Sierpiński Medal of the Warsaw University, the Medal of the Comission for National Education and the Alfred Jurzykowski Foundation Award.

Władysław Orlicz was closely associated with this journal, he published in it 54 of his papers starting from volume 1. In 1936 he became a member of the editorial board and in 1962 the editor-in-chief.

For a more detailed account of the work of Władysław Orlicz we refer to his two-volume *Collected Papers* published by PWN–Polish Scientific Publishers, Warsaw 1988.

Władysław Orlicz passed away suddenly in Poznań while proofreading one of his papers. The mathematical community lost the senior Polish mathematician, the very last representative of the **Banach School**.

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On pointwise ergodic theorems for positive operators

by

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Abstract. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and T a positive linear contraction on $L_1(\mu)$. By approximation T can be extended to the space of all nonnegative measurable functions. Suppose V is a positive measurable function and $0 < e \in L_1(\mu)$. Assuming that T is conservative, the following is proved: If $1 < p < \infty$ then $\lim_n (\sum_{i=0}^n T^i f)/(\sum_{i=0}^n T^i e)$ exists and is finite almost everywhere for all $0 \leqslant f \in L_p(V d\mu)$ if and only if $\sup_{n\geqslant 0} (\sum_{i=0}^n T^i V^{1-p'})/(\sum_{i=0}^n T^i e) < \infty$ almost everywhere, where $1/p + 1/p' = 1$. This generalizes a recent result of Martín-Reyes and de la Torre concerning measure preserving transformations on a finite measure space. Related results are also proved.

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Recently, Martín-Reyes and de la Torre ([8]–[10]) considered operators T, acting on measurable functions and having the form

$$Tf(x) = f(\tau x)$$

where $\tau\colon X \to X$ is a measure preserving transformation, and studied the problem of identifying those positive and measurable functions V such that for each $f \in L_p(V d\mu)$, the limit $\lim_n (\sum_{i=0}^n T^i f)/(\sum_{i=0}^n T^i e)$ exists and is finite a.e. on X, where $0 < e \in L_1(\mu)$ is fixed arbitrarily. Under the hypothesis that $\mu(X) < \infty$, Martín-Reyes and de la Torre succeeded in characterizing such functions V; they proved, with $e \equiv 1$, that (i) if $1 < p < \infty$ then such functions V are those which satisfy

$$\sup_{n\geqslant 0} \Big(\sum_{i=0}^n T^i V^{1-p'}\Big)\Big/\Big(\sum_{i=0}^n T^i e\Big) < \infty \quad \text{a.e. on } X \tag{1}$$

where $1/p + 1/p' = 1$, (ii) if $p = 1$ then such functions V are those which satisfy

$$\inf_{n\geqslant 0} V(\tau^n x) > 0 \quad \text{a.e. on } X. \tag{2}$$

Then the author [13] considered (more general) operators T of the form

$$Tf(x) = h(x) f(\tau x)$$

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where h is a positive measurable function and $\tau\colon X\to X$ is a null preserving transformation, and proved that if T is a conservative linear contraction on $L_1(\mu)$, where $\mu(X)=\infty$ may happen, then (2) is also equivalent to the a.e. convergence of the ratios $(\sum_{i=0}^n T^i f)/(\sum_{i=0}^n T^i e)$ for each $f\in L_1(Vd\mu)$. Thus a generalization was obtained for $p=1$.

In this paper we shall consider (most) general operators T, satisfying $T\geqslant 0$ and $\|T\|_1\leqslant 1$, and continue the investigation. Similar results will be proved.

2. Preliminaries and results. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and let $M^+(\mu)$ denote the space of all nonnegative extended real valued measurable functions on X. As usual, two functions f and g in $M^+(\mu)$ are not distinguished provided that $f=g$ a.e. on X. Let T be a positive linear contraction on $L_1(\mu)$; thus $\|Tf\|_1\leqslant\|f\|_1$ for all $f\in L_1(\mu)$ and $TL_1^+(\mu)\subset L_1^+(\mu)$, where $L_1^+(\mu)$ denotes the space of all nonnegative functions in $L_1(\mu)$. In order to extend the domain of T to $M^+(\mu)$, fix any $f\in M^+(\mu)$ and take $f_n\in L_1^+(\mu)$, $n=1,2,\ldots$, such that $f_n\uparrow f$ a.e. on X. We then define

$$Tf=\lim_n Tf_n \quad \text{a.e. on } X.$$

It is easily checked that by this process T can be uniquely extended to an operator on $M^+(\mu)$ satisfying $T(f+g)=Tf+Tg$ and $T(\alpha f)=\alpha Tf$ for all $f, g\in M^+(\mu)$ and constants α, $0\leqslant\alpha<\infty$. Similarly, the adjoint operator T^* of T, which acts on $L_\infty(\mu)$, can be extended to $M^+(\mu)$. In the sequel, T and T^* will be understood to be defined on $M^+(\mu)$ in this manner. For simplicity we shall use the following notation:

$$R_0^n(f,e)=\Big(\sum_{i=0}^n T^i f\Big)/\Big(\sum_{i=0}^n T^i e\Big) \quad\text{and}\quad M_0^\infty(f,e)=\sup_{n\geqslant 0} R_0^n(f,e)$$

for $f\in M^+(\mu)$ and $0<e\in L_1(\mu)$.

