This is because weakly null sequences in B lift to weakly null sequences in A. Indeed, let $q\colon A\to B$ be the quotient map and (b_n) weakly null in B with $q(a_n)=b_n$ and (a_n) bounded in A. Let (u_n) be a countable approximate unit for $\ker q$ and put $c_n=(1-u_n)a_n$. Then $q(c_n)=b_n$ and (c_n) is weakly null in A. For the latter, let p be the support projection in A^{**} for $\ker q$, so that $p(1-u_n)\to 0$ strongly in A^{**}, which implies $p(1-u_n)a_n\to 0$ strongly. Hence, for $f\in A^*$, we have $f(c_n)=f(pc_n)+f((1-p)c_n)=f(p(1-u_n)a_n)+f((1-p)a_n)\to 0$. Now we have:

THEOREM. *A separable C^*-algebra A has the Dunford–Pettis property if and only if A^* has this property.*

If A has the property, then using the lemma and the proof of Theorem 7, A is type I. Moreover, A has only finite-dimensional irreducible representations for otherwise $K(l_2)$ shows up in a quotient of A. Hence A^{**} is type I finite (cf. Theorem 1 in Hamana's paper).

References

[1] C. A. Akemann, P. G. Doods and J. L. B. Gamlen, *Weak compactness in the dual space of a C^*-algebra*, J. Funct. Anal. 10 (1972), 446–450.

[2] K. Andrews, *Dunford–Pettis sets in the space of Bochner integrable functions*, Math. Ann. 241 (1979), 35–41.

[3] J. Arazy, *Linear topological classification of matroid C^*-algebras*, Math. Scand. 52 (1983), 89–111.

[4] J. Bourgain, *New Classes of L^p-Spaces*, Lecture Notes in Math. 889, Springer, Berlin 1981.

[5] A. Connes, *Classification of injective factors*, Ann. of Math. 104 (1976), 73–115.

[6] J. Diestel, *A survey of results related to the Dunford–Pettis property*, Contemp. Math. 2 (1980), 15–60.

[7] I. Dobrakov, *On representation of linear operators on $C_0(T, X)$*, Czechoslovak Math. J. 21 (1971), 13–30.

[8] N. Dunford and B. J. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc. 47 (1940), 323–392.

[9] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. 5 (1953), 129–173.

[10] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, Academic Press, 1979.

[11] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer, Berlin 1971.

[12] M. Takesaki, *Theory of Operator Algebras I*, Springer, Berlin 1979.

[13] S. K. J. Tsui, *Decompositions of linear maps*, Trans. Amer. Math. Soc. 230 (1977), 87–112.

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Interpolation of compact operators by Goulaouic procedure

by

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Abstract. We show that the classical Lions–Peetre compactness theorems for Banach spaces (which are the main tools for proving all known compactness results in interpolation theory) fail in the locally convex case. We also prove a positive result assuming compactness of the operator in both sides.

1. Setting of the problem. Motivated by certain problems in the theory of partial differential equations, Goulaouic studied in [6] and [7] a procedure for extending any interpolation functor for Banach couples to more general couples of locally convex spaces. Let us briefly review this procedure.

A (Hausdorff) locally convex space E is said to be the *strict projective limit* of the family of Banach spaces $(E_i)_{i\in I}$ if the following conditions are satisfied:

1) $E=\bigcap_{i\in I}E_i$.

2) E is equipped with the projective limit topology.

3) For each $i\in I$, E is dense in E_i.

4) The family $(E_i)_{i\in I}$ is directed, i.e. given any finite subset $J\subset I$, there exists $k\in I$ such that for all $j\in J$ the embedding $E_k\hookrightarrow E_j$ is continuous.

We then write $E=\varprojlim_{i\in I}=E_i$.

