STUDIA MATHEMATICA 97 (1) (1990)

Stationary perturbations based on Bernoulli processes

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Abstract. It is shown that if random perturbations of an endomorphism by diffeomorphisms are based on a Bernoulli process with a finite number s of states, $s \geqslant 3$, then the iterations of the perturbed endomorphism form a weakly mixing process. For $s = 2$, the above statement is not true.

Consider the one-parameter family $\{T_\varepsilon\}_{\varepsilon\in(a,b)}$ of transformations of the interval $[0, 1]$ into itself such that

$$T_\varepsilon^{-1}(y) = (1-\varepsilon)y+\varepsilon g(y), \tag{1}$$

where $g \in C^2[0, 1]$, $g(0) = 0$, $g(1) = 1$, and $a = (1-\sup g')^{-1}$, $b = (1-\inf g')^{-1}$. Moreover, assume that there exists exactly one point y_0 for which $g'(y_0) = 1$ and $g'(y) < 1$ for $y < y_0$ (the case $g'(y) > 1$ for $y < y_0$ is similar).

Let $\sigma\colon X^{\mathbf{N}} \to X^{\mathbf{N}}$ be the one-sided $(p_1, \ldots, p_s)$-Bernoulli shift. Here $X = \{1, \ldots, s\}$ and $\mathbf{N} = \{1, 2, \ldots\}$. Such a transformation will be called a *Bernoulli process.* Let T be an endomorphism of the Lebesgue space $([0, 1], \mathscr{B}, m)$. Using the process σ we will randomly perturb the endomorphism T by s elements of the family (1). Namely, we take s functions $T_{\varepsilon_1}, \ldots, T_{\varepsilon_s}$, $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$, and we define the transformation

$$\bar{T}(x, y) = \big(\sigma(x),\ T_{\varepsilon_{x(1)}} \circ T(y)\big).$$

In addition we postulate that T preserves the product measure, which is equivalent to $\sum_{i=1}^{s} \varepsilon_i p_i = 0$. The proof of this fact may be found in [3].

According to that paper the Bernoulli process σ and the diffeomorphisms (1) generate more random perturbations in the case of $s \geqslant 3$ than in the case of $s = 2$.

In the present paper we prove the following theorems.

THEOREM 1. *If $s \geqslant 3$ and σ is a Bernoulli process, then the endomorphism $\bar{T}$ is weakly mixing for any positive nonsingular endomorphism T.*

THEOREM 2. *If $s = 2$ and σ is a (p_1, p_2)-Bernoulli shift, then for every pair $(\varepsilon_1, \varepsilon_2)$ such that $\varepsilon_1 p_1 + \varepsilon_2 p_2 = 0$, there exists an automorphism T such that $\bar{T}$ is not ergodic.*

1985 *Mathematics Subject Classification*: Primary 28D05; Secondary 60A10.

In connection with Theorem 1, we note that in [1], [4] there are described the ergodic properties of the endomorphisms $\bar{T}(x, y) = (\sigma(x), T_{x(1)}(y))$, where $T_1, \ldots, T_s$ are Lasota–Yorke type transformations and σ is a Bernoulli shift. In this paper we use some ideas from [4].

Before we start the proofs we make some additional considerations. Let σ be a Bernoulli shift. Let $T_1, \ldots, T_s$ be measurable transformations of the space $(Y, \mathscr{B}, \nu)$ such that the transformation

$$\bar{T}_0(x, y) = (\sigma(x), T_{x(1)}(y))$$

preserves the product measure $\mu\times\nu$. Here μ denotes the $(p_1, \ldots, p_s)$-Bernoulli measure. In this paper we assume that relations between sets hold modulo a set of measure zero.

LEMMA (S. Pelikan). *Let $A \subset X^{\mathbf{N}}\times Y$ be an invariant set of positive $\mu\times\nu$ measure ($\bar{T}_0 A \subset A$). Then there exists a set $B \subset Y$ such that $A = X^{\mathbf{N}}\times B$.*

COROLLARY 1. *Let $\bar{T}_0^{-1} A = A$. Then there exists a set B such that $A = X^{\mathbf{N}}\times B$ and $T_i^{-1} B = B$ for $i = 1, \ldots, s$.*

The following relations hold for transformations from the family (1):

$$(T_\beta^{-1} T_\varepsilon)'(x) = 1 \Leftrightarrow x = x_\varepsilon := T_\varepsilon^{-1}(y_0), \quad \text{if } \beta \neq \varepsilon,$$

(2) $\quad (T_\beta^{-1} T_\varepsilon)'(x) > 1 \quad$ if $x < x_\varepsilon$ when $\varepsilon > \beta$ and if $x > x_\varepsilon$ when $\varepsilon < \beta$.