T is called *conservative* if $\sum_{i=0}^\infty T^i g=\infty$ a.e. on X for some $0<g\in L_1(\mu)$, and $E\in\mathscr{F}$ is called *invariant* (under T) if $T^*1_E=1_E$, where 1_E denotes the indicator function of E. If there exists no invariant set E with $\mu(E)>0$ and $\mu(X\backslash E)>0$, then T is called *ergodic*. We also consider a null preserving transformation $\tau\colon X\to X$. (By definition τ is *null preserving* if it is a measurable transformation from X to X such that $A\in\mathscr{F}$ and $\mu(A)=0$ imply $\mu(\tau^{-1}A)=0$.) τ is called *conservative* if there exists no $E\in\mathscr{F}$ such that $\tau^{-1}E\subset E$ and $\mu(E\backslash\tau^{-1}E)>0$, and *ergodic* if $E\in\mathscr{F}$ and $\tau^{-1}E=E$ imply $\mu(E)=0$ or $\mu(X\backslash E)=0$. As is known, if we define an operator $T\colon L_1(\mu)\to L_1(\mu)$ by the relation

$$\int_A Tf\,d\mu=\int_{\tau^{-1}A} f\,d\mu \quad (f\in L_1(\mu),\ A\in\mathscr{F})$$

then T becomes a positive linear contraction on $L_1(\mu)$; T is conservative (resp. ergodic) if and only if τ is conservative (resp. ergodic). This T will be referred to as the *operator associated with* τ; it is clear that $T^*f(x)=f(\tau x)$. (For more detailed discussions on these matters we refer the reader to Krengel's book [7].)

We are now in a position to state the first result.

THEOREM 1. *Let T be a positive and conservative linear contraction on $L_1(\mu)$. Let V be a positive measurable function on X and $0<e\in L_1(\mu)$. If $1<p<\infty$ then the following are equivalent:*

(a) *For each $f\in L_p^+(Vd\mu)$, $\lim_n R_0^n(f,e)$ exists and is finite a.e. on X.*

(b) *$M_0^\infty(V^{1-p'},e)<\infty$ a.e. on X, where $1/p+1/p'=1$.*

If $p=1$ then (a) *implies*

(c) *$\inf_{n\geqslant 0} T^{*n}V>0$ a.e. on X,*

and conversely (a) *is implied by*

(d) *$V\geqslant\inf_{n\geqslant 1}T^{*n}V>0$ a.e. on X.*

Proof. Let $1<p<\infty$.

(a)$\Rightarrow$(b). We may assume that $\int e\,d\mu=1$. Then the operator $S\colon L_1(ed\mu)\to L_1(ed\mu)$ defined by

$$Sf=e^{-1}T(ef) \quad (f\in L_1(ed\mu))$$

is a positive and conservative linear contraction on $L_1(ed\mu)$. Clearly, the adjoint operator S^* of S is identical with the adjoint operator T^* of T. We now apply the Neveu–Chacon identification theorem of the limit in the Chacon–Ornstein ratio ergodic theorem (see e.g. [7], Chapter 3) to infer that

$$\begin{aligned}\lim_n R_0^n(V^{1-p'},e)&=\lim_n\Big(\sum_{i=0}^n T^iV^{1-p'}\Big)/\Big(\sum_{i=0}^n T^ie\Big)\\&=\lim_n\Big(\sum_{i=0}^n S^i(e^{-1}V^{1-p'})\Big)/\Big(\sum_{i=0}^n S^i1\Big)\\&=E\{e^{-1}V^{1-p'}\,|\,(X,\mathscr{I},ed\mu)\}\quad\text{a.e. on } X,\end{aligned}$$

where $\mathscr{I}$ and $E\{\cdot\,|\,(X,\mathscr{I},ed\mu)\}$ denote, respectively, the σ-field of all invariant subsets of X and the conditional expectation operator with respect to the measure space $(X,\mathscr{I},ed\mu)$. It follows that

$$\{x\colon M_0^\infty(V^{1-p'},e)(x)=\infty\}=\{x\colon \lim_n R_0^n(V^{1-p'},e)(x)=\infty\}\in\mathscr{I}.$$

Hence we may and do suppose that $\{M_0^\infty(V^{1-p'},e)=\infty\}=X$, when $\mu(\{M_0^\infty(V^{1-p'},e)=\infty\})>0$; and from this we derive a contradiction as follows.

In fact, by a standard argument we may assume, without loss of generality, that $(X, \mathscr{F}, \mu)$ is a separable measure space. Thus, $(X, \mathscr{F}, e d\mu)$ is a separable probability space, and by the isomorphism theorem (see e.g. Halmos [5], p. 173) together with the theory of Rokhlin [11], the measure algebra associated with $(X, \mathscr{F}, e d\mu)$ is isomorphic to the measure algebra associated with a Lebesgue space $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$. (Roughly speaking, $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$ is a Lebesgue space if and only if $\tilde{X}$ is a compact metric space and $(\tilde{X}, \tilde{\mathscr{F}}, \tilde{\mu})$ is the completion of a probability space $(\tilde{X}, \tilde{\mathscr{B}}, \tilde{\mu})$, where $\tilde{\mathscr{B}}$ stands for the Borel subsets of $\tilde{X}$.)

Since the measure algebra isomorphism has no influence upon the a.e. convergence of a sequence of functions, we may assume below that

$(X, \mathscr{F}, e d\mu)$ is a Lebesgue space.

