in Section 5 we give the short proof of the Corollaries.

Since sharp function estimates are important in Section 4, we give a characterization of $\dot{F}^{pq}$-spaces in terms of sharp functions as an appendix (Section 6), including a variant of the John–Nirenberg inequality.

In the following we shall always denote by c an abstract constant which may depend on the dimension n and the values of p, q above, unless otherwise stated. The precise value of c may change from line to line. For $p \in [1, \infty]$ the conjugate index p' is defined by $1/p+1/p' = 1$.

2. Multipliers on Triebel–Lizorkin spaces. We first recall the definition of the homogeneous Besov and Triebel–Lizorkin spaces (see Triebel [26]). Let φ be defined as in Section 1 and let R_k denote the operator defined by convolution with $\mathscr{F}^{-1}[\varphi(2^k\cdot)]$.

For $\alpha \in \mathbf{R}$, $0 < p, q \leqslant \infty$ the *homogeneous Besov space* $\dot{B}^p_{\alpha q}$ is defined as a subspace of S'/P (tempered distributions modulo polynomials) and the norm is given by

$$\|f\|_{\dot{B}^p_{\alpha q}} = \|\{2^{k\alpha} R_k f\}\|_{l^q(L^p)}.$$

Similarly, for $\alpha \in \mathbf{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$ the *homogeneous Triebel–Lizorkin space* $\dot{F}^{pq}_\alpha$ is defined as the space of all $f \in S'/P$ such that

$$\|f\|_{\dot{F}^{pq}_\alpha} = \|\{2^{k\alpha} R_{-k} f\}\|_{L^p(l^q)}$$

is finite. In particular, recall $\dot{F}^{p2}_0 = L^p$, $1 < p < \infty$, $\dot{F}^{p2}_0 = H^p$, $0 < p \leqslant 1$. In [15], Frazier and Jawerth have shown that a reasonable definition of $\dot{F}^{\infty q}_\alpha$ is obtain by setting

$$\|f\|_{\dot{F}^{\infty q}_\alpha} = \sup_Q \Big(|Q|^{-1} \int_Q \sum_{k \leqslant L(Q)} |2^{k\alpha} R_{-k} f(y)|^q \, dy\Big)^{1/q},$$

where the supremum is taken over all cubes Q and $L(Q) = \log_2$(sidelength of Q).

By $M(\dot{B}^p)$ and $M(\dot{F}^{pq})$ we denote the spaces of Fourier multipliers on the homogeneous Besov and Triebel–Lizorkin spaces with the usual operator norm. The notation is justified, because the space of multipliers on $\dot{B}^p_{\alpha q}$ does not depend on α and q and the space of multipliers on $\dot{F}^{pq}_\alpha$ does not depend on α (see [26, pp. 120 ff., 241]).

Our first theorem extends the H^1 result by Baernstein and Sawyer [1].

THEOREM 2.1. *Suppose* $0 < p, q \leqslant \infty$. *Let* w *be an increasing positive function such that*

$$\sum_{l>0} w(l)^{-pq/(p-q)} \leqslant 1. \tag{2.1}$$

Let

$$A(p, w, \sigma) = \Big(\sup_{t>0} \sum_{l>0} \|\varphi m(t\cdot) * \hat{\Psi}_l\|^{\sigma}_{M(\dot{B}^p)} w(l)^{\sigma}\Big)^{1/\sigma}, \tag{2.2}$$

$$D(q, \sigma) = \Big(\sup_{t>0} \sum_{l>0} \|\varphi m(t\cdot) * \hat{\Psi}_l\|^{\sigma}_{M(\dot{B}^q)}\Big)^{1/\sigma}. \tag{2.3}$$

Then

(a) $\|m\|_{M(\dot{F}^{pq})} + \|m\|_{M(\dot{F}^{p'q'})} \leqslant c[\|m\|_\infty + A(p, w, 1)]$ *if* $1 < p \leqslant q \leqslant p'$,

(b) $\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + A(p, w, p)]$ *if* $0 < p \leqslant 1$, $p \leqslant q \leqslant \infty$,

(c) $\|m\|_{M(\dot{F}^{pq})} + \|m\|_{M(\dot{F}^{p'q'})} \leqslant c[\|m\|_\infty + A(p, w, 1) + D(q, 1)]$
if $1 < q \leqslant p \leqslant p'$,

(d) $\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + A(p, w, 1) + D(q, q)]$ *if* $0 < q \leqslant 1$, $1 \leqslant p \leqslant \infty$,

(e) $\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + A(p, w, p) + D(q, q)]$ *if* $0 < q < p \leqslant 1$.

Remark. A variant for $1 < p < q < p'$ already appears in [22], which reads in our notation

$$\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + \sum_{l>0} \sup_{t>0} \|\varphi m(t\cdot) * \hat{\Psi}_l\| \, l^{|1/p - 1/q|}].$$

The second theorem generalizes Theorem 1.1.

THEOREM 2.2. *Let* $0 < p, q, r \leqslant \infty$, $\sigma \leqslant 1$ *and*

$$B(p, r, \sigma) = \Big(\sup_{t>0} \sum_{l>0} 2^{ln|1/p - 1/r|\sigma} \|\varphi m(t\cdot) * \hat{\Psi}_l\|^{\sigma}_{M(\dot{B}^r)}\Big)^{1/\sigma} \tag{2.4}$$

and $D(q, \sigma)$ *as in* (2.3). *Then we have the estimates*

(a) $\|m\|_{M(\dot{F}^{pq})} + \|m\|_{M(\dot{F}^{p'q'})} \leqslant c[\|m\|_\infty + B(p, r, 1)]$ *if* $1 \leqslant p < r \leqslant q \leqslant p'$,

(b) $\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + B(p, r, p)]$ *if* $0 < p \leqslant 1$, $p < r \leqslant q \leqslant \infty$,

(c) $\|m\|_{M(\dot{F}^{pq})} + \|m\|_{M(\dot{F}^{p'q'})} \leqslant c[\|m\|_\infty + B(p, r, 1) + D(q, 1)]$
if $1 < q < r < p \leqslant p'$,

(d) $\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + B(p, r, 1) + D(q, q)]$ *if* $0 < q \leqslant 1$, $1 \leqslant p \leqslant \infty$,

(e) $\|m\|_{M(\dot{F}^{pq})} \leqslant c[\|m\|_\infty + B(p, r, p) + D(q, q)]$ *if* $0 < q \leqslant r < p \leqslant 1$.

Since for $1 \leqslant p \leqslant \infty$, $M^p \subset M(\dot{B}^p)$, we recover Theorem 1 and H^p results ($p < r = 1$) of [1] if we take $q = 2$.

The formulation involving $M(\dot{B}^r)$ instead of M_r ($1 \leqslant r \leqslant \infty$) seems to be more natural since it gives a unified theorem for all values of p, q, r. We finish this section with two technical lemmas which explain that for $1 \leqslant r \leqslant \infty$ the $M(\dot{B}^r)$-condition on $\varphi m(t\cdot) * \hat{\Psi}_l$ is not really more general than the M_r-condition. Neither lemma is required for the proof of Theorem 1.1.

First we collect some error estimates which were used in Carbery [4]. For a convolution operator T with kernel K we denote by T^l the operator with kernel

$$K^l = K\Psi_l. \tag{2.5}$$

Then we have

LEMMA 2.3 (Carbery [4]). *Let* T *be a convolution operator with kernel* $K = \mathscr{F}^{-1}[m]$. *Then the kernel of* $R_{j+k}(TR_{j+k})^{j+l}$ *is* $2^{-(j+k)n} K_{ijkl}(2^{-j-k}\cdot)$, *where for each fixed* ϱ *and* N, $(1+|x|)^{\varrho} |K_{ijkl}(x)|$ *is bounded by*

(a) $c_{\varrho,N} 2^{(k-i)n} (1+2^{l-k})^{-N} \|m\|_\infty$ *if* $i > k+5$,

(b) $c_{\varrho,N} 2^{(i-k)n} 2^{(i-l)N} \|m\|_\infty$ *if* $i < k-5$, $i \leqslant l$,

(c) $c_{\varrho} 2^{(l-i)n} \|m\|_\infty$ *if* $i > l$.

The same bounds are obtained for the norm of $R_{j+k}(TR_{j+k})^{j+l}$ *as a bounded operator on* $\dot{B}^p_{0p}$ $(0 < p < \infty)$.

The proof consists in a straightforward analysis of

$$\hat{K}_{ijkl}(\xi) = \varphi(\xi) \int \varphi(2^{i-k} y)\, m(2^{-j-k} y)\, \hat{\Psi}_{l-k}(\xi - y)\, dy$$

and its derivatives (see [4]).