Let $s \geqslant 3$. For a sequence $\varepsilon_1, \ldots, \varepsilon_s$ such that $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$, $1 \leqslant i, j \leqslant 3$, we will assume without loss of generality that $\varepsilon_2 < \varepsilon_1 < \varepsilon_3$. By (2) we have

$$(T_{\varepsilon_3}^{-1} T_{\varepsilon_1})'(x) > 1 \quad \text{for } x > x_{\varepsilon_1}, \qquad (T_{\varepsilon_2}^{-1} T_{\varepsilon_1})'(x) > 1 \quad \text{for } x < x_{\varepsilon_1}.$$

We define the auxiliary transformation φ: $[x_{\varepsilon_3}, x_{\varepsilon_2}] \to [x_{\varepsilon_3}, x_{\varepsilon_2}]$ by

$$\varphi(x) = \begin{cases} T_{\varepsilon_2}^{-1} T_{\varepsilon_1}(x) & \text{for } x \in [x_{\varepsilon_3}, x_{\varepsilon_1}], \\ T_{\varepsilon_3}^{-1} T_{\varepsilon_1}(x) & \text{for } x \in (x_{\varepsilon_1}, x_{\varepsilon_2}]. \end{cases}$$

We have $\varphi'(x) > 1$ for $x \neq x_{\varepsilon_1}$ and $\varphi'(x_{\varepsilon_1}^-) = 1$.

LEMMA 2. *The transformation φ is ergodic with respect to the Lebesgue measure, i.e. the equality $\varphi^{-1} D = D$ implies $D = [x_{\varepsilon_3}, x_{\varepsilon_2}]$ for every measurable set D of positive measure.*

Proof. Observe that $\inf(\varphi^2(x))' > 1$. Indeed, assume conversely that $(\varphi^2)'(x^-) = 1$ for some x. Then $\varphi'(x^-) = \varphi'(\varphi(x)^-) = 1$, which is possible only if $x = x_{\varepsilon_1}$ and $\varphi(x_{\varepsilon_1}) = x_{\varepsilon_1}$. The last equality contradicts the definition of φ. Let $\varphi^{-1} D = D$ for some set D of positive measure. By Lemma 2 from [2], there exists a nonempty interval $I \subset D$. By the definition of φ, it is easy to see that there exists n such that $x_{\varepsilon_1} \in \varphi^n(I)$. Hence we conclude that there exists exactly one φ-invariant set, which implies $D = [x_{\varepsilon_3}, x_{\varepsilon_2}]$. ∎

Proof of Theorem 1. It is sufficient to show that the transformation $\bar{T}\times\bar{T}$ is ergodic. By the definition of $\bar{T}$,

$$\bar{T}\times\bar{T}((x, y), (u, v)) = (\sigma\times\sigma(x, u), T_{\varepsilon_{x(1)}}\circ T\times T_{\varepsilon_{u(1)}}\circ T(y, v)).$$

Let A be a $\bar{T}\times\bar{T}$-invariant set. Then by the Bernoulli property of the process $\sigma\times\sigma$ and by Corollary 1, $A = X^{\mathbf{N}}\times X^{\mathbf{N}}\times B$ where

$$(T^{-1}\times T^{-1})(T_{\varepsilon_i}^{-1}\times T_{\varepsilon_j}^{-1})(B) = B \quad \text{for } i, j = 1, 2, 3.$$

Let $D = (T\times T)B$. Then by positive nonsingularity of T we get

$$(T_{\varepsilon_j}^{-1} T_{\varepsilon_m}\times T_{\varepsilon_i}^{-1} T_{\varepsilon_n})(D) = D \quad \text{for } i, j, n, m = 1, 2, 3.$$

Hence

(3) $\quad (I\times T_{\varepsilon_j}^{-1} T_{\varepsilon_1})(D) = D, \quad (T_{\varepsilon_j}^{-1} T_{\varepsilon_1}\times I)(D) = D \quad$ for $j = 2, 3$.