Then the ergodic decomposition of S can be considered (see Section 11.11 of [6] and §3 of [11]). That is, there exists a countable family $\{E_i\}$ of invariant sets such that if ξ denotes the decomposition of X induced by $\{E_i\}$, i.e., $C \in \xi$ has the form

$$C = \bigcap_i E_i(\varepsilon_i)$$

where $\varepsilon_i = \pm 1$, $E_i(1) = E_i$ and $E_i(-1) = X \backslash E_i$, then:

(i) The factor space $(X/\xi, \mathscr{F}_\xi, \mu_\xi)$ of $(X, \mathscr{F}, e d\mu)$ with respect to ξ is a Lebesgue space.

(ii) To a.e. $C \in X/\xi$, with respect to μ_ξ, there corresponds a Lebesgue measure μ_C on C such that if $A \in \mathscr{F}$ then $A \cap C$ is measurable with respect to μ_C for a.e. $C \in X/\xi$, and the function $h(C) = \mu_C(A \cap C)$ is measurable with respect to μ_ξ and satisfies, for all $Z \in \mathscr{F}$ of the form $Z = \iota^{-1}(Z/\xi)$, where $Z/\xi \in \mathscr{F}_\xi$ and $\iota\colon X \to X/\xi$ denotes the canonical mapping,

$$\int_{A \cap Z} e\, d\mu = \int_{Z/\xi} h(C)\, d\mu_\xi(C) = \int_{Z/\xi} \mu_C(A \cap C)\, d\mu_\xi(C).$$

(iii) To a.e. $C \in X/\xi$ there corresponds a conservative and ergodic positive linear contraction S_C on $L_1(C, \mu_C)$ such that if $f \in L_1(X, e d\mu)$ and $g \in L_\infty(X, e d\mu)$ then, for a.e. $C \in X/\xi$,

$$(Sf)_C = S_C f_C \quad \text{and} \quad (S^* g)_C = S_C^* g_C$$

where $(Sf)_C$, $(S^* g)_C$ and f_C denote, respectively, the restrictions of Sf, $S^* g$ and f to C.

Since $X = \{M_0^\infty(V^{1-p'}, e) = \infty\}$ by assumption, we see in virtue of the Chacon–Ornstein theorem that

$$\int_C e^{-1} V^{1-p'} d\mu_C = \infty \quad \text{for a.e. } C \in X/\xi.$$

Using this we see without difficulty that there exists an $F \in \mathscr{I}$, with $\mu(F) > 0$, and a countable family $\{A_n\}$ of disjoint sets in $\mathscr{F}$ such that

$$1 \leqslant \int_{A_n \cap C} e^{-1} V^{1-p'} d\mu_C < \infty \quad \text{for all } C \in F/\xi.$$

Write

$$B_n = A_n \cap F, \quad B_\infty = \bigcup_{n=1}^\infty B_n.$$

Since $\mu_C(B_n \cap C) > 0$ for all $C \in F/\xi$, it follows that $\mu(B_n) > 0$.

Define

$$h_n(C) = \int_{B_n \cap C} e^{-1} V^{1-p'} d\mu_C \quad (C \in F/\xi),$$

and let $f \geqslant 0$ be the function on X defined by $f = 0$ on $X \backslash B_\infty$ and

$$f^p(x) V(x) = h_n^{-1}(C) n^{-p} V^{1-p'}(x) \quad \text{for } x \in B_n \cap C.$$

Then we have

$$\begin{aligned}
\int_X f^p V d\mu &= \sum_{n=1}^\infty \int_{B_n} f^p V e^{-1} e\, d\mu \\
&= \sum_{n=1}^\infty \int_{F/\xi} \Big(\int_{B_n \cap C} h_n^{-1}(C) n^{-p} e^{-1} V^{1-p'} d\mu_C \Big) d\mu_\xi \\
&= \sum_{n=1}^\infty \int_{F/\xi} n^{-p} d\mu_\xi = \sum_{n=1}^\infty n^{-p} \int_F e\, d\mu < \infty.
\end{aligned}$$

Next we apply the relations $f = f V^{1/p} V^{-1/p}$ and

$$(f V^{1/p})^p = f^p V = h_n^{-1}(C) n^{-p} V^{1-p'} = h_n^{-1}(C) n^{-p} (V^{-1/p})^{p'}$$

on $B_n \cap C$ for all $C \in F/\xi$ to get

$$\begin{aligned}
\int_{B_n \cap C} f e^{-1} d\mu_C &= \Big(\int_{B_n \cap C} f^p V e^{-1} d\mu_C \Big)^{1/p} \Big(\int_{B_n \cap C} V^{1-p'} e^{-1} d\mu_C \Big)^{1/p'} \\
&= \Big(h_n^{-1}(C) n^{-p} \int_{B_n \cap C} e^{-1} V^{1-p'} d\mu_C \Big)^{1/p} \big(h_n(C)\big)^{1/p'} \\
&= n^{-1} \big(h_n(C)\big)^{1/p'} \geqslant n^{-1} \quad \text{for } C \in F/\xi,
\end{aligned}$$

where the last inequality is due to the fact that $h_n(C) \geqslant 1$ for all $C \in F/\xi$. Consequently

$$\int_C f e^{-1} d\mu_C = \sum_{n=1}^\infty \int_{B_n \cap C} f e^{-1} d\mu_C \geqslant \sum_{n=1}^\infty n^{-1} = \infty \quad (C \in F/\xi),$$

and, since S_C is conservative and ergodic, we have for $C \in F/\xi$

$$\lim_n R_0^n(f, e) = \lim_n \Big(\sum_{i=0}^n S^i(fe^{-1})\Big)/\Big(\sum_{i=0}^n S^i 1\Big)$$
$$= \lim_n \Big(\sum_{i=0}^n S_C^i(fe^{-1})_C\Big)/\Big(\sum_{i=0}^n S_C^i 1_C\Big) = \infty \qquad \mu_C\text{-a.e. on } C.$$

Hence $\lim_n R_0^n(f, e) = \infty$ μ-a.e. on F, and this contradicts (a), because $f \in L_p^+(Vd\mu)$.