As a consequence of Lemma 2.3 we have

LEMMA 2.4. *Suppose* m *is supported in* $\{\xi;\ 1/4 \leqslant |\xi| \leqslant 4\}$. *Then for* $l > 0$, $0 < p, q \leqslant \infty$, $p \leqslant r$, *we have the inequality*

$$\|\mathscr{F}^{-1}[(m * \hat{\Psi}_l)\hat{f}]\|_{\dot{F}^{pq}_0} \leqslant c_{pqN}[2^{ln(1/p - 1/r)} \|m * \hat{\Psi}_l\|_{M(\dot{B}^r)} + 2^{-lN} \|m\|_\infty] \, \|f\|_{\dot{B}^p_{0\infty}}.$$

Before proving this lemma we have to mention one more important tool, the Calderón decomposition formula. The following version is proved in [14]. For each $M \in \mathbf{N}$ one can choose $\eta, \varphi \in S$ with the following properties:

(i) $\hat{\eta}$ has compact support in $\{x;\ |x| \leqslant 10^{-n}\}$ and

$$\int \hat{\eta}(x) x^{\alpha}\, dx = 0, \quad |\alpha| \leqslant M, \tag{2.6}$$

(ii) φ has compact support in $\{\xi;\ 1/2 < |\xi| < 2\}$,

(iii) $\sum_{k\in\mathbf{Z}} \eta(2^k \xi)\, \varphi(2^k \xi) = 1$, $\xi \neq 0$.

If we define $[P_k f]^\wedge = \eta(2^k\cdot)\hat{f}$, $[R_k f]^\wedge = \varphi(2^k\cdot)\hat{f}$ then

$$\sum_{k\in\mathbf{Z}} P_k R_k f = f \tag{2.7}$$

where the convergence is in S'/P and also in $\dot{F}^{pq}_{\alpha}$. A few more notations: Let $\tilde{\varphi}$

be a function compactly supported in $\{\xi;\ 1/4 \leqslant |\xi| \leqslant 4\}$ which equals 1 in $\{\xi;\ 1/2 \leqslant |\xi| \leqslant 2\}$. Then we define $[\tilde{R}_k f]^\wedge = \tilde{\varphi}(2^k \xi)\hat{f}(\xi)$, hence we have

$$\tilde{R}_k R_k = R_k. \tag{2.8}$$

We can use R_k or $\tilde{R}_k$ in the definition of Besov and Triebel–Lizorkin spaces.

Finally, if $Q_0 = \{x;\ 0 \leqslant x_i < 1,\ i = 1, \ldots, n\}$ is the unit cube, then for $l \in \mathbf{Z}$, $\gamma \in \mathbf{Z}^n$ let

$$Q_{l,\gamma} = \{x;\ \gamma - 2^{-l} x \in Q_0\}. \tag{2.9}$$

For fixed l the $Q_{l,\gamma}$ clearly form a mesh of cubes of sidelength 2^l. We denote the characteristic function of $Q_{l,\gamma}$ by $\chi_{l,\gamma}$, hence $\sum_\gamma \chi_{l,\gamma} = 1$.

Proof of Lemma 2.4. We assume $p < \infty$ and leave the analogous proof for $p = \infty$ to the reader. We first estimate $\|T^l P_k R_k f\|_p$ if $|k| \leqslant 10$. Since the convolution kernel of P_k ($|k| \leqslant 10$) is supported in the unit cube, we have by Hölder's inequality

$$\|T^l P_k R_k f\|_p^p \leqslant c \sum_{\sigma \in \mathbf{Z}^n} \|T^l P_k (R_k f) \chi_{l,\sigma}\|_r^p\, 2^{ln(1-p/r)} \tag{2.10}$$

$$\leqslant c \sum_{\gamma \in \mathbf{Z}^n} \|T^l P_k (R_k f) \chi_{0,\gamma}\|_r^p\, 2^{ln(1-p/r)}.$$

For the second inequality simply write $\chi_{l,\sigma} = \sum_{Q_{0,\gamma} \subset Q_{l,\sigma}} \chi_{0,\gamma}$ and use the embedding $l^{p/r} \subset l^1$. We introduce one more Littlewood–Paley decomposition and estimate

$$\sum_\gamma \|P_{k+\nu} R_{k+\nu} T^l P_k [(R_k f)\chi_{0,\gamma}]\|_r^p \leqslant c \|R_{k+\nu} T^l\|_{\dot{B}^r \to \dot{B}^r}^p \sum_\gamma \|P_{k+\nu} \tilde{R}_{k+\nu} P_k [(R_k f)\chi_{0,\gamma}]\|_r^p$$

where $\dot{B}^r \to \dot{B}^r$ indicates that we take the operator norm in $\dot{B}^r_{0r}$.

Since all moments of $\hat{\varphi}$ and all moments up to order M of $\hat{\eta}$ vanish, it is easy to see (by Taylor's formula) that the convolution kernel $H_{k,\nu}$ of $P_{k+\nu}\tilde{R}_{k+\nu}P_k$ is bounded by

$$c_{M,N} 2^{-\nu M} 2^{-(k+\nu)n} (1 + 2^{-(k+\nu)}|x|)^{-N} \tag{2.11}$$

if $\nu \geqslant 0$, and by

$$c_{K,N} 2^{\nu K} 2^{-kn} (1 + 2^{-k}|x|)^{-N} \tag{2.12}$$

if $\nu \leqslant 0$; here $K, N = 0, 1, 2, \ldots$

We have the estimate

$$\sum_\gamma \|P_{k+\nu}\tilde{R}_{k+\nu} P_k (R_k f)\chi_{0,\gamma}\|_r^p \tag{2.13}$$

$$\leqslant c \sum_\gamma \Big(\int \Big[\int_{Q_{0,\gamma}} |H_{k,\nu}(x-y)|\,dy\Big]^r dx\Big)^{p/r} \Big[\sup_{z \in Q_{0,\gamma}} |R_k f(z)|\Big]^p.$$

It is straightforward to check that

$$\Big(\int \Big[\int_{Q_{0,\gamma}} |H_{k,\nu}(x-y)|\,dy\Big]^r dx\Big)^{1/r}$$

is bounded by $2^{-\nu M}(1 + 2^{(k+\nu)n(-1+1/r)})$ if $\nu \geqslant 0$ and by $2^{\nu K}(1 + 2^{kn(-1+1/r)})$ if $\nu \leqslant 0$.

For $|k| \leqslant 10$ we may use the Plancherel–Pólya inequality (see Triebel [26, p. 19]) to get

$$\sum_\gamma \sup_{z \in Q_{0,\gamma}} |R_k f(z)|^p \leqslant c \|R_k f\|_p^p. \tag{2.14}$$

Clearly $\|R_{k+\nu} T^l\|_{\dot{B}^r \to \dot{B}^r} \leqslant C \|m * \hat{\Psi}_l\|_{M(\dot{B}^r)}$, which we use for $|\nu| \leqslant 10$. For $|\nu| \geqslant 10$ we have by Lemma 2.3

$$\|R_{k+\nu} T^l\|_{\dot{B}^r \to \dot{B}^r} \leqslant c_N 2^{-lN} 2^{-\nu n} \|m\|_\infty$$

and if we put (2.10)–(2.14) together and sum over all ν we get the estimate

$$\|T^l P_k R_k f\|_r^p \leqslant c_N [2^{ln(1/p-1/r)} \|m * \hat{\Psi}_l\|_{M(\dot{B}^r)} + 2^{-lN}\|m\|_\infty]^p \|R_k f\|_p^p \tag{2.15}$$

for $|k| \leqslant 10$. By similar considerations using Lemma 2.3 again we get for $|k| \geqslant 10$

$$\|T^l P_k R_k f\|_p \leqslant c_N \min(2^{-kn}, 2^{kn}) 2^{-lN} \|m\|_\infty \|R_k f\|_p \tag{2.16}$$

and the conclusion of Lemma 2.4 immediately follows by (2.15) and (2.16).

3. Estimates in $\dot{F}^{pq}$ ($p < q$). We shall have to use some variant of Calderón–Zygmund theory. In our approach we use a maximal operator introduced by Peetre [20] which replaces the Hardy–Littlewood maximal operator in standard Calderón–Zygmund theory. Let (using the notation in Section 2, (2.5))

$$\mathcal{N}^k f(x) = \sup_{|z| \leqslant 10\sqrt{n}2^k} |\tilde{R}_k f(x+z)|, \tag{3.1}$$

$$\mathcal{N}_q f(x) = |\{\mathcal{N}^k f(x)\}|_{l^q}. \tag{3.2}$$

Then Peetre showed that

$$\|f\|_{\dot{F}_0^{pq}} = \|\mathcal{N}_q f\|_p \tag{3.3}$$

if $0 < p < \infty$, $0 < q \leqslant \infty$. In fact, $\mathcal{N}_q f$ is dominated by a sequence of Hardy–Littlewood maximal operators (see Fefferman and Stein [11], Peetre [20]). For the limiting case $p = \infty$ one has the equivalence

$$\|f\|_{\dot{F}_0^{\infty q}} \approx \Big(\sup_Q |Q|^{-1} \int_Q \sum_{k \leqslant L(Q)} |\mathcal{N}^k f(y)|^q\,dy\Big)^{1/q}. \tag{3.4}$$

Peetre's proof can be used to show this equivalence as well (see also

Lemma 6.4 below). We shall have to consider the level sets

$$\Omega_\mu = \{x;\ \mathcal{N}_q f(x) > 2^\mu\} \tag{3.5}$$

for $\mu \in \mathbf{Z}$, and, following R. Fefferman [13], also the expanded sets

$$\tilde{\Omega}_\mu = \{x;\ M\chi_\Omega(x) > 1/2\},$$

where χ_Ω denotes the characteristic function of Ω and M the Hardy–Littlewood maximal operator. Of course $|\tilde{\Omega}| \leqslant c|\Omega|$. For $\mu \in \mathbf{Z}$, $\mathcal{A}_\mu$ will denote the collection of all dyadic cubes Q with the property $|Q \cap \Omega_\mu| > |Q|/2$ but $|Q \cap \Omega_{\mu+1}| \leqslant |Q|/2$. We have the following elementary lemma which is—in a slightly different version—contained in [8], where instead of $\mathcal{N}_q f$ an area integral is used.

