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Stable rank of holomorphic function algebras

by

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Abstract. We calculate the stable rank of stable subalgebras of $A(K)$.

Introduction. The concept of the stable rank of a ring, introduced by H. Bass [1], has been very useful in treating some problems in algebraic K-theory. In a series of papers G. Corach and F. D. Suárez calculated the stable rank of many Banach algebras. Among them are the well-known algebras $A(K)$, where $A(K)$ is the Banach algebra of all continuous complex-valued functions on a compact set K of the plane C which are analytic in the interior K^0 of K. In this paper we restrict ourselves mainly to subalgebras of $A(K)$, where K has a "good" boundary. For these algebras we calculate the stable rank. It is worth mentioning that the algebras may bear no topology at all. Many subalgebras of the disc algebra $A(\bar{\mathbf{D}})$ satisfy our conditions, for example, W^+, $A^\infty(\bar{\mathbf{D}})$ and $A_\alpha(\bar{\mathbf{D}})$ (for definition, see below).

This paper presents material from the author's thesis. In a forthcoming paper we will study the subalgebras of the disc algebra more closely.

§1. It is well known that the group of units in a Banach algebra is open. Unfortunately, this feature is lost in the general case of a topological algebra. Therefore we define:

A topological algebra A is called a *Q-algebra* if the set of units, A^{-1}, is open in A.

In this paper we consider complex, commutative Q-algebras with unit element being denoted by 1.

Given a Q-algebra A, an element $a \in A^n$ is called *unimodular* if there exists $b \in A^n$ such that

$$\langle b, a\rangle := \sum_{i=1}^{n} b_i a_i = 1.$$

We denote by $U_n(A)$ the set of unimodular elements of A^n. Finally, $a = (a_1, \ldots, a_n) \in U_n(A)$ is called *reducible* if there exist $x_1, \ldots, x_{n-1}$ in A such that

$$(a_1 + x_1 a_n, \ldots, a_{n-1} + x_{n-1} a_n) \in U_{n-1}(A).$$

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The *stable rank* of A, denoted by $\operatorname{sr}(A)$, is the least integer n such that every $a \in U_{n+1}(A)$ is reducible.

From the theory of the stable rank of Q-algebras we mention the following fact; see [10, p. 18, Korollar 1], or [4, Proposition 1]. (The proof given there is also valid for Q-algebras.)

PROPOSITION 1.1. *Suppose that A is a Q-algebra, γ: $[0, 1] \to \mathbf{C}$ is a continuous curve and let Γ: $[0, 1] \to U_2(A)$, $\Gamma(t) := (a-\gamma(t), b)$ such that $(a-\gamma(0), b)$ is reducible. Then $(a-\gamma(1), b)$ is also reducible.*

§ 2. Let $C(K)$ resp. $R(K)$ denote the Banach algebras of all continuous complex-valued functions on a compact set $K \subset \mathbf{C}$, resp. of all those functions in $C(K)$ which can be approximated uniformly by rational functions with poles off K. Let $\|f\|_K = \sup_{z\in K} |f(z)|$.

A subalgebra A of $A(K)$ is called *stable* if A contains all the polynomials and if $(f-f(z_0))/(z-z_0) \in A$ whenever $f \in A$ and $z_0 \in K^0$. Many of the well-known examples of subalgebras of $A(K)$ have this property, e.g.,

$$W^+ := \{\sum_{n=0}^{\infty} a_n z^n\colon \sum_{n=0}^{\infty} |a_n| < \infty\};$$

$$A^\infty(K) := \{f \in C(K)\colon f \text{ analytic in } K^0,\ \forall n \exists g_n \in C(K)\colon g_n|_{K^0} = f^{(n)}\};$$

$$A_\alpha(K) := \{f \in C(K)\colon f \text{ analytic in } K^0 \text{ and satisfies a Hölder–Lipschitz condition on } K \text{ of order } \alpha\}\ (0 < \alpha \leqslant 1).$$

A subalgebra A of $A(K)$ is called *inversionally closed* if a function $f \in A$ is invertible whenever it has no zero in K. (Note that no invertible function in A can have a zero in K. We require the converse.)

All algebras mentioned above are inversionally closed and stable.