Let now (A_0, A_1) be a (compatible) couple of locally convex spaces (meaning that they are continuously embedded in a Hausdorff topological vector space). We say that (A_0, A_1) is the *strict projective limit* of the family $(A_{0,i}, A_{1,j})_{(i,j)\in I\times J}$ of Banach couples provided that the following conditions hold:

1) $A_0=\varprojlim_{i\in I}A_{0,i}$, $A_1=\varprojlim_{j\in J}A_{1,j}$.

2) All spaces $A_{0,i}$, $A_{1,j}$ are continuously embedded in a common Hausdorff topological vector space $\mathcal{A}$.

3) For each $(i,j)\in I\times J$, $A_0\cap A_1$ is dense in $A_{0,i}\cap A_{1,j}$ (the norm in $A_{0,i}\cap A_{1,j}$ being $\max\{\|a\|_{A_{0,i}}, \|a\|_{A_{1,j}}\}$).

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If this is the case, we write

$$(A_0, A_1) = \varprojlim_{i,j} (A_{0,i}, A_{1,j}).$$

Any interpolation functor for Banach couples F can be extended to projective limit couples by defining the *interpolated space* as the projective limit of the family $\{F(A_{0,i}, A_{1,j})\}_{(i,j)\in I\times J}$:

$$F(A_0, A_1) = \varprojlim_{i,j} F(A_{0,i}, A_{1,j}).$$

As an example, consider the Schwarz classical space $\mathcal{D}_{L_p}$ of all infinitely differentiable complex functions f defined in $\mathbf{R}^n$, with $D^\alpha f \in L_p$ for every multi-index α. Then we have

$$(*) \qquad (\mathcal{D}_{L_{p_0}}, \mathcal{D}_{L_{p_1}})_{\theta,p} = \mathcal{D}_{L_p}.$$

Here $1 \leqslant p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $(\ ,\ )_{\theta,p}$ denotes the real interpolation method (see [11] and [12] for details on this method).

In general, if $(A_0, A_1) = \varprojlim_{i,j} (A_{0,i}, A_{1,j})$ then *the topology of* $(A_0, A_1)_{\theta,p}$ is defined by the family of norms

$$r_{i,j}(a) = \Big[\int_0^\infty (t^{-\theta} K_{i,j}(t, a))^p \, dt/t\Big]^{1/p}$$

where $K_{i,j}$ is the *Peetre K-functional* associated to the couple $(A_{0,i}, A_{1,j})$, i.e.

$$K_{i,j}(t, a) = \inf\{\|a_0\|_{A_{0,i}} + t\|a_1\|_{A_{1,j}}:\ a = a_0 + a_1,\ a_0 \in A_{0,i},\ a_1 \in A_{1,j}\}.$$

Besides the Riesz type formula $(*)$, Goulaouic derived in [6] and [7] many other properties of this interpolation procedure, but there is no result there (nor in the subsequent literature) on the stability of compact operators for this procedure. Accordingly, we study this problem here.

The behaviour of compactness under interpolation is a very natural question for applications of interpolation theory to other branches of analysis and thus has received attention from the beginning of abstract interpolation theory. The first result in this direction was obtained in 1960 by M. A. Krasnosel'skii [10] for the case of L_p-spaces. Other contributions are due to Lions–Peetre [11] and Hayakawa [8], among others. But in fact, the question whether Krasnosel'skii's result holds true in abstract interpolation does not have a complete answer yet.

Quite recently new approaches to some classical results have been developed in [2]–[5], also yielding new compactness theorems. Surprisingly, the following result established in 1964 by Lions and Peetre [11] plays a main role in the proofs of all (new and old) compactness theorems.

LIONS–PEETRE LEMMA. *Let $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, let (A_0, A_1) be a Banach couple and let B be a Banach space. Assume that T is a linear operator.*

(i) *If $T: A_0 \to B$ is compact and $T: A_1 \to B$ is continuous, then $T: (A_0, A_1)_{\theta,q} \to B$ is compact.*

(ii) *If $T: B \to A_0$ is compact and $T: B \to A_1$ is continuous, then $T: B \to (A_0, A_1)_{\theta,q}$ is compact.*

(In fact, Lions and Peetre showed that this is true for any interpolation method of exponent θ and not only for the real method.)