LEMMA 3.1. *Let Ω_μ be as in* (3.5). *For $s \geqslant q$ we have*

$$\sum_k \sum_{\substack{Q \in \mathcal{A}_\mu \\ L(Q)=k}} \|(R_k f)\chi_Q\|_s^q 2^{kn(1-q/s)} \leqslant 2^{1+\mu q} |\Omega_\mu|. \tag{3.6}$$

Proof. If $L(Q) = k$ we clearly have for all $x \in Q$

$$|R_k f(x)| \leqslant \inf_{y \in Q} |\mathcal{N}^k f(y)|$$

and consequently for $Q \in \mathcal{A}_\mu$

$$\|(R_k f)\chi_Q\|_s^q 2^{kn(1-q/s)} \leqslant 2 \int_{Q \cap \Omega_\mu} |\mathcal{N}^k f(y)|^q\, dy.$$

Hence the left-hand side of (3.6) is dominated by

$$2\sum_k \int_{\Omega_\mu} |\mathcal{N}^k f(y)|^q\, dy = 2\int_{\Omega_\mu} [\mathcal{N}_q f(y)]^q\, dy \leqslant 2^{1+\mu q}|\Omega_\mu|. \ \blacksquare$$

We may assume that the Littlewood–Paley operators defining $\dot{F}_0^{pq}$ are given by $R_k P_k$, and for fixed k we decompose

$$TR_k P_k f = TR_k P_k(\tilde{R}_k f) = \sum_{L(Q)=k} TR_k P_k[(\tilde{R}_k f)\chi_Q].$$

Setting $e_Q = (\tilde{R}_k f)\chi_Q$ we introduce a vector-valued function

$$F(f) = \{e_Q\} \tag{3.7}$$

where Q runs over all dyadic cubes.

For each Q we define, again following R. Fefferman [13], an expanded cube as follows: If $Q \in \mathcal{A}_\mu$, then $S(Q)$ is the unique dyadic cube containing Q in the Whitney decomposition of $\tilde{\Omega}_\mu$ (see Stein [24], p. 167). With these definitions we can write down our basic splitting

$$TR_k f = G^k(F(f)) + H^k(F(f)) \tag{3.8}$$

where

$$G^k(F) = \sum_{l \in \mathbf{Z}} (TR_k)^{k+l} \sum_{\substack{L(S(Q)) < k+l \\ L(Q)=k}} P_k e_Q, \tag{3.9}$$

$$H^k(F) = \sum_{l \in \mathbf{Z}} (TR_k)^{k+l} \sum_{\substack{L(S(Q)) \geqslant k+l \\ L(Q)=k}} P_k e_Q. \tag{3.10}$$

(Recall that the kernel of $(TR_k)^{k+l}$ is $\mathcal{F}^{-1}[\varphi(2^k\cdot)m]\Psi_l$.) We shall consider the vector-valued functions $G(F) = \{G^k(F)\}$ etc. in $L^p(l^q)$. Note that the map $f \to G(F(f))$ is not linear, but $F \to G(F)$ is (the same remark applies to H). Lemma 3.1 suggests considering the functions F in vector-valued weighted Lebesgue spaces $X(p, q)$ and $X_\infty(p, q)$:

$$\|F\|_{X(p,q)} = \Big(\sum_\mu |\Omega_\mu|^{1-p/q}\Big(\sum_{Q \in \mathcal{A}_\mu} \|e_Q\|_q^q\Big)^{p/q}\Big)^{1/q},$$

which we use for $q \geqslant 1$, and

$$\|F\|_{X_\infty(p,q)} = \Big(\sum_\mu |\Omega_\mu|^{1-p/q}\Big(\sum_k \sum_{\substack{L(Q)=k \\ Q \in \mathcal{A}_\mu}} 2^{kn}\|e_Q\|_\infty^q\Big)^{p/q}\Big)^{1/q},$$

which we can use for all p, q.

Remark. The only reason we introduce $X(p, q)$ is to avoid the slight complication which arises if one interpolates "L^∞-type spaces" by the complex method.

By Lemma 3.1 we have

$$\|F(f)\|_{X(p,q)}^p \leqslant \|F(f)\|_{X_\infty(p,q)}^p \leqslant c\sum_\mu 2^{\mu p}|\{x;\ \mathcal{N}_q f(x) > 2^\mu\}|$$
$$\leqslant c\|\mathcal{N}_q f\|_p^p \leqslant c'\|f\|_{\dot{F}_0^{pq}}^p.$$

Now Theorems 2.1 and 2.2 follow in the case $1 \leqslant p < q \leqslant \infty$ from the following three propositions and an application of Lemma 2.4 to replace M^p-conditions by $M(\dot{B}^p)$-conditions.

PROPOSITION 3.2. *Suppose $q \geqslant 1$, $0 < p \leqslant q$, and*

$$\sup_{t>0} \sum_{l>0} \|\varphi m(t\cdot) * \hat{\Psi}_l\|_{M^q} \leqslant \tilde{D}(q, 1) < \infty.$$

Then

$$\|H(F)\|_{L^p(l^q)} \leqslant c[\|m\|_\infty + \tilde{D}(q, 1)]\,\|F\|_{X(p,q)}. \tag{3.11}$$

PROPOSITION 3.3. *Suppose* $1 < p \leqslant q$, w *increasing as in* (2.1) *and*

$$\sup_{t>0} \sum_{l>0} \|\varphi m(t\cdot) * \hat{\Psi}_l\|_{M^p} w(l) = \tilde{A}(p, w, 1) < \infty .$$

Then

$$\|G(F)\|_{L^p(l^q)} \leqslant c\tilde{A}(p, w, 1)\, \|F\|_{X(p,q)}. \tag{3.12}$$

PROPOSITION 3.4. *Suppose* $1 \leqslant p < r \leqslant q$ *and*

$$\sup_{t>0} \sum_{l>0} \|\varphi m(t\cdot) * \hat{\Psi}_l\|_{M^r} = \tilde{B}(p, r, 1) < \infty .$$

Then

$$\|G(F)\|_{L^p(l^q)} \leqslant c\tilde{B}(p, r, 1)\, \|F\|_{X(p,q)}. \tag{3.13}$$

Proof of Proposition 3.2. We first assume that $0 < p \leqslant 1 \leqslant q \leqslant \infty$. For $\mu \in \mathbf{Z}$ let $b_k^\mu = \sum e_Q$, where the sum is extended over all $Q \in \mathscr{A}_\mu$ with $L(Q) = k$. Also let $b_{kl}^\mu = \sum e_Q$, where the sum is extended over all $Q \in \mathscr{A}_\mu$ with $L(Q) = k$, $L(S(Q)) \geqslant k+l$. Then

$$\|H(F)\|_{L^p(l^q)}^p \leqslant \sum_\mu \Big\|\Big(\sum_k \Big|\sum_l (TR_k)^{k+l} P_k b_{kl}^\mu\Big|^q\Big)^{1/q}\Big\|_p^p .$$

We observe that for every l, $(TR_k)^{k+l} P_k b_{kl}^\mu$ is supported in $\tilde{\Omega}_\mu$. Hence by Hölder's inequality

$$\|H(F)\|_{L^p(l^q)}^p \leqslant c \sum_\mu |\Omega_\mu|^{1-p/q} \Big(\sum_k \Big\|\sum_l (TR_k)^{k+l} P_k b_{kl}^\mu\Big\|_q^q\Big)^{p/q} \tag{3.14}$$

since $|\tilde{\Omega}_\mu| \leqslant c|\Omega_\mu|$. Observe that the L^q-operator norm of $(TR_k)^{k+l}$ is just the M^q-norm of $\varphi m(2^{-k}\cdot) * \hat{\Psi}_l$. Since $b_{kl}^\mu = b_k^\mu$ for $l \leqslant 0$ and

$$\sum_{l\leqslant 0} (TR_k)^{k+l} = TR_k - \sum_{l>0} (TR_k)^{k+l},$$

we obtain

$$\Big\|\sum_l (TR_k)^{k+l} P_k b_{kl}^\mu\Big\|_q \leqslant \sum_{l>0} \|\varphi m(2^{-k}\cdot) * \hat{\Psi}_l\|_{M^q} \|P_k b_{kl}^\mu\|_q + \|\varphi m(2^{-k}\cdot)\|_{M^q} \|P_k b_k^\mu\|_q$$
$$+ \sum_{l>0} \|\varphi m(2^{-k}\cdot) * \hat{\Psi}_l \|P_k b_{kl}^\mu\|_q .$$

Of course, for every l, we have

$$\|P_k b_{kl}^\mu\|_q^q \leqslant \sum_{\substack{Q\in\mathscr{A}_\mu \\ L(Q)=k}} \|e_Q\|_q^q$$

and the same with b_k^μ instead of b_{kl}^μ. Now

$$\|\varphi m(2^{-k}\cdot)\|_{M^q} \leqslant c\big(\|m\|_\infty + \tilde{D}(q, 1)\big) \tag{3.15}$$

and we obtain (3.11) for $0 < p \leqslant 1$.