The proof for $A^\infty(K)$ is trivial, for W^+ we refer to [9], p. 301, whereas the proof for $A_\alpha(K)$ is standard, but requires some effort.

THEOREM 2.1. *Every stable and inversionally closed subalgebra of $R(K)$ has stable rank one.*

Proof. We have to show that every unimodular element (f, g) is reducible, that is, we have to show that a function $h \in A$ exists such that $f+hg$ is invertible in A. Since A is inversionally closed it is enough to show that $f+hg$ has no zero in K.

Since the algebra $R(K)$ has stable rank one (see [3] or [10]), there exists $k \in R(K)$ and a zero-free function $u \in R(K)$ such that

$$f+kg = u.$$

Now approximate k by rational functions $r_n \in R(K)$. Obviously we have

$$f+r_n g = u+(r_n-k)g.$$

Since r_n has poles off K, we have $r_n = p_n \cdot q_n^{-1}$, where p_n and q_n are polynomials and q_n has no zeros in K. On the other hand, A is inversionally closed and contains all polynomials, so $r_n \in A$.

There exists $\delta > 0$ such that for all $z \in K$

$$|u(z)| \geqslant \delta,$$

because u has no zero in K. Now we choose N so large that

$$\|(r_N-k)g\|_K < \delta/2$$

and obtain for all $z \in K$

$$|f(z)+r_N(z)g(z)| \geqslant \delta/2.$$

Taking the function h to be r_N we are done. ∎

Remark. In Theorem 2.1 it is enough to assume that A contains all polynomials and is inversionally closed.

It is natural to ask whether the assumption $A \subset R(K)$ can be replaced by $A \subset A(K)$. In this general setting the problem is unsolved, but we present an affirmative answer if K has a "good" boundary.

To this end we need the following proposition, whose proof may be found in [7], Theorem 1, or [10], Theorem 2.8.

PROPOSITION 2.2. *Let $K \subset \mathbf{C}$ be compact such that $R(\partial K) = C(\partial K)$ and let $A \subset A(K)$ be stable and inversionally closed. Then the element $(f, g) \in A^2$ is unimodular iff f and g have no common zero in K.*

Proof. Of course we have only to show that if f and g have no common zero in K, then (f, g) is unimodular.

To this end consider $\alpha, \beta \in C(K)$,

$$\alpha(z) := \frac{\overline{f(z)}}{|f(z)|^2+|g(z)|^2}, \quad \beta(z) := \frac{\overline{g(z)}}{|f(z)|^2+|g(z)|^2}.$$

Obviously, we have $\alpha f+\beta g = 1$. In particular, these functions are continuous on ∂K, so by assumption, we can approximate them by rational functions r_k $(k = 1, 2)$. For all $z \in \partial K$ the following inequality holds:

$$|r_1(z)f(z)+r_2(z)g(z)| \geqslant |\alpha(z)f(z)+\beta(z)g(z)| - \|(r_1(z)-\alpha(z))f_1(z)+(r_2(z)-\beta(z))g(z)\|_{\partial K}.$$

Now the second term on the right-hand side can be made arbitrarily small and the first is identically 1. So, for sufficiently good approximations, the function $r_1 f+r_2 g$ has no zero on ∂K. Let $r_k = p_k/q_k$, where p_k and q_k are polynomials such that q_k has no zero on ∂K. Define $\tilde{q} := q_1 q_2$ and consider the function $H := (\tilde{q}r_1)f+(\tilde{q}r_2)g$. The stability of A gives $H \in A$ and, moreover, the function

H does not vanish on ∂K. This implies that H has only finitely many zeros at all. (This is a consequence of the identity theorem for analytic functions. Note that the interior of K may be disconnected!) By assumption the algebra A is stable and inversionally closed. So there exists a polynomial $s \in A$ and an invertible function $u \in A^{-1}$ such that $H = su$, i.e.,

$$(1) \qquad pf + qg = su,$$

where we have abbreviated $p := \tilde{q} r_1$ and $q := \tilde{q} r_2$ (these are also polynomials).