(b) $\Rightarrow$ (a). Put

$$A_N = \{x\colon \lim_n R_0^n(V^{1-p'}, e)(x) \leqslant N\} \qquad \text{for } N \geqslant 1.$$

Since $A_N \in \mathscr{I}$ and since (b) implies $A_N \uparrow X$, we may suppose without loss of generality that $A_N = X$. Then, since

$$\int_X V^{1-p'} d\mu = \int_X E\{e^{-1} V^{1-p'} | (X, \mathscr{F}, ed\mu)\} ed\mu \leqslant \int_X N\, ed\mu < \infty,$$

the Hölder inequality yields that if $f \in L_p^+(Vd\mu)$ then

$$\int_X f\, d\mu \leqslant \Big(\int_X f^p V d\mu\Big)^{1/p} \Big(\int_X V^{1-p'} d\mu\Big)^{1/p'} < \infty,$$

which together with the Chacon–Ornstein theorem proves (a).

Let $p = 1$.

(a) $\Rightarrow$ (c). We use the operator S on $L_1(ed\mu)$ introduced in the above proof of (a) $\Rightarrow$ (b). Write

$$W = \inf_{n \geqslant 0} S^{*n} V.$$

It is clear that $W \geqslant \min\{V, S^* W\}$. Thus, if we let $A = \{W = 0\}$, then any $f \in L_1^+(A, ed\mu)$ satisfies

$$0 \leqslant \int_X (Sf) W ed\mu = \int_X f(S^* W) ed\mu = \int_A f(S^* W) ed\mu \leqslant \int_A f W ed\mu = 0,$$

so that $Sf \in L_1^+(A, ed\mu)$. Since S is conservative, this implies $A \in \mathscr{I}$. Therefore we may and do suppose that $W = 0$ on X when $\mu(A) > 0$; and from this we derive a contradiction.

In fact, we first notice that for a.e. $C \in X/\xi$ (with respect to μ_ξ)

$$\inf_{n \geqslant 0} S_C^{*n} V_C = \inf_{n \geqslant 0} (S^{*n} V)_C = 0 \qquad \mu_C\text{-a.e. on } C.$$

Hence the function

$$h_n(C) = \mu_C(\{V < n^{-1}\} \cap C) \qquad (C \in X/\xi)$$

satisfies $h_n(C) > 0$ for μ_ξ-a.e. $C \in X/\xi$, so that if f_n denotes the function on X defined by

$$f_n(x) = n^{-1} (h_n(C))^{-1} 1_{\{V < 1/n\}}(x) \qquad (x \in C \in X/\xi),$$

then we have

$$\int_X f_n ed\mu = \int_{X/\xi} \Big(\int_C f_n d\mu_C\Big) d\mu_\xi(C) = 1/n,$$

$$\int_X f_n V ed\mu \leqslant \int_{X/\xi} \Big(n^{-1} \int_C f_n d\mu_C\Big) d\mu_\xi(C) = 1/n^2.$$

Therefore the function $f = \sum_{n=1}^\infty f_n$ satisfies $f \in L_1^+(Ved\mu)$ and $f \notin L_1^+(ed\mu)$. It follows from the Neveu–Chacon identification theorem that

$$\lim_n R_0^n(ef, e) = \lim_n \Big(\sum_{i=0}^n S^i f\Big)/\Big(\sum_{i=0}^n S^i 1\Big) = \infty \qquad \text{a.e. on } X,$$

and this contradicts (a).

(d) $\Rightarrow$ (a). (d) implies that the function $W = \inf_{n \geqslant 0} T^{*n} V$ satisfies

$$T^* W \leqslant \inf_{n \geqslant 1} T^{*n} V = \inf_{n \geqslant 0} T^{*n} V = W \quad \text{and} \quad W > 0 \text{ a.e. on } X.$$

Since T is conservative, W is measurable with respect to $\mathscr{I}$ (see e.g. [7], p. 116). Thus the set $A_N = \{W \geqslant 1/N\}$ is in $\mathscr{I}$ for each $N \geqslant 1$. Since $A_N \uparrow X$, we may suppose without loss of generality that $A_N = X$. Then $f \in L_1^+(Vd\mu)$ implies $f \in L_1^+(\mu)$, so that the Chacon–Ornstein theorem proves (a).

Corollary. *Let T be the positive linear contraction on $L_1(\mu)$ associated with a null preserving and conservative transformation $\tau\colon X \to X$. Let V be a positive measurable function on X and $0 < e \in L_1(\mu)$. Then the following are equivalent:*

(a) *For each $f \in L_1^+(Vd\mu)$, $\lim_n R_0^n(f, e)$ exists and is finite a.e. on X.*
(b) $\inf_{n \geqslant 0} V(\tau^n x) > 0$ *a.e. on X.*

Proof. Since τ is conservative, so is T, and further $V(x) \geqslant \inf_{n \geqslant 1} V(\tau^n x)$ for a.e. $x \in X$. Thus the corollary follows from Theorem 1, because $T^{*n} V(x) = V(\tau^n x)$.