Let $D_y = \{v: (y, v)\in D\}$. Then $T_{\varepsilon_2}^{-1} T_{\varepsilon_1} D_y = T_{\varepsilon_3}^{-1} T_{\varepsilon_1} D_y = D_y$, for a.e. y, which implies $\varphi^{-1}(D_y \cap [x_{\varepsilon_3}, x_{\varepsilon_2}]) = D_y \cap [x_{\varepsilon_3}, x_{\varepsilon_2}]$ for a.e. y. By Lemma 2, we have $D_y \supset [x_{\varepsilon_1}, T_{\varepsilon_2}^{-1} T_{\varepsilon_1}(x_{\varepsilon_1})]$. Since $T_{\varepsilon_2}^{-1} T_{\varepsilon_1} D_y = D_y$, we get $D_y = [0, 1]$ for a.e. y. Consequently, $D = E\times[0, 1]$ for some set E. By applying the equalities (3) to the set E, we get $E = [0, 1]$ and hence $D = [0, 1]\times[0, 1]$, which yields $B = [0, 1]\times[0, 1]$. ∎

Proof of Theorem 2. Let $\varepsilon_1 p_1 + \varepsilon_2 p_2 = 0$. We will find a set B such that $T_{\varepsilon_1}^{-1} B = T_{\varepsilon_2}^{-1} B = TB$ for some automorphism T. From the definition (1) we see that it is easy to construct a set B_0 such that $T_{\varepsilon_2} T_{\varepsilon_1}^{-1} B_0 = B_0$ and $0 < m(B_0) < 1$. Let B be such a set B_0. Observe that $m(T_{\varepsilon_1}^{-1} B) = m(B)$. Indeed,

$$m(B) = \mu\times m[\bar{T}^{-1}(X^{\mathbf{N}}\times B)] = \mu\times m[A_1\times T_{\varepsilon_1}^{-1} B] + \mu\times m[A_2\times T_{\varepsilon_2}^{-1} B]$$
$$= p_1 m(T_{\varepsilon_1}^{-1} B) + p_2 m(T_{\varepsilon_2}^{-1} B) = m(T_{\varepsilon_1}^{-1} B).$$

Let T be any automorphism of the interval $[0, 1]$ such that $TB = T_{\varepsilon_1}^{-1} B$. Then the set $X^{\mathbf{N}}\times B$ is $\bar{T}$-invariant. ∎

Let $I(x) = x$ for every $x\in[0, 1]$.

THEOREM 3. *If a process σ satisfies the assumptions of Theorem 2 and $T = I$, then the endomorphism $\bar{T}$ is ergodic.*

Proof. Let $\bar{T}^{-1} A = A$. Then $A = X^{\mathbf{N}}\times B$ and $T_{\varepsilon_1}^{-1} B = T_{\varepsilon_2}^{-1} B$, where $\varepsilon_1 p_1 + \varepsilon_2 p_2 = 0$. We define the auxiliary transformation ψ: $[x_{\varepsilon_1}, x_{\varepsilon_2}] \to [x_{\varepsilon_1}, x_{\varepsilon_2}]$ as follows:

$$\psi(x) = \begin{cases} T_{\varepsilon_2}^{-1}(x) & \text{for } x\in[x_{\varepsilon_1}, y_0], \\ T_{\varepsilon_1}^{-1}(x) & \text{for } x\in(y_0, x_{\varepsilon_2}]. \end{cases}$$

Here $\psi'(x) > 1$ for $x \neq y_0$ and $\psi'(y_0^-) = 1$. The next part of the proof is identical with the argument used for the sets D_y in the proof of Theorem 1. ∎

EXAMPLE. For $g(y) = y^2$ we obtain the family of perturbations by parabolas $T_\varepsilon(x) = (2\varepsilon)^{-1}\{[(1-\varepsilon)^2+4\varepsilon x]^{1/2}+\varepsilon-1\}$ where $\varepsilon\in(-1, 1)$.