The aim of this note is to show that the Lions–Peetre Lemma fails for the Goulaouic procedure. We also prove a positive result of Hayakawa type.

2. The counterexample. First let us recall the definition of the echelon space of order $p \geqslant 1$.

Let $(a_{m,n})$ be an infinite real matrix such that

$$0 < a_{m,n} < a_{m+1,n}, \qquad m, n = 1, 2, \ldots$$

The space $l_p[a_{m,n}]$ consists of all sequences $\xi = (\xi_n)$ of scalars such that for every $m \in \mathbf{N}$

$$\nu_m(\xi) = \|\xi\|_{l_p(a_{m,n})} = \Big(\sum_{n=1}^\infty (a_{m,n}|\xi_n|)^p\Big)^{1/p} < \infty,$$

and its topology is defined by the sequence of norms ν_m. See [9], [13], and [1] for details on these spaces.

In order to see that the Lions–Peetre Lemma (i) fails for the Goulaouic procedure, take

$$a_{m,n} := (m/(m+1))^n, \qquad m, n = 1, 2, \ldots,$$

and let T be the identity operator $T\xi = \xi$. Note that

$$a_{m,n} < a^2_{2m+1,n} < 1, \qquad m, n = 1, 2, \ldots$$

Thus the restrictions $T: l_2 \to l_2[a_{m,n}]$ and $T: l_2[a^2_{m,n}] \to l_2[a_{m,n}]$ are continuous. In addition,

$$\sum_{n=1}^\infty a_{m,n}/a_{m+1,n} < \infty, \qquad m = 1, 2, \ldots$$

Hence the Fréchet space $l_2[a_{m,n}]$ is nuclear (see, e.g., [13], Chap. II, § 3,4(1)) and consequently any bounded subset of $l_2[a_{m,n}]$ is relatively compact. This implies that $T: l_2 \to l_2[a_{m,n}]$ is compact.

Nevertheless, $T: (l_2, l_2[a^2_{m,n}])_{1/2,2} \to l_2[a_{m,n}]$ is not compact. Indeed, the couple $(l_2, l_2[a^2_{m,n}])$ is the strict projective limit of the sequence of Banach couples $(l_2, l_2(a^2_{m,n}))_{m\in\mathbf{N}}$. Therefore, using [12], Thm. 1.18.5, we obtain

$$(l_2, l_2[a^2_{m,n}])_{1/2,2} = \varprojlim_{m\in\mathbf{N}} (l_2, l_2(a^2_{m,n}))_{1/2,2} = \varprojlim_{m\in\mathbf{N}} l_2(a_{m,n}) = l_2[a_{m,n}].$$

And clearly the identity map of $l_2[a_{m,n}]$ is not compact.

Next we show that the Lions–Peetre Lemma (ii) also fails in the locally convex case. Take now

$$a_{m,n} := 1+(n+1)^{2m}, \quad m, n = 1, 2, \ldots,$$

and let again $T\xi = \xi$. Since $\lim_{n\to\infty}(a_{1,n}/a_{2,n}) = 0$, the embedding from $l_2(a_{2,n})$ into $l_2(a_{1,n})$ is compact. Hence $T\colon l_2[a_{m,n}] \to l_2$ is a compact operator. Moreover,

$$a_{m,n}^2 < 2a_{4m,n}, \quad m, n = 1, 2, \ldots,$$

so that $T\colon l_2[a_{m,n}] \to l_2[a_{m,n}^2]$ is continuous. But anew

$$T\colon l_2[a_{m,n}] \to (l_2, l_2[a_{m,n}^2])_{1/2,2} = l_2[a_{m,n}]$$

is not compact.

3. A positive result. We close this note by proving that under the hypothesis of compactness in both sides, the interpolated operator is also compact.