If $1 \leqslant p \leqslant q$ we apply interpolation, using an argument in [7]. Let $b_{kl} = \sum_{\mu\in\mathbf{Z}} b_{kl}^\mu$. Then we have as above

$$\|H(F)\|_{L^q(l^q)}^q = \sum_k \Big\|\sum_{l\in\mathbf{Z}} (TR_k)^{k+l} P_k b_{kl}\Big\|_q^q \leqslant c[\|m\|_\infty + \tilde{D}(q, 1)]^q \sum_k \sum_{L(Q)=k} \|e_Q\|_q^q,$$

hence (3.11) for $p = q$. We may interpolate between the weighted Lebesgue spaces $X(1, q)$ and $X(q, q)$ and this finishes the proof of Proposition 3.2.

Proof of Proposition 3.3. Let $a_l^{k\mu} = \sum e_Q$ where the sum is extended over all $Q \in \mathscr{A}_\mu$ with $L(Q) = k$, $k \leqslant L(S(Q)) < k+l$ and let $a_l^k = \sum_{\mu\in\mathbf{Z}} a_l^{k\mu}$. Since $l^p \subset l^q$ we have

$$\|G(F)\|_{L^p(l^q)} \leqslant \Big(\sum_k \Big\|\sum_{l>0} (TR_k)^{k+l} P_k a_l^k\Big\|_p^p\Big)^{1/p} \leqslant A(p, w, 1)\cdot I \tag{3.16}$$

where

$$I^p = \sum_k \sup_{l>0} w(l)^{-p} \|P_k a_l^k\|_p^p .$$

For each Whitney cube S in $\tilde{\Omega}_\mu$ let $a_S^{k\mu} = \sum e_Q$ where the sum is extended over all $Q \in \mathscr{A}_\mu$ with $L(Q) = k$, $S(Q) = S$. Then we clearly have

$$I^p \leqslant \sum_k \sup_{l>0} w(l)^{-p} \sum_\mu \sum_{k\leqslant L(S)\leqslant k+l} \|a_S^{k\mu}\|_p^p \leqslant \sum_\mu \sum_S \sum_k w\big(L(S)-k\big)^{-p} \|a_S^{k\mu}\|_p^p$$

where we have used the monotonicity of w. We apply Hölder's inequality for the k-summation and, using (2.1), we get

$$I^p \leqslant \sum_\mu \sum_S \Big(\sum_k \|a_S^{k\mu}\|_p^q\Big)^{p/q} \leqslant \sum_\mu \sum_S |S|^{1-p/q} \Big(\sum_k \|a_S^{k\mu}\|_q^q\Big)^{p/q} .$$

Now

$$\sum_{S\in W(\tilde{\Omega}_\mu)} |S| \leqslant |\tilde{\Omega}_\mu|$$

where we sum up over the family $W(\tilde{\Omega}_\mu)$ of all Whitney cubes in $\tilde{\Omega}_\mu$. Applying once more Hölder's inequality we see that

$$I^p \leqslant \sum_\mu |\tilde{\Omega}_\mu|^{1-p/q} \Big(\sum_k \sum_S \|a_S^{k\mu}\|_q^q\Big)^{p/q} \leqslant \|F\|_{X(p,q)}^p ,$$

which proves the proposition.

Proof of Proposition 3.4. Let a_l^k be as in the proof of Proposition 3.3. As in Section 2, we denote by $\chi_{k+l,\sigma}$ the characteristic function of a dyadic cube

$Q_{k+l,\sigma}$ with sidelength 2^{k+l}. We proceed as in the proof of Proposition 3.3 to obtain

$$\|G(F)\|_{L^p(l^q)}^p \leqslant \sum_k \Big[\sum_{l>0} \Big(\sum_\sigma \|(TR_k)^{k+l} P_k[a_l^k \chi_{k+l,\sigma}]\|_p^p\Big)^{1/p}\Big]^p \tag{3.17}$$

where we have used the fact that $(TR_k)^{k+l} P_k[a_l^k \chi_{k+l,\sigma}]$ is supported on a dilate of $Q_{k+l,\sigma}$ with comparable sidelength. We apply Hölder's inequality to dominate the right-hand side of (3.17) by

$$\sum_k \Big[\sum_{l>0} 2^{(k+l)n(1/p-1/r)} \Big(\sum_\sigma \|(TR_k)^{k+l} P_k[a_l^k \chi_{k+l,\sigma}]\|_r^p\Big)^{1/p}\Big]^p$$

$$\leqslant [B(p, r, 1)]^p \sum_k 2^{kn(1-p/r)} I_k^p$$

where

$$I_k^p = \sup_{l>0} \sum_\sigma \|P_k a_l^k \chi_{k+l,\sigma}\|_r^p.$$

Since $l^{p/r} \subset l^1$ we have (using the same notation as in the proof of Proposition 3.3)

$$I_k^p \leqslant \sup_{l>0} \sum_\mu \sum_{\substack{S \in W(\tilde{\Omega}_\mu) \\ k \leqslant L(S) \leqslant k+l}} \|a_S^{k\mu}\|_r^p \leqslant \sum_\mu \sum_{j \geqslant k} \sum_{\substack{L(S)=j \\ S \in W(\tilde{\Omega}_\mu)}} 2^{jn(1/r-1/q)p} \|a_S^{k\mu}\|_q^p$$

and furthermore

$$\sum_k 2^{kn(1-p/r)} I_k^p \leqslant \sum_j 2^{jn(1-p/r)} \sum_\mu \sum_{\substack{L(S)=j \\ S \in W(\tilde{\Omega}_\mu)}} \sum_{k \leqslant j} 2^{(k-j)n(1-p/r)} \|a_S^{k\mu}\|_q^p$$

$$\leqslant c_{pr} \sum_\mu \sum_{S \in W(\tilde{\Omega}_\mu)} \Big(\sum_k \|a_S^{k\mu}\|_q^q\Big)^{p/q} |S|^{1-p/q} \leqslant c'_{pr} \|F\|_{X(p,q)}^p$$

where the last inequality was already shown in the proof of Proposition 3.3. ∎

We now state the analogues of Propositions 3.2–3.4 which cover the cases $p \leqslant 1$ or $q \leqslant 1$ of Theorems 2.1 and 2.2 (which follow by means of Lemma 3.1). We use the same notation as in Theorems 2.1 and 2.2.

PROPOSITION 3.5. *Let* $0 < p \leqslant q \leqslant 1$. *Then*

$$\|H(F)\|_{L^p(l^q)} \leqslant c\big(\|m\|_\infty + D(q, q)\big) \|F\|_{X_\infty(p,q)}.$$

PROPOSITION 3.6. *Let* $p \leqslant \min(q, 1)$, *and let* w *be increasing as in* (2.1). *Then*

$$\|G(F)\|_{L^p(l^q)} \leqslant c\big(\|m\|_\infty + A(p, w, p)\big) \|F\|_{X_\infty(p,q)}.$$

PROPOSITION 3.7. *Let* $p < r \leqslant q$, $0 < p \leqslant 1$. *Then*

$$\|G(F)\|_{L^p(l^q)} \leqslant cB(p, r, p) \|F\|_{X_\infty(p,q)}.$$

The proofs of these estimates are quite similar to the proofs of Propositions 3.2–3.4, so we only sketch the necessary modifications. The following lemma will be needed:

LEMMA 3.8. *Let* Γ *be a collection of dyadic cubes* Q *with sidelength* 2^k *which are contained in an open set* E. *For each* Q *let* e_Q *be a bounded function supported in* Q *and let* $b = \sum_{Q \in \Gamma} e_Q$. *Then we have for* $0 < p \leqslant 1$

$$\|R_{k+\nu} P_k b\|_p^p \leqslant c_\varepsilon 2^{-|\nu|\varepsilon} |E|^{1-p/q} \Big(\sum_{Q \in \Gamma} \|e_Q\|_\infty^q 2^{kn}\Big)^{p/q}$$

if $\varepsilon = M - n(1/p-1) > 0$, M *as in* (2.6).

Proof. The convolution kernel $I_{k\nu}$ of $R_{k+\nu} P_k$ satisfies the bounds (2.11) if $\nu \geqslant 0$ and (2.12) if $\nu \leqslant 0$ and so we have

$$\|R_{k+\nu} P_k b\|_p^p \leqslant c \sum_{Q \in \Gamma} \|e_Q\|_\infty^p \int \Big[\int_Q |I_{k\nu}(x-y)|\, dy\Big]^p dx,$$

which is bounded by $c_K 2^{\nu K p} \sum_Q \|e_Q\|_\infty^p 2^{kn}$ for $\nu \leqslant 0$ and every $K \in \mathbf{N}$ and by $c 2^{-\nu M p} 2^{\nu n(1-p)} \sum_Q \|e_Q\|_\infty^p 2^{kn}$ for $\nu \geqslant 0$. Since $L(Q) = k$ and since different cubes with the same size are disjoint we obtain by an application of Hölder's inequality

$$\sum_Q \|e_Q\|_\infty^p 2^{kn} \leqslant c\Big(\sum_Q |Q|\Big)^{1-p/q} \Big(\sum_Q \|e_Q\|_\infty^q 2^{kn}\Big)^{p/q} \leqslant c|E|^{1-p/q} \Big(\sum_Q \|e_Q\|_\infty^q 2^{kn}\Big)^{p/q},$$

which proves the lemma.