Remark. In Theorem 1, the implications (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (d) for $p = 1$ do not hold; and if T is not assumed to be conservative, the equivalence (a) $\Leftrightarrow$ (b) for $1 < p < \infty$ and the implication (a) $\Rightarrow$ (c) for $p = 1$ do not hold. To show this, we give the following examples.

Example 1. We consider the measure space $(X, \mathscr{F}, \mu)$, where X is the nonnegative integers, $\mathscr{F}$ is the subsets of X, and μ is defined by $\mu(\{k\}) = 2^{-k}$

for each $k \geqslant 0$. Let T be the operator defined by

$$Tf(k) = 2^{-1}(f(0)+f(k+1)) \quad (k \geqslant 0).$$

Then we see that $\|Tf\|_1 = \|f\|_1$ for all $f \in L_1^+(\mu)$ and $T1 = 1$; thus T is a positive and conservative linear contraction on $L_1(\mu)$. Further, if W is the function on X defined by $W(0) = 1$ and $W(k) = 0$ for $k \geqslant 1$, then an elementary calculation shows that

$$\lim_n T^{*n} W(k) = 2^{-1} \quad \text{for each } k \geqslant 0.$$

It follows that the function V on X defined by $V(k) = 10^{-k}$ $(k \geqslant 0)$ satisfies $\inf_{n \geqslant 0} T^{*n} V(k) > 0$ for each $k \geqslant 0$, and (c) holds. (Incidentally we note that T maps $L_1^+(Vd\mu)$ into $L_1^+(Vd\mu)$.) But (a) does not hold for $p = 1$, because the function f in $L_1^+(Vd\mu)$ defined by $f(k) = 10^k$ for each $k \geqslant 0$ satisfies

$$\lim_n R_0^n(f, 1)(0) \geqslant \lim_n (n+1)^{-1} T^n f(0) \geqslant \lim_n (n+1)^{-1} 5^n = \infty.$$

Next, let $V(0) = 2^{-1}$ and $V(k) = 1$ for all $k \geqslant 1$. Then we have $V(0) < \inf_{n \geqslant 1} T^{*n} V(0)$. Thus (d) does not hold. But, since $L_1(Vd\mu) = L_1(\mu)$ as sets of functions, the Chacon–Ornstein theorem implies that (a) holds for $p = 1$.

Example 2. Let $1 < p < \infty$. We consider the measure space $(X, \mathscr{F}, \mu)$, where X is the integers, $\mathscr{F}$ is the subsets of X, and μ is defined by

$$\mu(\{k\}) = \begin{cases} 1 & \text{if } k \leqslant 1, \\ k^p & \text{if } k > 1. \end{cases}$$

Define

$$Tf(k) = f(k+1) \quad (k \in X).$$

T becomes a positive linear contraction on $L_1(\mu)$ which is not conservative. Let $V = 1$ on X and $0 < e \in L_1(\mu)$ be any function. Then

$$M_0^\infty(V^{1-p'}, e)(k) = \sup_{n \geqslant 1} n\Big(\sum_{i=0}^{n-1} e(k+i)\Big)^{-1} = \infty \quad (k \in X),$$

and so (b) does not hold. But the Hölder inequality yields

$$\sum_{k=1}^\infty f(k) \leqslant \Big(\sum_{k=1}^\infty f^p(k) k^p\Big)^{1/p} \Big(\sum_{k=1}^\infty k^{-p'}\Big)^{1/p'} < \infty$$

for all $f \in L_p^+(Vd\mu)$. Hence (a) holds.

Example 3. Let X and $\mathscr{F}$ be the same as in Example 2, but we consider here μ to be the counting measure. Then the operator T in Example 2 becomes an invertible positive isometry of $L_1(\mu)$ which is not conservative either. Define

$V(k) = 1$ if $k \geqslant 0$ and $V(k) = |k|^{-1}$ if $k < 0$. Then

$$\lim_n T^{*n} V(k) = \lim_n V(k-n) = 0 \quad (k \in X).$$

Thus (c) does not hold. But, clearly, (a) holds for $p = 1$.

Although in the above theorem the assumption of T's being conservative is not omitted, if T is invertible and the ratios

$$R_{-m}^n(f, e) = \Big(\sum_{i=-m}^n T^i f\Big) \Big/ \Big(\sum_{i=-m}^n T^i e\Big) \quad \text{with } m, n \geqslant 0$$

are considered, then the assumption is not necessary. That is, we have

Theorem 2 (cf. [8], [13]). *Let T be an invertible positive isometry of $L_1(\mu)$. Let V be a positive measurable function on X and $0 < e \in L_1(\mu)$. If $1 < p < \infty$ then the following are equivalent:*

(a) *For each $f \in L_p^+(Vd\mu)$, the limit*

$$R_{-\infty}^\infty(f, e)(x) = \lim_{m,n \to \infty} R_{-m}^n(f, e)(x)$$

exists and is finite a.e. on X.