THEOREM. *Let the couples (A_0, A_1) and (B_0, B_1) be the strict projective limits of the families of Banach couples $(A_{0,i}, A_{1,j})_{(i,j)\in I\times J}$ and $(B_{0,s}, B_{1,z})_{(s,z)\in S\times Z}$, respectively. Assume that T is a linear operator such that $T\colon A_k \to B_k$ compactly for $k = 0, 1$. Then if $0 < \theta < 1$ and $1 \leq q \leq \infty$, $T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also compact.*

Proof. Find $i \in I$ and $j \in J$ such that $T\colon (A_0, \|\ \|_{A_{0,i}}) \to B_0$ and $T\colon (A_1, \|\ \|_{A_{1,j}}) \to B_1$ are compact. Put

$$U = \{a \in (A_0, A_1)_{\theta,q}\colon r_{i,j}(a) \leq 1\}.$$

We are going to show that $T(U)$ is precompact in $(B_0, B_1)_{\theta,q}$.

Given any $s \in S$, $z \in Z$ and $\varepsilon > 0$, by the density of A_0 in $A_{0,i}$, A_1 in $A_{1,j}$ and $A_0 \cap A_1$ in $A_{0,i} \cap A_{1,j}$, we can extend T to an operator $\tilde{T}$ such that $\tilde{T}\colon A_{0,i} \to B_{0,s}$ and $\tilde{T}\colon A_{1,j} \to B_{1,z}$ are compact, and $\tilde{T}|_{A_0+A_1} = T$. Then, using [2], Thm. 3.1 (the extended version of Hayakawa's result), we see that

$$\tilde{T}\colon (A_{0,i}, A_{1,j})_{\theta,q} \to (B_{0,s}, B_{1,z})_{\theta,q}$$

is compact. It follows that

$$T\colon ((A_0, A_1)_{\theta,q}, r_{i,j}) \to (B_{0,s}, B_{1,z})_{\theta,q}$$

is also compact. Hence, there exists a finite set $\{a_1, \ldots, a_n\} \subset U$ such that

$$T(U) \subset \bigcup_{k=1}^{n} \left(T(a_k) + \{b \in (B_{0,s}, B_{1,z})_{\theta,q}\colon r_{s,z}^*(b) \leq 1\}\right).$$

Finally, if $b \in T(U) - T(a_k)$ then $b \in (B_0, B_1)_{\theta,q}$ and therefore

$$T(U) \subset \bigcup_{k=1}^{n} \left(T(a_k) + \{b \in (B_0, B_1)_{\theta,q}\colon r_{s,z}^*(b) \leq 1\}\right).$$

This completes the proof.

References

[1] F. Cobos, *A new class of perfect Fréchet spaces*, Math. Nachr. 120 (1985), 203–216.

[2] F. Cobos, D. E. Edmunds and A. J. B. Potter, *Real interpolation and compact linear operators*, J. Funct. Anal. 88 (1990), 351–365.

[3] F. Cobos and D. L. Fernandez, *On interpolation of compact operators*, Ark. Mat. 27 (1989), 211–217.

[4] F. Cobos and J. Peetre, *Interpolation of compactness using Aronszajn–Gagliardo functors*, Israel J. Math. 68 (1989), 220–240.

[5] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, preprint.

[6] C. Goulaouic, *Prolongements de foncteurs d'interpolation et applications*, Ann. Inst. Fourier (Grenoble) 18 (1968), 1–98.

[7] —, *Interpolation entre des espaces localement convexes définis à l'aide de semi-groupes; cas des espaces de Gevrey*, ibid. 19 (1970), 269–278.

[8] K. Hayakawa, *Interpolation by real method preserves compactness of operators*, J. Math. Soc. Japan 21 (1969), 189–199.

[9] G. Köthe, *Topological Vector Spaces I*, Springer, Berlin 1969.

[10] M. A. Krasnosel'skii, *On a theorem of M. Riesz*, Soviet Math. Dokl. 1 (1960), 229–231.

[11] J. L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. 19 (1964), 5–68.

[12] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam 1978.

[13] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Math. Stud. 67, Amsterdam 1982.

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