Proof of Proposition 3.5. We begin with the inequality (3.14) which also holds for $q < 1$. Then

$$\Big\|\sum_{l \in \mathbf{Z}} (TR_k)^{k+l} P_k b_{kl}^\mu\Big\|_q^q \leqslant \sum_{l>0} \sum_\nu \|(TR_k)^{k+l} R_{k+\nu} \tilde{R}_{k+\nu} P_k b_{kl}^\mu\|_q^q$$

$$+ \sum_{|\nu| \leqslant 5} \|TR_k R_{k+\nu} \tilde{R}_{k+\nu} P_k b_k^\mu\|_q^q$$

$$+ \sum_{l>0} \sum_\nu \|(TR_k)^{k+l} R_{k+\nu} \tilde{R}_{k+\nu} P_k b_k^\mu\|_q^q$$

$$\leqslant c[\|m\|_\infty + D(q, q)]^q \sum_\nu \big[\|\tilde{R}_{k+\nu} P_k b_k^\mu\|_q^q$$

$$+ \sup_{l>0} \|\tilde{R}_{k+\nu} P_k b_{kl}^\mu\|_q^q\big].$$

This is by Lemma 3.8 (for the case $p=q$) dominated by

$$c[\|m\|_\infty + D(q,q)]^q \sum_{L(Q)=k} \|e_Q\|_\infty^q 2^{kn}$$

and the rest of the proof is the same as for Proposition 3.2.

Proof of Proposition 3.6. With the notation as in the proof of Proposition 3.3 we have

$$\|G(F)\|^p_{L^p(l^q)} \leqslant \Big(\sum_k \sum_{l>0} \|(TR_k)^{k+l} P_k a_l^k\|_p^p\Big)^{1/p}$$

$$\leqslant \sum_k \sum_{l>0} \sum_\nu \|(TR_k)^{k+l} R_{k+\nu}\|^p_{M(\dot B^p)} \| R_{k+\nu} P_k a_l^k\|_p^p$$

$$\leqslant c[A(p,w,p)]^p \sum_\nu I_\nu^p$$

where

$$I_\nu^p = \sum_k \sup_{l>0} w(l)^{-p} \|R_{k+\nu} P_k a_l^k\|_p^p.$$

The estimate for I in the proof of Proposition 3.3 can be used to get the inequality

$$I_\nu^p \leqslant c \sum_\mu \sum_S \Big(\sum_k \|R_{k+\nu} P_k a_S^{k\mu}\|_q^q\Big)^{p/q}$$

and if we use Lemma 3.8 for $E=S$ we can proceed exactly as in Proposition 3.3 and finish the proof. ∎

In order to prove Proposition 3.7 one modifies the proof of Proposition 3.4 in exactly the same manner. We omit the details.

4. $\dot F^{pq}$-estimates, $p>q$. In this section we shall give proofs of Theorems 2.1 and 2.2 in the case $p>q$. If $1<q<p\leqslant\infty$ we may of course use duality and the results in Section 3. We however prefer a direct approach to cover all cases; and, more important, a direct proof turns out to be useful to obtain estimates for non-convolution operators as well (see Remark (a) in Section 5 below). Our estimates are based on the inequality

$$\|g\|_{\dot F^{pq}} \leqslant c\,\|\mathcal{N}_q^\# g\|_p \tag{4.1}$$

if $g\in\dot F^{pq}$, for the sharp function

$$\mathcal{N}_q^\# g(x) = \sup_{x\in S}\Big(|S|^{-1}\int_S \sum_{k\leqslant L(S)} |P_k R_k g(y)|^q\,dy\Big)^{1/q}$$

where the supremum is taken over all cubes S containing x. (4.1) is a special

case of Proposition 6.1 below (for $p=\infty$ (4.1) with equality is just the definition of $\dot F_0^{\infty q}$).

Instead of estimating Tf we shall estimate $\mathcal{N}_q^\# Tf$. In order to apply (4.1) for $q<p<\infty$ we have to assume that *a priori* $Tf\in\dot F_0^{pq}$. However, the class of Schwartz functions f for which $\hat f$ is compactly supported in $\mathbf{R}^n\setminus\{0\}$ is dense in $\dot F_0^{pq}$ if $p,q<\infty$. Since the hypotheses of Theorems 2.1 and 2.2 imply boundedness on the Besov space $\dot B_{0p}^p$, we know that *a priori* for these functions Tf is in $\dot F_0^{pq}$ (with bounds depending on the support of f), hence we can use (4.1).

Our basic estimate (somehow dual to (3.8)) is

$$\mathcal{N}_q^\# Tf(x) \leqslant c[G_q^\# f(x) + H_q^\# f(x)] \tag{4.2}$$

where

$$G_q^\# f(x) = \Big(\sup_{x\in S} |S|^{-1}\int_S \sum_{k\leqslant L(S)} \Big|\sum_{l>L(S)-k} (TR_k)^{k+l} P_k R_k f\Big|^q dy\Big)^{1/q}, \tag{4.3}$$

$$H_q^\# f(x) = \Big(\sup_{x\in S} |S|^{-1}\int_S \sum_{k\leqslant L(S)} \Big|\sum_{l\leqslant L(S)-k} (TR_k)^{k+l} P_k R_k f\Big|^q dy\Big)^{1/q}. \tag{4.4}$$

The proof of Theorems 2.1 and 2.2 will be finished by the following three propositions, because by Minkowski's inequality we have the embedding $\dot F_0^{pq} \subset \dot B_{0p}^p$ for $p\geqslant q$.

PROPOSITION 4.1. *Let $p>q>0$, $\sigma=\min(q,1)$. Then*

$$\|H_q^\# f\|_p \leqslant c[\|m\|_\infty + D(q,\sigma)]\,\|f\|_{\dot F_0^{pq}}.$$

PROPOSITION 4.2. *Let $p>q>0$, $\sigma=\min(p,1)$, w increasing as in* (2.1). *Then*

$$\|G_q^\# f\|_p \leqslant cA(p,w,\sigma)\,\|f\|_{\dot B_{0p}^p}.$$

PROPOSITION 4.3. *Let $p>r\geqslant q$, $\sigma=\min(p,1)$. Then*

$$\|G_q^\# f\|_p \leqslant cB(p,r,\sigma)\,\|f\|_{\dot B_{0p}^p}.$$

Remark. A proof of the $\dot F^{pq}$-estimates for $1\leqslant p'\leqslant q\leqslant p\leqslant\infty$ which does not rely on Section 3 follows by observing that in this case $D(q,1)$ is dominated by $\|m\|_\infty + A(p,w,1)$ or $\|m\|_\infty + B(p,r,1)$.

Proof of Proposition 4.1. We first assume $q>1$ and the proposition will follow by real interpolation between the cases $p=q$ and $p=\infty$. The appropriate interpolation theorem including the case $p=\infty$ is proved in Frazier and Jawerth [15].

For $p=q$ we have a pointwise estimate involving the Hardy–Littlewood maximal operator M:

$$H_q^{\#} f(x) \leqslant \big(M(\sum_k |TR_k P_k \tilde{R}_k f|^q)\big)^{1/q} + \big(M(\sum_k [\sum_{l>0} |(TR_k)^{k+l} P_k \tilde{R}_k f|]^q)\big)^{1/q}$$

and conclude the weak type inequality

$$\lambda |\{x;\ H_q^{\#} f(x) > \lambda\}|^{1/q} \leqslant c [\|m\|_\infty + D(q, 1)] \, \|f\|_{\dot{F}_0^{qq}}. \tag{4.5}$$

(Remark: We could have used $\mathcal{N}^{\#}_{q,\gamma}$, $\gamma < q$, defined in (6.1) below, to obtain a strong type inequality.) For $p = \infty$ we observe that for a fixed dilate $\tilde{S}$ of S the function

$$\sum_{l \leqslant L(S)-k} (TR_k)^{k+l} P_k [(\tilde{R}_k f) \chi_{\mathbf{R}^n \setminus \tilde{S}}]$$

vanishes on S (we have to dilate S by a factor $5\sqrt{n}$, say). Hence $\|H_q^{\#} f\|_\infty$ can be estimated by

$$\sup_S |S|^{-1} \Big(\sum_{k \leqslant L(S)} \Big\| \sum_{l \leqslant L(S)-k} (TR_k)^{k+l} P_k [(\tilde{R}_k f) \chi_{\tilde{S}}] \Big\|_q^q \Big)^{1/q}.$$

Now

$$\Big\| \sum_{l \leqslant L(S)-k} \ldots \Big\|_q^\sigma \leqslant c \|TR_k P_k [(\tilde{R}_k f) \chi_{\tilde{S}}]\|_q^\sigma + c \sum_{l>0} \|(TR_k)^{k+l} P_k [(\tilde{R}_k f) \chi_{\tilde{S}}]\|_q^\sigma$$

$$\leqslant (\|m\|_\infty + D(q, \sigma))^\sigma \sum_v \|P_{k+v} R_{k+v} P_k [(R_k f) \chi_{\tilde{S}}]\|_q^\sigma.$$

By Lemma 3.8

$$\sum_{k \leqslant L(S)} \Big(\sum_v \|P_{k+v} R_{k+v} P_k [(R_k f) \chi_{\tilde{S}}]\|_q^\sigma \Big)^{q/\sigma} \leqslant c \sum_{k \leqslant L(S)} \sum_{\substack{Q \subset S \\ L(Q)=k}} 2^{kn} \|(\tilde{R}_k f) \chi_Q\|_\infty^q$$

$$\leqslant c \sum_{k \leqslant L(S)} \int_S [\mathcal{N}^k f(y)]^q \, dy \leqslant c \|f\|^q_{\dot{F}_0^{\infty q}}$$

by (3.4). This finishes the proof.