(b) *The maximal function $M_{-\infty}^\infty(V^{1-p'}, e)(x) = \sup_{m,n \geqslant 0} R_{-m}^n(V^{1-p'}, e)(x)$ is finite a.e. on X.*

If $p = 1$ then (a) *is equivalent to*

(c) $\inf_{-\infty < n < \infty} T^{*n} V > 0$ *a.e. on X.*

Sketch of proof. Since T is an invertible positive isometry of $L_1(\mu)$, T^{-1} is also positive and for any $f \in M^+(\mu)$

$$R_{-\infty}^\infty(f, e)(x) = E\{e^{-1} f \mid (X, \mathscr{I}, ed\mu)\}(x) \quad \text{a.e. on } X \text{ (cf. [12]).}$$

Upon using this and the ergodic decomposition of the invertible S, S being introduced in the proof of Theorem 1, the implication (a) $\Rightarrow$ (b) for $1 < p < \infty$ follows as in Theorem 1. The proofs of the other implications are also similar to those of the corresponding parts of Theorem 1. We omit the details.

In the following theorem we do not assume that T is invertible or conservative. Before stating the theorem, let us recall that if $0 < p < 1$ then $L_p(\mu) = \{f\colon \int |f|^p d\mu < \infty\}$ is a complete linear metric space with the metric d_p defined by

$$d_p(f, g) = \int |f-g|^p d\mu = \|f-g\|_p^p;$$

and if $\mu(X) < \infty$ then $L_0(\mu) = \{f\colon |f| < \infty \text{ a.e. on } X\}$ is a complete linear metric space with the metric d_0 defined by

$$d_0(f, g) = \int \frac{|f-g|}{1+|f-g|} d\mu,$$

and a sequence $\{f_n\}$ in $L_0(\mu)$ converges in measure to an $f \in L_0(\mu)$ if and only if $\lim_n d_0(f_n, f) = 0$.

THEOREM 3. *Let T be a positive linear contraction on $L_1(\mu)$. Let V be a positive measurable function on X and $0 < e \in L_1(\mu)$. If $0 < p < \infty$ then the following are equivalent:*

(a) *For any $f \in L_p^+(V d\mu)$, $\lim_n R_0^n(f, e)$ exists and is finite a.e. on X.*
(b) *$M_0^\infty(f, e) < \infty$ a.e. on X for all $f \in L_p^+(V d\mu)$.*
(c) *There exists a positive measurable function U on X such that*

$$\int_{\{M_0^\infty(f,e) > \lambda\}} U\, d\mu \leqslant \lambda^{-p} \int_X f^p V\, d\mu \quad (\lambda > 0,\ f \in L_p^+(V d\mu)).$$

(d) *There exists a positive measurable function U on X such that*

$$\liminf_n \int_{\{R_0^n(f,e) > \lambda\}} U\, d\mu \leqslant \lambda^{-p} \int_X f^p V\, d\mu \quad (\lambda > 0,\ f \in L_p^+(V d\mu)).$$

Proof. As is easily seen, we may and do suppose that $\mu(X) < \infty$ (cf. the proof of Theorem 1).

(a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (a). We first notice that for each $n \geqslant 0$ the mapping $f \to R_0^n(f, e)$ from $L_p^+(V d\mu)$ to $L_0(\mu)$ is continuous. In fact, if this is not the case, then there exists a sequence $\{f_k\}$ in $L_p^+(V d\mu)$, with $\lim_k \int f_k^p V\, d\mu = 0$, such that the sequence $\{R_0^n(f_k, e)\}$ does not converge in measure to the zero function. Thus (if necessary, take a subsequence of $\{f_k\}$) we can choose two positive reals ε and δ so that

$$\mu(\{R_0^n(f_k, e) > \varepsilon\}) > \delta \quad \text{for all } k \geqslant 1.$$

Here we may suppose without loss of generality that the function $f(x) = \sum_{k=1}^\infty f_k(x)$ is in $L_p^+(V d\mu)$. Then, writing $E_k = \{R_0^n(f_k, e) > \varepsilon\}$, we get

$$R_0^n(f, e)(x) = \sum_{k=1}^\infty R_0^n(f_k, e)(x) \geqslant \varepsilon\Big(\sum_{k=1}^\infty 1_{E_k}(x)\Big);$$

but, since $\delta < \mu(E_k) \leqslant \mu(X) < \infty$ for all $k \geqslant 1$, the function $h(x) = \sum_{k=1}^\infty 1_{E_k}(x)$ must satisfy $h = \infty$ on a set of positive measure, which contradicts (b).

By virtue of this fact, we see that the mapping $f \to M_0^\infty(f, e)$ from $L_p^+(V d\mu)$ to $L_0(\mu)$ is continuous at the zero function of $L_p^+(V d\mu)$ (see e.g. [4], p. 10), which is equivalent to saying that there exists a positive decreasing function $C(\lambda)$, defined for $\lambda > 0$ and tending to zero as $\lambda \uparrow \infty$, such that for all $f \in L_p^+(V d\mu)$ with $\|f\|_p \leqslant 1$,

$$\mu(\{M_0^\infty(f, e) > \lambda\}) \leqslant C(\lambda) \quad (\lambda > 0).$$

The proof of (b) $\Rightarrow$ (a) is now a routine matter, since $L_1^+(\mu) \cap L_p^+(V d\mu)$ is a dense subspace of $L_p^+(V d\mu)$ and for each $f \in L_1^+(\mu) \cap L_p^+(V d\mu)$, $\lim_n R_0^n(f, e)$ exists and is finite a.e. on X by the Chacon–Ornstein theorem. We may omit the details. (See e.g. [3], pp. 2–4.)

(b) $\Rightarrow$ (c). This follows from Nikishin's theorem (see e.g. [2], p. 536).