To finish the proof we show that we can get rid of all the zeros of s in equation (1). If $\zeta \in K$ is a zero of the polynomial s, we write $s(z) = (z-\zeta)\tilde{s}(z)$. Since H does not vanish on ∂K, ζ lies in the interior of K. Note that at least one of the functions f and g does not vanish at ζ, say f. Using the stability of A again, there exists $k \in A$ such that

$$(2) \qquad f(z)/f(\zeta) - 1 = (z-\zeta)k(z) \quad (z \in K).$$

Multiplying (1) with k yields

$$pkf + qkg = sku = (z-\zeta)k\tilde{s}u.$$

Together with (2) this implies the existence of $\tilde{p}, \tilde{q} \in A$, not necessarily polynomials, such that

$$\tilde{p}f + \tilde{q}g = \tilde{s}u.$$

After finitely many steps the right-hand side is invertible in A. ∎

Remark. Proposition 2.2 is a special case of the so called "Nullstellensatz" for these algebras; see [7] or [10].

THEOREM 2.3. *Let $K \subset \mathbf{C}$ be compact such that $R(\partial K) = C(\partial K)$ and let $A \subset A(K)$ be stable and inversionally closed. Then its stable rank is one.*

Remark. The condition on K is rather mild, since by the well-known Hartogs–Rosenthal theorem $R(\partial K) = C(\partial K)$ if ∂K has two-dimensional Lebesgue measure zero. Also A may not be topologically complete.

Proof. We have to show that every unimodular element (f, g) is reducible. Since A is inversionally closed, this is equivalent to the existence of $h \in A$ such that $f + hg$ has no zero in K. For $g = 0$ there is nothing to prove, so we suppose $g \neq 0$.

Step 1: $f(z) = z - \lambda$. Consider the algebra A endowed with the topology of uniform convergence on K. Since A is inversionally closed, the result is a Q-algebra. By Proposition 1.1 it is sufficient to exhibit a path $\Gamma\colon [0, 1] \to U_2(A)$,

$$\Gamma(t) = (z - \gamma(t), g),$$

with $\Gamma(0) = (z-\lambda, g)$ and reducible $\Gamma(1)$.

Let G be the component of K^0 which contains λ. By the maximum principle there exists $\zeta \in \partial G$ with $g(\zeta) \neq 0$. Obviously, $\zeta \in \partial K$ and Proposition 2.2 yields $(z-\zeta, g) \in U_2(A)$.

Because of the continuity of g at ζ there exists a convex neighborhood U of ζ such that g never vanishes in $U \cap G$. Since ζ is a boundary point of G and of K, there exist $\zeta' \in G \cap U$ and $\zeta'' \in U \cap [\mathbf{C} \setminus K]$. Now the algebra A is inversionally closed, in particular we have $1/(z-\zeta'') \in A$, since $z - \zeta'' \in A$.

Let Z denote the set of zeros of g in G. Then it is well known that $G \setminus Z$ is open and connected. So there exists a path $\tilde{\gamma}$ joining λ and ζ' which avoids the zeros of g, that is,

$$(z - \tilde{\gamma}(t), g) \in U_2(A)$$

by Proposition 2.2. Now we join ζ' and ζ'' by a line segment. The composite path with $\tilde{\gamma}$ will be called γ. Now, the line segment joining ζ' and ζ'' is in U, in which g never vanishes. Thus

$$(z - \gamma(t), g) \in U_2(A).$$

Since $z - \zeta''$ is invertible in A, it is clear that $(z-\zeta'', g)$ is reducible.

Proposition 1.1 yields the assertion.

Step 2: f is a polynomial. This is standard, and we refer to [3], p. 630, Step 4, or [10]. (It is done by factoring f into linear factors and then multiplying the results of Step 1.)

Step 3: $f \in A$. Since (f, g) is unimodular, there exist $\alpha, \beta \in A$ such that

$$\alpha f + \beta g = 1.$$

We multiply this identity with g and add it to the former. The result is that the element (f, g^2) is unimodular. Now we use (f, g^2) rather than (f, g).

By the proof of Proposition 2.2 there exist polynomials $p, q, s \in A$ and $u \in A^{-1}$ such that

$$(1) \qquad pf + qg^2 = su.$$

Also it is shown there that the polynomial s has all its zeros in K^0.