Proof of Proposition 4.2. We use the assumptions (2.1) on w and Hölder's inequality to dominate $G_q^{\#} f(x)$ by

$$\Big(\sup_{x \in S} |S|^{-1} \int_S \Big(\sum_{k \leqslant L(S)} |w(L(S)) - k|^p \Big[\sum_{l > L(S)-k} |(TR_k)^{k+l} P_k \tilde{R}_k f| \Big]^p \Big)^{q/p} \Big)^{1/q}.$$

Since w is increasing and the Hardy–Littlewood maximal operator is bounded on $L^{p/q}$ we get

$$\|G_q^{\#} f\|_p \leqslant c \Big(\sum_k \Big\| \sum_{l>0} w(l) |(TR_k)^{k+l} P_k R_k f| \Big\|_p^p \Big)^{1/p}$$

which in the case $p \geqslant 1$ is bounded by

$$c \sup_{t>0} \sum_l w(l) \|\varphi m(t\cdot) * \hat{\Psi}_l\|_{M^p} \Big(\sum_k \|R_k f\|_p^p \Big)^{1/p}$$

and the proposition follows by applying Lemma 2.4.

For $p \leqslant 1$ the argument is similar and uses the same kind of modification which already occurred in the proofs of Propositions 3.6 and 4.1.

Proof of Proposition 4.3. Again we only consider the case $p > 1$ and leave the minor modifications for the case $q < p \leqslant 1$ to the reader. We apply Hölder's inequality to see that, for $\varepsilon > 0$, $G_q^{\#} f(x)$ is bounded by

$$c \sup_{x \in S} |S|^{-1+\varepsilon q/n} \Big(\sum_{k \leqslant L(S)} 2^{-k\varepsilon p} \Big(\int_S \Big| \sum_{l \geqslant L(S)-k} (TR_k)^{k+l} P_k R_k f \Big|^q \Big)^{p/q} \Big)^{1/p}.$$

Let

$$M_{\delta,k,l} g(x) = \sup_{x \in S} \sup_{k \leqslant L(S) \leqslant k+l} |S|^{-1+\delta/n} \int_S |g(y)| \, dy.$$

Then by Minkowski's inequality

$$\|G_q^{\#} f\|_p^p \leqslant c \sum_k 2^{-k\varepsilon p} \Big[\sum_{l>0} \|M_{\varepsilon q,k,l}(|(TR_k)^{l+k} P_k \tilde{R}_k f|^q)\|_{p/q}^{1/q} \Big]^p. \tag{4.6}$$

We observe that if h is a function supported in some cube of sidelength $c_0 2^{k+l}$, then $M_{\varepsilon q,k,l}[(TR_k)^{l+k} h]$ vanishes outside a cube with the same center and comparable sidelength $c_1 2^{l+k}$. Furthermore, it is easy to see that

$$M_{\delta,k,l} g(x) \leqslant c|x|^{\delta-n} * |g|$$

uniformly in k and l. So if we choose $\varepsilon = n(1/r - 1/p)$ the theorem about fractional integration tells us that $M_{\varepsilon q,k,l}$ is bounded from $L^{r/q}$ to $L^{p/q}$ with a bound independent of k and l. Therefore we obtain the estimate (denoting by $\chi_{k+l,\gamma}$ the characteristic functions of cubes with sidelength 2^{k+l})

$$\|M_{\varepsilon q,k,l}(|(TR_k)^{k+l} P_k \tilde{R}_k f|^q)\|_{p/q}^{1/q} \leqslant c \Big(\sum_\gamma \| |(TR_k)^{k+l} P_k [(\tilde{R}_k f) \chi_{k+l,\gamma}]|^q \|_{r/q}^{p/q} \Big)^{1/p}$$

$$\leqslant c \Big(\sum_\gamma \|(TR_k)^{k+l}\|^p_{L^r \to L^r} \|(\tilde{R}_k f) \chi_{k+l,\gamma}\|_r^p \Big)^{1/p}$$

$$\leqslant c 2^{(k+l)n(1/r-1/p)} \|\varphi m(2^{-k}\cdot) * \hat{\Psi}_l\|_{M^r} \|R_k f\|_p$$

where we have applied Hölder's inequality for the last step.

Going back to (4.6) we get by Lemma 2.4 (and since $\varepsilon = n(1/r - 1/p)$)

$$\|G_q^{\#} f\|_p^p \leqslant cB(p, r, 1) \sum_k \|\tilde{R}_k f\|_p^p \leqslant cB(p, r, 1) \|f\|_{\dot{B}^p_{0p}}. \ \blacksquare$$

5. Multiplier criteria. We want to prove Corollary 1.2. In order to verify the hypothesis of Theorem 1.1 we have to show for $1 \leqslant p \leqslant 2$, $1/q = 1/p - 1/2$ the inequality

$$\|K_l * f\|_p \leqslant c 2^{ln(1/p-1/2)} \|m * \hat{\Psi}_l\|_q \|f\|_p \tag{5.1}$$

where $\hat{K}_l = m * \hat{\Psi}_l$, and m is supported in $\{\xi;\ 1/4 < |\xi| < 4\}$. We may assume that f is supported in a cube of sidelength 2^l, because then $K_l * f$ is supported in a slightly larger cube of size $c2^{ln}$. By Hölder's inequality

$$\|K_l * f\|_p \leqslant c 2^{ln(1/p-1/2)} \|K_l * f\|_2.$$

Using Plancherel's theorem and Hölder's inequality we obtain

$$\|K_l * f\|_2^2 \leqslant c \int |m * \hat{\Psi}_l(\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi \leqslant c \|m * \hat{\Psi}_l\|_q^2 \|\hat{f}\|_{p'}^2$$

and (5.1) follows by an application of the Hausdorff–Young inequality.

In order to prove Corollary 1.2 the straightforward use of the Hausdorff–Young inequality is replaced by an application of the L^2-restriction theorem for the Fourier transform (see Tomas [25]). This argument is due to Fefferman and Stein (see [10], for the variants needed here see [9] or [21]).

Remarks. (a) The methods of this paper may be used to prove certain endpoint estimates for pseudodifferential operators with weak regularity properties. For example operators with symbol of type $S^{-m}_{\varrho,\varrho}$, $m = n(1-\varrho)|1/p - 1/2|$, are bounded on the inhomogeneous space F^{pq}_0 if $p \leqslant q \leqslant p'$ (see Miyachi [19] for related results and further references, cf. also Carbery and Seeger [6]).

(b) Corollary 1.2 is a best possible estimate for H^1. More refined L^p-estimates are discussed in a subsequent paper. A variant of Theorem 1.1 involving parabolic dilations can be used to give endpoint results for the class of quasiradial multipliers considered in [21].

(c) It would be interesting to find an analogue of Theorem 1 on product domains. To give an example, let $\theta \in C^\infty(\mathbf{R})$ vanish near 0 and $\theta(\gamma) = 1$, $|\gamma| \geqslant 1$. Define $m(\xi_1, \xi_2) = \theta(\xi_1)\theta(\xi_2)|\xi_1 \xi_2|^{-b} \exp(i|\xi_1 \xi_2|^a)$. Is it true that m is a multiplier on $L^p(\mathbf{R}^2)$ if $b/a = |1/p - 1/2|$, $1 < p < \infty$?

6. Appendix: Sharp maximal operators on $\dot{F}^{pq}$. We consider the operator

$$\mathcal{N}^{\#}_{q,\gamma} f(x) = \Big(\sup_{Q \ni x} |Q|^{-1} \int_Q \Big(\sum_{k \leqslant L(Q)} |R_k f(y)|^q\Big)^{\gamma/q} dy\Big)^{1/\gamma}. \tag{6.1}$$

Note that, by definition, $\|f\|_{\dot{F}^{\infty q}_0} = \|\mathcal{N}^{\#}_{q,q} f\|_\infty$. The following proposition is a variant of the sharp function estimate of Fefferman and Stein [12], and, in fact, the latter is used to prove it.

Proposition 6.1. *Suppose $0 < p, q \leqslant \infty$. For every $f \in \dot{F}^{pq}_0$ we have the a priori estimate*

$$\|f\|_{\dot{F}^{pq}_0} \leqslant c \|\mathcal{N}^{\#}_{q,\gamma} f\|_p.$$

A partial converse is

Proposition 6.2. *Suppose $0 < \gamma < p \leqslant \infty$, $0 < q \leqslant \infty$. Then for all $f \in \dot{F}^{pq}_0$*

$$\|\mathcal{N}^{\#}_{q,\gamma} f\|_p \leqslant c \|f\|_{\dot{F}^{pq}_0}.$$

A limiting estimate for the case $p = \infty$ is

Proposition 6.3. *Suppose $0 < r \leqslant q < \infty$. Then there exist positive constants ε, c_0, c_1 depending only on n and r, such that for all nonconstant $f \in \dot{F}^{\infty q}_0$ we have the estimate*

$$c_0 \leqslant \sup_Q \Big(\log |Q|^{-1} \int_Q \exp\Big[\varepsilon^q \sum_{k \leqslant L(Q)} |R_k f(y)|^q / \|f\|^q_{\dot{F}^{\infty q}_0}\Big] dy\Big)^{1/q} \leqslant c_1.$$

Proposition 6.3 may be considered as a variant of the John–Nirenberg inequality. Related results have been obtained by Frazier and Jawerth [15]. We do not use their approach, but it is conceivable that their technique also proves Propositions 6.2 and 6.3.