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (b). Let $f \in L_p^+(V d\mu)$. We apply the Neveu–Chacon identification theorem to infer that for a.e. x in the set $E = \{M_0^\infty(f, e) = \infty\}$,

$$M_0^\infty(f, e)(x) = \lim_n R_0^n(f, e)(x) = \infty.$$

It follows that for any $\lambda > 0$

$$E \subset \liminf_n \{R_0^n(f, e) > \lambda\};$$

and thus (d), together with Fatou's lemma, implies

$$\int_E U\, d\mu \leqslant \liminf_n \int_{\{R_0^n(f,e) > \lambda\}} U\, d\mu \leqslant \lambda^{-p} \int_X f^p V\, d\mu.$$

Therefore, letting $\lambda \uparrow \infty$, $\int_E U\, d\mu = 0$ and $\mu(E) = 0$. This proves (b).

Let us now apply Theorem 3 to null preserving transformations $\tau\colon X \to X$ in a finite measure space. For simplicity we set

$$A_n f(x) = n^{-1} \sum_{i=0}^{n-1} f(\tau^i x), \qquad MAf(x) = \sup_{n \geqslant 1} |A_n f(x)|.$$

THEOREM 4 (cf. [10]). *Let $(X, \mathscr{F}, \mu)$ be a finite measure space and let $\tau\colon X \to X$ be a null preserving transformation. If $0 < p < \infty$, then the following are equivalent:*

(a) *For any $f \in L_p(\mu)$, $\lim_n A_n f$ exists and is finite a.e. on X.*
(b) *$MAf < \infty$ a.e. on X for all $f \in L_p(\mu)$.*
(c) *There exists a positive measurable function U on X such that*

$$\int_{\{MAf > \lambda\}} U\, d\mu \leqslant \lambda^{-p} \int_X |f|^p\, d\mu \quad (\lambda > 0,\ f \in L_p(\mu)).$$

(d) *There exists a positive measurable function U on X such that*

$$\sup_{n \geqslant 1} \int_{\{|A_n f| > \lambda\}} U\, d\mu \leqslant \lambda^{-p} \int_X |f|^p\, d\mu \quad (\lambda > 0,\ f \in L_p(\mu)).$$

(e) *For every $E \in \mathscr{F}$, $\lim_n n^{-1} \sum_{i=0}^{n-1} \mu(\tau^{-i} E)$ exists, and further there exists a positive measurable function U on X such that*

$$\liminf_n \int_{\{|A_n f| > \lambda\}} U\, d\mu \leqslant \lambda^{-p} \int_X |f|^p\, d\mu \quad (\lambda > 0,\ f \in L_p(\mu)).$$

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are obvious. (b) $\Rightarrow$ (c) follows from Nikishin's theorem, because (b) implies that the mapping $f \to MAf$ from $L_p(\mu)$ to $L_0(\mu)$ is continuous at the zero function of $L_p(\mu)$.

(d) ⇒ (e). We may suppose that $0 < U \leqslant 1$ on X. We first notice, using the Marcinkiewicz interpolation theorem (see e.g. [2], pp. 148–150 for a proof, which is also valid for the case $0 < p < 1$), that if (d) holds then to each r, $p < r < \infty$, there corresponds a constant $C_r > 0$ such that

$$\sup_{n \geqslant 1} \int |A_n f|^r U \, d\mu \leqslant C_r \int |f|^r \, d\mu \qquad (f \in L_r(\mu)).$$

Here we may take r satisfying $1 < r < \infty$. Then, since $0 < U \leqslant 1$ on X, the Hölder inequality shows that for all $f \in L_\infty(\mu)$ and $n \geqslant 1$

$$\int |A_n f| \, U \, d\mu \leqslant \left(\int |A_n f|^r U \, d\mu\right)^{1/r} \left(\int U \, d\mu\right)^{1/r'} \leqslant \left(C_r \int |f|^r \, d\mu\right)^{1/r} (\mu(X))^{1/r'}.$$

On the other hand, given an $\varepsilon > 0$ there exists $\delta > 0$ so that $\mu(\{U < \delta\}) < \varepsilon$. Then we have

$$\begin{aligned}\int \Big(n^{-1} \sum_{i=0}^{n-1} 1_{\tau^{-i}E}\Big) d\mu &\leqslant \delta^{-1} \int \Big(n^{-1} \sum_{i=0}^{n-1} 1_{\tau^{-i}E}\Big) U \, d\mu + \varepsilon \\ &\leqslant \delta^{-1} C_r^{1/r} (\mu(X))^{1/r'} (\mu(E))^{1/r} + \varepsilon;\end{aligned}$$

and consequently

$$\lim_{\mu(E) \to 0} \sup_{n \geqslant 1} n^{-1} \sum_{i=0}^{n-1} \mu(\tau^{-i}E) = 0.$$

It follows that the set $\{n^{-1} \sum_{i=0}^{n-1} T^i 1: \; n \geqslant 1\}$ is weakly sequentially compact in $L_1(\mu)$, where T denotes the positive linear contraction on $L_1(\mu)$ associated with τ. By a mean ergodic theorem (cf. e.g. [1], p. 661), $n^{-1} \sum_{i=0}^{n-1} T^i 1$ converges to a function in $L_1(\mu)$ with respect to the norm topology. This proves the first part of (e); the second part is trivial.