Now suppose $p(\zeta) = g(\zeta) = 0$. By the identity above we have $s(\zeta) = 0$, in particular $\zeta \in K^0$. Dividing (1) by $z - \zeta$, we arrive at

$$(2) \qquad \frac{p}{z-\zeta} f + q \frac{g}{z-\zeta} g = \frac{s}{z-\zeta} u.$$

Using the stability of A we know that all the quotients in (2) are elements of A. Now (2) is exactly of the form of (1), only the degree of the polynomial s is reduced.

Repeating this process, we get the identity

$$(3) \qquad pf + \tilde{h} g = su$$

such that $p, \tilde{h}, s \in A$, p and g have no common zero in K and u is invertible in A.

Step 4: The polynomial p "vanishes" in (3). Since p and g have no common zero in K the second step implies the existence of $k, v \in A$, $v \in A^{-1}$ such that $p+kg = v$. Putting this in (3) yields

$$f+Hg = su\frac{1}{v} \tag{4}$$

with $H \in A$ chosen appropriately.

Step 5: The polynomial s in (4) "vanishes". By this equation s and g can have no common zero in K. (Such a zero would be a common zero of f and g, contradicting $\alpha f+\beta g = 1$.) Now the second step implies the existence of $h, w \in A$, $w \in A^{-1}$ such that $s+lg = w$. Together with (4) this yields

$$f+\left(H+lu\frac{1}{v}\right)g = u\frac{1}{v}w.$$

Since the right-hand side is invertible, we are done. ∎

Note that there exist compact sets K with $R(\partial K) = C(\partial K)$, but $A(K) \neq R(K)$; see [12], p. 72, Example 9.8.

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Remarks on singular convolution operators

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Abstract. We prove some endpoint estimates for singular convolution operators. For example, let m be a bounded function such that for some suitable $C_0^\infty(\mathbf{R}^n\setminus\{0\})$-function φ, $\varphi m(t\cdot)$ is in the Besov space $B^q_{\alpha 1}$, uniformly in $t>0$. Then m is a Fourier multiplier on $L^p(\mathbf{R}^n)$ if $\alpha = n(1/p-1/2) > n/q$, $1<p\leqslant 2$, and on H^1 if $\alpha = n/2$, $2<q\leqslant\infty$. If m is radial we may replace $B^q_{n/2,1}$ by $B^2_{n/2,1}$.

1. Introduction. The purpose of this paper is to prove endpoint estimates for some classes of multiplier transformations on $L^p(\mathbf{R}^n)$ and other function spaces. We consider a convolution operator T, defined by $Tf = \mathscr{F}^{-1}[m\mathscr{F}f]$, where $\mathscr{F}f$ (or $\hat{f}$) denotes the Fourier transform of f. The M^p-multiplier norm of m is defined as the norm of T as a bounded operator on $L^p(\mathbf{R}^n)$.

To formulate a theorem let us introduce some notation. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be supported in $\{\xi;\ 1/4<|\xi|<4\}$ and positive in $\{\xi;\ 1/2<|\xi|<2\}$. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be supported in $\{x;\ |x|\leqslant 1\}$ and be equal to 1 in $\{x;\ |x|\leqslant 1/2\}$. Define $\Psi_l(x) = \psi(2^{-l}x)-\psi(2^{-l+1}x)$. Then Ψ_l is supported in $\{x;\ 2^{l-2}\leqslant |x|\leqslant 2^l\}$. Ψ_l will be used to decompose the convolution kernel $\mathscr{F}^{-1}[m]$.

THEOREM 1.1. *Suppose that* $1<p<r\leqslant 2$ *and*

$$\|m\|_\infty + \sup_{t>0}\sum_{l>0} 2^{ln(1/p-1/r)}\|\varphi m(t\cdot)*\hat{\Psi}_l\|_{M^r} \leqslant A. \tag{1.1}$$

Then m is a multiplier in M^p, *and* $\|m\|_{M^p}\leqslant cA$. *If* $p=1<r\leqslant 2$ *we have the conclusion that T is a bounded operator on the Hardy space* H^1.

This result can be considered as an endpoint version of Hörmander's multiplier theorem [16] (compare (1.2) below). It extends a result by Baernstein and Sawyer [1] who proved that the condition

$$\|m\|_\infty^p + \sup_{t>0}\sum_{l>0} 2^{ln(1-p)}\|\varphi m(t\cdot)*\hat{\Psi}_l\|_1^p < \infty$$

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