Final remarks

I. It is not difficult to see that the above results may be presented, in a more general form, as follows.

Let $\{T_\varepsilon\}_{\varepsilon\in(a,b)}$ be a family of transformations of the interval $[0, 1]$ into itself such that $0\in(a, b)$ and $T_0 = I$, $T_\varepsilon\in C^2[0, 1]$, $T_\varepsilon(0) = 0$, $T_\varepsilon(1) = 1$, $T_\varepsilon'(x) > 0$ for every $x\in[0, 1]$ and $\varepsilon\in(a, b)$. Moreover, assume that there exists $y_0\in(0, 1)$ such that for every $\beta, \varepsilon\in(a, b)$

$$(T_\beta^{-1})'(y) > (T_\varepsilon^{-1})'(y) \quad \text{for } y < y_0 \text{ when } \beta < \varepsilon \ (\varepsilon < \beta),$$
$$(T_\beta^{-1})'(y) > (T_\varepsilon^{-1})'(y) \quad \text{for } y > y_0 \text{ when } \varepsilon < \beta \ (\beta < \varepsilon).$$

We define the transformation $\bar{T}(x, y) = (\sigma(x), T_{\varepsilon_{x(1)}}\circ T(y))$, where T is a positively nonsingular endomorphism and $\varepsilon_1, \ldots, \varepsilon_s$, $\varepsilon_i\neq\varepsilon_j$ for $i\neq j$, satisfy $\sum_{i=1}^{s} p_i(T_{\varepsilon_i}^{-1})'(y) = 1$. Then the transformation $\bar{T}$ preserves the product measure and

a) $\bar{T}$ is weakly mixing for $s\geqslant 3$,

b) for every pair $(\varepsilon_1, \varepsilon_2)$ such that $\sum_{i=1}^{2} p_i(T_{\varepsilon_i}^{-1})'(y) = 1$, there exists an automorphism T such that $\bar{T}$ is not ergodic.

II. In this paper due to the assumption of Lemma 1 we only consider a Bernoulli process σ. We now show that if σ is an aperiodic Markovian process then the assertion of Lemma 1 is not true in general. Let $\sigma\colon X^{\mathbf{N}}\to X^{\mathbf{N}}$ be the one-sided shift preserving a Borel measure η, and let $T_1, \ldots, T_s$ be measurable transformations of the space $(Y, \mathscr{B}, \nu)$ such that the transformation

$$\bar{T}(x, y) = (\sigma(x), T_{x(1)}(y))$$

preserves the product measure $\eta\times\nu$. Let $f(x)\in L_1(\eta)$ and $g(y)\in L_1(\nu)$. Then

$$(f\cdot g)(\bar{T}(x, y)) = \sum_{i=1}^{s} 1_{A_i}(x) f(\sigma(x)) g(T_i(y)).$$

The equality $(f\cdot g)(\bar{T}(x, y)) = f(x)g(y)$ holds iff

$$f(\sigma(x)) = \sum_{i=1}^{s} 1_{A_i}(x)\lambda_i f(x), \tag{4}$$

where $\lambda_1, \ldots, \lambda_s$ are the numbers such that $g(T_i(y)) = \lambda_i^{-1} g(y)$ for $i = 1, \ldots, s$.

By (4) we obtain an example of an aperiodic Markovian process which does not satisfy Lemma 1. Let σ be a Bernoulli process, for $s = 2$. The partition $\{A_{11}, A_{12}, A_{21}, A_{22}\}$, where $A_{ij} = A_i\cap\sigma^{-1}(A_j)$, defines an aperiodic Markovian process for $s = 4$. Let Y be the set $\{-1, 1\}$ with the measure ν such that $\nu(-1) = \nu(1) = \frac{1}{2}$. We define $T_1(y) = T_4(y) = y$ and $T_2(y) = T_3(y) = -y$ for every $y\in Y$. Let $f(x) = 1_A(x) - 1_{A^c}(x)$, where $A = A_{11}\cup A_{12}$, and $g(y) = y$ for $y\in Y$. Then $(f\cdot g)(\bar{T}(x, y)) = f(x)g(y)$ and the set

$$\{(x, y)\colon f(x)g(y) = 1\} = A\times\{1\}\cup A^c\times\{-1\}$$

is $\bar{T}$-invariant.

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Received March 29, 1989
Revised version July 18, 1989 (2551)