(e) ⇒ (a). To prove this, define

$$\nu(E) = \lim_n n^{-1} \sum_{i=0}^{n-1} \mu(\tau^{-i}E) \qquad (E \in \mathscr{F}).$$

By the Vitali–Hahn–Saks theorem, ν is a (countably additive) finite measure, absolutely continuous with respect to μ and invariant under τ. Then we write

$$Y = \{x: \; (d\nu/d\mu)(x) > 0\}, \qquad Z = X \setminus Y.$$

Since $\nu(\tau^{-1}Z) = \nu(Z) = 0$, we have $\nu(Y \cap \tau^{-1}Z) = 0 = \mu(Y \cap \tau^{-1}Z)$. It follows that $\mu(Y \setminus \tau^{-1}Y) = 0$. Thus by neglecting a null set we may suppose that $Y \subset \tau^{-1}Y$; and so τ can be considered to be a measure preserving transformation of the measure space (Y, ν).

We now consider $V = d\mu/d\nu$ (which is defined on Y), $e = 1_Y$, and T_Y on $L_1(Y, \nu)$ defined by

$$T_Y f(x) = f(\tau x) \qquad (x \in Y, \; f \in L_1(Y, \nu)),$$

and apply Theorem 3 to see that for every $f \in L_p(Y, \mu)$ the limit $\lim_n A_n f(x)$ exists and is finite a.e. on Y.

To finish the proof, put $A = \lim_n \tau^{-n} Y$. Since $x \in A$ implies $\tau^n x \in Y$ for all sufficiently large n, it suffices to show that $\mu(X \setminus A) = 0$. In fact, we have

$$\mu(X \setminus A) = \lim_n n^{-1} \sum_{i=0}^{n-1} \mu(\tau^{-i}(X \setminus A)) = \nu(X \setminus A) = 0,$$

completing the proof.

PROPOSITION. *Let $(X, \mathscr{F}, \mu)$ be a nonatomic finite measure space and let $\tau: X \to X$ be an ergodic null preserving transformation. Then to each p, $0 < p < 1$, there corresponds an $f \in L_p(\mu)$ such that for almost all $x \in X$, the averages $A_n f(x)$ fail to converge to a finite limit.*

Proof. Since τ is ergodic and $E(f)$ is invariant under τ, $E(f)$ being the set of all $x \in X$ at which $\lim_n A_n f(x)$ exists and is finite, it follows that $\mu(E(f)) = 0$ or $\mu(X \setminus E(f)) = 0$. Thus it suffices to prove the existence of an $f \in L_p(\mu)$ for which $\mu(E(f)) < \mu(X)$. If such an f exists in $L_\infty(\mu)$, there is nothing to prove. Therefore we suppose below that $\lim_n A_n f$ exists and is finite a.e. on X for all $f \in L_\infty(\mu)$. Then, clearly, $\nu(E) = \lim_n n^{-1} \sum_{i=0}^{n-1} \mu(\tau^{-i}E)$ exists for all $E \in \mathscr{F}$, and τ can be considered to be an ergodic measure preserving transformation of the measure space (Y, ν), where $Y = \{x: \; (d\nu/d\mu)(x) > 0\}$.

Take $\delta > 0$ and $E \in \mathscr{F}$, with $E \subset Y$, such that $\nu(E) > 0$ and $(d\nu/d\mu)(x) > \delta$ for $x \in E$. Since μ is nonatomic and $0 < p < 1$, there exists an $f \in L_p^+(E, \mu)$ such that $\int_E f \, d\mu = \infty$. Then $\int_E f \, d\nu = \infty$, and hence by the Birkhoff individual ergodic theorem, $\lim_n A_n f(x) = \infty$ a.e. on Y because τ is ergodic on (Y, ν). This completes the proof.

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Stable rank of holomorphic function algebras

by

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Abstract. We calculate the stable rank of stable subalgebras of $A(K)$.

Introduction. The concept of the stable rank of a ring, introduced by H. Bass [1], has been very useful in treating some problems in algebraic K-theory. In a series of papers G. Corach and F. D. Suárez calculated the stable rank of many Banach algebras. Among them are the well-known algebras $A(K)$, where $A(K)$ is the Banach algebra of all continuous complex-valued functions on a compact set K of the plane C which are analytic in the interior K^0 of K. In this paper we restrict ourselves mainly to subalgebras of $A(K)$, where K has a "good" boundary. For these algebras we calculate the stable rank. It is worth mentioning that the algebras may bear no topology at all. Many subalgebras of the disc algebra $A(\bar{\mathbf{D}})$ satisfy our conditions, for example, W^+, $A^\infty(\bar{\mathbf{D}})$ and $A_\alpha(\bar{\mathbf{D}})$ (for definition, see below).

This paper presents material from the author's thesis. In a forthcoming paper we will study the subalgebras of the disc algebra more closely.

§1. It is well known that the group of units in a Banach algebra is open. Unfortunately, this feature is lost in the general case of a topological algebra. Therefore we define:

A topological algebra A is called a *Q-algebra* if the set of units, A^{-1}, is open in A.

In this paper we consider complex, commutative Q-algebras with unit element being denoted by 1.

Given a Q-algebra A, an element $a \in A^n$ is called *unimodular* if there exists $b \in A^n$ such that

$$\langle b, a\rangle := \sum_{i=1}^{n} b_i a_i = 1.$$

We denote by $U_n(A)$ the set of unimodular elements of A^n. Finally, $a = (a_1, \ldots, a_n) \in U_n(A)$ is called *reducible* if there exist $x_1, \ldots, x_{n-1}$ in A such that

$$(a_1 + x_1 a_n, \ldots, a_{n-1} + x_{n-1} a_n) \in U_{n-1}(A).$$

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