We shall use the following pointwise estimate: Let

$$\tilde{\mathcal{N}}^{\#}_{q,\gamma,\varrho} f(x) = \left(\sup_{Q \ni x} |Q|^{-1} \int_Q \left(\sum_{k \leqslant L(Q)} \left[\sup_z \frac{|R_k f(y-z)|}{(1 + 2^{-k}|z|)^\varrho}\right]^q dy\right)^{\gamma/q}\right)^{1/\gamma}.$$

Lemma 6.4. *If $\varrho > \max\{n, 2n/q, 2n/\gamma\}$, then for almost all $x \in \mathbf{R}^n$ we have*

$$\tilde{\mathcal{N}}^{\#}_{q,\gamma,\varrho} f(x) \leqslant c \mathcal{N}^{\#}_{q,\gamma} f(x).$$

Proof. We use Peetre's mean value inequality [17]

$$|R_k f(y-z)| \leqslant c\Big((2^k\delta)^{-n} \int_{Q(y-z, 2^k\delta)} |R_k f(w)|^r dw\Big)^{1/r} + c 2^k \delta \sup_{|y-z-u| \leqslant 2^k \delta} |\nabla R_k f(u)| \tag{6.2}$$

where $Q(z, t)$ is the cube with center z and sidelength t. This is valid for all $r > 0$, and we use it for $r < \min(q, \gamma)$ such that $n(1/r + 1/\gamma) < \varrho$.

Since $\nabla R_k f = \nabla R_k \tilde{R}_k f$ we have as in [20], [26] the straightforward inequality

$$2^k \sup_z \sup_{|y-z-u| \leqslant 2^k\delta} \frac{|\nabla R_k f(u)|}{(1 + 2^{-k}|z|)^\varrho} \leqslant c \sup_{z \in \mathbf{R}^n} \frac{|R_k f(y-z)|}{(1 + 2^{-k}|z|)^\varrho}$$

and, on choosing $\delta > 0$ sufficiently small, the left-hand side of (6.2) is dominated by $c \sum_{l \geqslant 0} 2^{-l\varrho} I_{k,l} f(y)$ where

$$I_{k,l} f(y) = \Big(\sup_{|z| \leqslant 2^{l+1+k}} 2^{-kn} \int_{Q(y-z, 2^k)} |R_k f(w)|^r dw\Big)^{1/r}.$$

Let us fix x and $Q \ni x$ and let $\tilde{Q}(l)$ be the cube one gets by expanding Q by a factor 2^{l+1}. Then we clearly have

$$I_{k,l} f(y) \leqslant c 2^{ln/r} \big(M[|(R_k f)\chi_{\tilde{Q}(l)}|^r]\big)^{1/r}.$$

Let $g_{kl} = (R_k f)\chi_{\tilde{Q}(l)}$ if $k \leqslant L(Q)$, and $g_{kl} = 0$ if $k > L(Q)$. Since $\gamma, q > r$ we may use the Fefferman–Stein inequality for sequences of maximal operators [11] and obtain

$$\int_Q \Big(\sum_{k \leqslant L(Q)} |I_{k,l} f(y)|^q\Big)^{\gamma/q} dy \leqslant c 2^{ln\gamma/r} \|\{M(|g_{k,l}|^r)\}\|^{\gamma/r}_{L^{\gamma/r}(l^{q/r})}$$

$$\leqslant c 2^{ln\gamma/r} \int_{\tilde{Q}(l)} \Big[\sum_{k \leqslant L(Q)} |R_k f|^q\Big]^{\gamma/q} dy$$

$$\leqslant c 2^{ln(1+\gamma/r)} [\mathcal{N}^{\#}_{q,\gamma} f(x)]^{\gamma} |Q|.$$

Hence we get

$$\tilde{\mathcal{N}}^{\#}_{q,\gamma,\varrho} f(x) \leqslant \sum_{l \geqslant 0} 2^{-l\varrho} \sup_{x \in Q} \Big(|Q|^{-1} \int_Q \Big(\sum_{k \leqslant L(Q)} |I_{k,l} f(y)|^q\Big)^{\gamma/q}\Big)^{1/\gamma}$$

$$\leqslant c \sum_{l \geqslant 0} 2^{-l\varrho} 2^{ln(1/\gamma + 1/r)} \tilde{\mathcal{N}}^{\#}_{q,\gamma} f(x).$$

Since the sum converges, the lemma is proved.

Proof of Proposition 6.1. Since $\mathcal{N}^{\#}_{q,\gamma} f(x) \leqslant \mathcal{N}^{\#}_{q,\delta} f(x)$ for $\gamma < \delta$ we may assume $\gamma < q$, $\gamma < p$. We apply the sharp function estimate by Fefferman and Stein [12] to the $l^{q/\gamma}$-valued function $\{|R_k f|^{\gamma}\}$ and obtain the inequality

$$\|f\|_{\dot{F}^{pq}} \leqslant c \Big\|\sup_{Q \ni x} |Q|^{-1} \int_Q \Big(\sum_{k \in \mathbf{Z}} \Big[|R_k f|^{\gamma} - |Q|^{-1} \int_Q |R_k f|^{\gamma} dz\Big]^{q/\gamma}\Big) dy\Big\|^{1/\gamma}_{p/\gamma}.$$

Using Taylor's formula we get the estimate

$$(6.3)\qquad |Q|^{-1} \int_Q \Big(\sum_{k \geqslant L(Q)} \Big[|R_k f|^{\gamma} - |Q|^{-1} \int_Q |R_k f(z)|^{\gamma} dz\Big]\Big)^{q/\gamma} dy$$

$$\leqslant c|Q|^{-1} \int_Q \sum_{k \geqslant L(Q)} \Big|\sup_{w \in Q} [(y - x_Q)\cdot \nabla] R_k f(w)\Big|^q dy$$

where x_Q is the center of Q. By a straightforward calculation using $R_k = \tilde{R}_k R_k$, (6.3) is dominated by

$$c_{\varrho} |Q|^{-1} \int_Q \Big(\sum_{k \geqslant L(Q)} 2^{(L(Q)-k)q} \sup_z \frac{|R_k f(y+z)|^q}{(1+2^{-k}|z|)^{\varrho}} dy\Big)^{1/q} \leqslant c_{\varrho} \tilde{\mathcal{N}}_{q,q,\varrho} f(x)$$

for $x \in Q$ and, if we choose ϱ large enough, the proof is finished by an application of Lemma 6.4.

Proof of Proposition 6.2. We clearly have for all $\gamma < p \leqslant \infty$

$$\|\mathcal{N}^{\#}_{q,\gamma} f\|_p \leqslant c \Big\|M\Big[\Big(\sum_k |R_k f|^q\Big)^{\gamma/q}\Big]\Big\|^{1/\gamma}_{p/\gamma} \leqslant c \|\{R_k f\}\|_{L^p(l^q)},$$

which gives the $\dot{F}^{pq}_0$ estimate for $p < \infty$. Now suppose $p = \infty$. We may assume that $\gamma > q$. Let $A = A_{n,q}$ be the best constant in the inequality

$$(6.4)\qquad \Big(\sup_Q |Q|^{-1} \int_Q \sum_{k \leqslant L(Q)} |\mathcal{N}^k f(y)|^q dy\Big)^{1/q} \leqslant A \|f\|_{\dot{F}^{\infty q}_0}$$

and $B_f = 2A \|f\|_{\dot{F}^{\infty q}_0}$.

For a cube Q let $E(Q, B_f)$ be the set of all $x \in Q$ such that

$$\sum_{k \leqslant L(Q)} |\mathcal{N}^k f(x)|^q > B^q_f.$$

Following Chang and Fefferman [8] we define a nested family of open sets $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \ldots$ as follows. Let $\Omega_0 = Q$. If Ω_ν is already defined, let $\{Q^\nu_i\}$ be the family of dyadic Whitney cubes in Ω_ν (as defined in Stein [24, p. 167]). We set

$$\Omega_{\nu+1} = \bigcup_i E(Q^\nu_i, B_f),$$

which is a subset of Ω_ν and open in Q. Then by Chebyshev's inequality

$$|\Omega_{\nu+1}| \leqslant B_f^{-q} \sum_i \int_{E(Q^\nu_i, B_f)} \sum_{k \leqslant L(Q^\nu_i)} |\mathcal{N}^k f|^q dy \leqslant B_f^{-q} \sum_i |Q^\nu_i| A^q \|f\|^q_{\dot{F}^{\infty q}_0} \leqslant 2^{-q} |\Omega_\nu|.$$

By iteration,

$$(6.5)\qquad |\Omega_\nu| \leqslant 2^{-q\nu} |Q|$$

for all $\nu \geqslant 0$.

We observe that each Whitney cube Q^ν_i is contained in a unique Whitney cube $Q^\mu_j = Q^\mu(i, \nu)$ in the decomposition of Ω_μ for $\mu < \nu$.

If we set $Q^0(i, \nu) = Q$ we obtain a nested sequence of cubes $Q^0(i, \nu) \supset Q^1(i, \nu) \supset \ldots \supset Q^\nu_i$. Now

$$(6.6)\qquad \Big(\int_Q \Big(\sum_{k \leqslant L(Q)} |R_k f(y)|^q\Big)^{\gamma/q} dy\Big)^{1/\gamma}$$

$$\leqslant \Big(\sum_{\nu=0}^{\infty} \sum_i \int_{Q^\nu_i \setminus E(Q^\nu_i)} \Big(\sum_{k \leqslant L(Q)} |R_k f(y)|^q\Big)^{\gamma/q} dy\Big)^{1/\gamma}$$

$$= \Big(\sum_{\nu=0}^{\infty} \sum_i \int_{Q^\nu_i \setminus E(Q^\nu_i)} \Big(\sum_{\mu=0}^{\nu} [I^{i,\nu}_\mu(y)]^q\Big)^{\gamma/q} dy\Big)^{1/\gamma}$$

where

$$I_\nu^{i,\nu}(y) = \Big(\sum_{k \leqslant L(Q_i^\nu)} |R_k f(y)|^q \Big)^{1/q}$$

and for $\mu = 0, 1, \ldots, \nu-1$

$$I_\mu^{i,\nu}(y) = \Big(\sum_{L(Q^{\mu+1}(i,\nu)) < k \leqslant L(Q^\mu(i,\nu))} |R_k f(y)|^q \Big)^{1/q}.$$

By Minkowski's inequality the term in (6.6) is dominated by

$$\Big(\sum_{\mu=0}^{\infty} \Big(\sum_i \int_{Q_i^\mu \setminus E(Q_i^\mu)} |I_\mu^{i,\mu}(y)|^\gamma \, dy \Big)^{q/\gamma} \Big)^{1/q} + \Big(\sum_{\mu=0}^{\infty} \Big(\sum_{\nu > \mu} \sum_i \int_{Q_i^\nu \setminus E(Q_i^\nu)} |I_\mu^{i,\nu}(y)|^\gamma \, dy \Big)^{q/\gamma} \Big)^{1/q}$$
$$=: \Big(\sum_{\mu=0}^{\infty} J_{\mu,1}^q \Big)^{1/q} + \Big(\sum_{\mu=0}^{\infty} J_{\mu,2}^q \Big)^{1/q}.$$

By definition of $E(Q_i^\nu)$ we have $I_\mu^{i,\mu}(y) \leqslant B_f$ in $Q_i^\mu \setminus E(Q_i^\mu)$ and consequently

$$\sum_{\mu=0}^{\infty} J_{\mu,1}^q \leqslant \sum_\mu \Big(\sum_i |Q_i^\mu| \, B_f^\gamma \Big)^{q/\gamma} \leqslant B_f^q \sum_\mu |\Omega_\mu|^{q/\gamma}. \tag{6.7}$$

In order to estimate $J_{\mu,2}^q$ we observe that

$$\bigcup_{\substack{\nu > \mu \\ Q_s^\nu \subset Q_i^\mu}} Q_s^\nu \setminus E(Q_s^\nu) = \bigcup_{Q_j^{\mu+1} \subset Q_i^\mu} Q_j^{\mu+1}.$$

Also if $y \in Q_s^\nu \setminus E(Q_s^\nu) \subset Q_j^{\mu+1} \subset Q_m^\mu$ we have $Q^{\mu+1}(i, \nu) = Q_j^{\mu+1}$, $Q^\mu(i, \nu) = Q_m^\mu$ and obtain

$$J_{\mu,2}^q \leqslant \Big(\sum_m \sum_{Q_j^{\mu+1} \subset Q_m^\mu} \int_{Q_j^{\mu+1}} \Big(\sum_{L(Q_j^{\mu+1}) \leqslant k \leqslant L(Q_m^\mu)} |R_k f(y)|^q \Big)^{\gamma/q} dy \Big)^{q/\gamma}.$$

If $Q_j^{\mu+1}$ is a Whitney cube which is contained in the Whitney cube Q_m^μ, then there is an $x_{j,\mu} \in Q_m^\mu \setminus \Omega_{\mu+1}$ such that $\operatorname{dist}(x_{j,\mu}, Q_j^{\mu+1}) \leqslant 4\sqrt{n}$ (see Stein [24], p. 167). Hence $R_k f(y) \leqslant \mathcal{N}^k f(x_{j,\mu})$ for $y \in Q_j^{\mu+1}$, $k \geqslant L(Q_j^{\mu+1})$, which implies

$$\Big(\sum_{L(Q_j^{\mu+1}) \leqslant k \leqslant L(Q_m^\mu)} |R_k f(y)|^q \Big)^{1/q} \leqslant B_f,$$

and furthermore

$$\sum_{\mu=0}^{\infty} J_{\mu,2}^q \leqslant c \sum_\mu \Big(\sum_j |Q_j^{\mu+1}| \, B_f^\gamma \Big)^{q/\gamma} \leqslant c B_f^q \sum_\mu |\Omega_{\mu+1}|^{q/\gamma}. \tag{6.8}$$

By (6.5), (6.7) and (6.8) we finally obtain

$$\Big(|Q|^{-1} \int_Q \Big(\sum_{k \leqslant L(Q)} |R_k f(y)|^q \Big)^{\gamma/q} dy \Big)^{1/\gamma} \leqslant c B_f |Q|^{-1/\gamma} \Big(\sum_{\mu=0}^{\infty} |\Omega_\mu|^{q/\gamma} \Big)^{1/q}$$
$$\leqslant cA \, \|f\|_{\dot F_0^{\infty q}} \Big(\sum_{\mu=0}^{\infty} 2^{-\mu q^2/\gamma} \Big)^{1/q} \leqslant C_{n,q,\gamma} \, \|f\|_{\dot F_0^{\infty q}}$$

where

$$C_{n,q,\gamma} \leqslant c_n A (1 + \gamma/q^2)^{1/q}.$$

The proof of Lemma 6.4 shows that $A = A_{n,q} \leqslant \tilde{C}_{n,r}$ for all $q \in [r, \infty]$, hence

$$C_{n,q,\gamma} \leqslant C_{n,r} \gamma^{1/q} \tag{6.9}$$

for all $r \leqslant q \leqslant \gamma \leqslant \infty$. ∎

Proof of Proposition 6.4. One half of the proposition is an immediate consequence of Jensen's inequality. The other half follows by Proposition 6.3 and a standard argument using the growth of the constant in (6.9), for $\gamma \to \infty$:

$$|Q|^{-1} \int_Q \exp \Big[\varepsilon^q \sum_{k \leqslant L(Q)} |R_k f(y)|^q / \|f\|_{\dot F_0^{\infty q}}^q \Big] dy$$
$$\leqslant \sum_{l=0}^{\infty} \frac{\varepsilon^{lq}}{l!} \|f\|_{\dot F_0^{\infty q}}^{-ql} |Q|^{-1} \int_Q \Big[\Big(\sum_{k \leqslant L(Q)} |R_k f(y)|^q \Big)^{1/q} \Big]^{lq} dy \leqslant \sum_{l=0}^{\infty} \frac{\varepsilon^{lq}}{l!} c_{n,r}^{lq} (1 + l^{1/q})^{lq},$$

which is bounded if $\varepsilon < c_{n,r}^{-1} e^{-1/q} \leqslant c_{n,r}^{-1} e^{-1/r}$, if $q \geqslant r$. ∎

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The convolution equation of Choquet and Deny on semigroups

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Abstract. We characterize the nonnegative solutions f of the convolution equation $f(x) = \int_S f(x+y)\,d\sigma(y)$, $\forall x \in S$, where S is a locally compact, separable, metrizable abelian semigroup with cancellation, and σ is a nonnegative measure. The technique is to identify the extreme rays of the cone of solutions. The case where S is a group was studied by Choquet and Deny.

§ 1. Introduction. Consider the convolution equation

$$\mu = \mu * \sigma \tag{1.1}$$

on a locally compact abelian group G, where σ, μ are regular Borel measures on G, $\sigma \geqslant 0$ is given, and μ is to be determined. Choquet and Deny [4] showed that if σ is a probability measure, and if the regularization of μ is bounded (i.e., $\mu * \varphi$ is bounded for any continuous function φ on G with compact support), then $\mu = f \cdot \omega$, where ω is the Haar measure on G, and f satisfies

$$f(x) = f(x+y), \quad \forall x \in G,\ y \in \operatorname{supp}\sigma,$$

i.e. f is a periodic function with periods $y \in \operatorname{supp}\sigma$. The equation in the form

$$f(x) = \int_G f(x-y)\,d\sigma(y), \quad \forall x \in G, \tag{1.2}$$

was later considered by Doob, Snell and Williamson by a simple martingale argument [7] (see also [15, p. 151]). The result has important applications in renewal processes [8].

The nonnegative measures μ satisfying (1.1) were characterized by Deny [6]: Suppose in addition G is metrizable and separable, and $\operatorname{supp}\sigma$ generates the group G. Then the extreme rays of the cone

$$H = \{\mu \geqslant 0 \colon \mu * \sigma = \mu\}$$

are of the form $\mu = cg \cdot \omega$, where $c > 0$ is a constant, and g is a nonnegative

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