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On transition multimeasures with values in a Banach space

by

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Abstract. In this paper we examine transition multimeasures, i.e., set-valued vector measures parametrized by a parameter in a measurable space. First we establish the existence of transition selectors. Then we define a set-valued integral with respect to a multimeasure and we show that it generates a new transition multimeasure, for which we obtain a characterization of its measure selectors. Then we allow the parameter of the transition multimeasure to vary over a Polish space and we obtain a set-valued version of Feller's property. Finally, we look at the action of the transition multimeasure on measures defined on the parameter space.

1. Introduction. The theory of multimeasures (set-valued measures) has its origins in mathematical economics and in particular in equilibrium theory for exchange economies with production, in which the coalitions and not the individual agents are the basic economic units (see Vind [25] and Hildenbrand [15]). Since then the subject of multimeasures has been developed extensively. Important contributions were made, among others, by Artstein [1], Costé [8], [9], Costé–Pallu de la Barrière [10], Drewnowski [12], Godet-Thobie [13], Hiai [14] and Pallu de la Barrière [17]. Further applications in mathematical economics can be found in Klein–Thompson [16] and Papageorgiou [19].

In this paper we study multimeasures parametrized by the elements of a measurable space (transition multimeasures). Such multimeasures turn out to be the appropriate tool to establish the existence of Markov temporary equilibrium processes in dynamic economies (see Blume [6]).

2. Preliminaries. In this section we establish our notation and terminology and we recall some basic facts from the theories of multifunctions and multimeasures that we will need in the sequel.

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X:\ \text{nonempty, closed, (convex)}\},$$
$$P_{(w)k(c)}(X) = \{A \subseteq X:\ \text{nonempty, } (w)\text{-compact, (convex)}\}.$$

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Also by $X^*_{w^*}$ we will denote the dual of X endowed with the weak* topology and by $P_{kc}(X^*_{w^*})$ we will denote the nonempty, w^*-compact and convex subsets of X^*.

If $A \in 2^X \setminus \{\emptyset\}$, we define $|A| = \sup\{\|\alpha\|: \alpha \in A\}$ (the *norm* of the set A), $\sigma(x^*, A) = \sup\{(x^*, \alpha): \alpha \in A\}$, $x^* \in X^*$ (the *support function* of A) and $d(x, A) = \inf\{\|x-\alpha\|: \alpha \in A\}$ (the *distance function* from A).

A multifunction $F: \Omega \to P_f(X)$ is said to be *measurable* if for all $x \in X$, $\omega \to d(x, F(\omega))$ is measurable. This definition is in fact equivalent to saying that there exist measurable functions $f_n: \Omega \to X$ s.t. for all $\omega \in \Omega$, $F(\omega) = \mathrm{cl}\{f_n(\omega)\}_{n\geq 1}$. Furthermore, if there exists a complete σ-finite measure $\mu(\cdot)$ on Σ, then both the above definitions are equivalent to saying that

$$\mathrm{Gr}\, F = \{(\omega, x) \in \Omega \times X: x \in F(\omega)\} \in \Sigma \times B(X),$$

$B(X)$ being the Borel σ-field of X (graph measurability; for details we refer to Wagner [26]).

By S^1_F we will denote the set of integrable selectors of $F(\cdot)$, i.e.

$$S^1_F = \{f(\cdot) \in L^1(X): f(\omega) \in F(\omega) \ \ \mu\text{-a.e.}\}.$$

This set may be empty. A straightforward application of Aumann's selection theorem tells us that if $F: \Omega \to P_f(X)$ is measurable and $\omega \to |F(\omega)|$ belongs in L^1_+ (such an $F(\cdot)$ is usually called *integrably bounded*), then $S^1_F \neq \emptyset$. Having this set we can define a set-valued integral for $F(\cdot)$ as follows:

$$\int_\Omega F(\omega)\, d\mu(\omega) = \{\int_\Omega f(\omega)\, d\mu(\omega): f \in S^1_F\}.$$

The vector-valued integrals of the right-hand side are defined in the sense of Bochner. This integral is known in the literature as *Aumann's integral*, since it was first introduced by Aumann [4] as the natural generalization of the Minkowski sum of sets.

Next let X be any Banach space. A *multimeasure* is a map $M: \Sigma \to 2^X \setminus \{\emptyset\}$ s.t. (i) $M(\emptyset) = \{0\}$, and (ii) for every $\{A_n\}_{n\geq 1} \subseteq \Sigma$ pairwise disjoint we have $M_n(\bigcup_{n\geq 1} A_n) = \sum_{n\geq 1} M(A_n)$. Depending on the way we interpret this infinite sum we get different types of multimeasures. However, all these definitions coincide when $M(\cdot)$ is $P_{wkc}(X)$-valued (see Proposition 3 of Godet-Thobie [13] and Pallu de la Barrière [17]). This fact can be viewed as the set-valued version of the well-known Orlicz–Pettis theorem (see Diestel–Uhl [11]). So for the needs of this work we can say that $M: \Sigma \to P_f(X)$ is a *multimeasure* (set-valued measure) if and only if for every $x^* \in X^*$, $A \to \sigma(x^*, M(A))$ is a signed measure. Similarly $M: \Sigma \to P_{kc}(X^*_{w^*})$ is an $X^*_{w^*}$*-valued multimeasure* if and only if for all $x \in X$, $A \to \sigma(x, M(A))$ is a signed measure.

If $M(\cdot)$ is a multimeasure and $A \in \Sigma$, then we define

$$|M|(A) = \sup_\pi \sum_k |M(A_k)|,$$

where the supremum is taken over all finite Σ-partitions $\pi = \{A_k\}^n_{k=1}$ of A.

If $|M|(\Omega) < \infty$, then $M(\cdot)$ is said to be *of bounded variation*. Also by S_M we will denote all vector measures $m: \Sigma \to X$ that are *selectors* of $M(\cdot)$, i.e. $m(A) \in M(A)$ for all $A \in \Sigma$.

Now let (Ω, Σ) and $(T, \mathscr{T})$ be measurable spaces and X a separable Banach space. A multivalued map $M: \Omega \times \mathscr{T} \to P_f(X)$ is said to be a *transition multimeasure* if

(1) for all $A \in \mathscr{T}$, $\omega \to M(\omega, A)$ is a measurable multifunction,
(2) for all $\omega \in \Omega$, $A \to M(\omega, A)$ is a multimeasure.

A *selector transition measure* or simply a *transition selector* of $M(\cdot, \cdot)$ is a map $m: \Omega \times \mathscr{T} \to X$ s.t.

(1) for all $A \in \mathscr{T}$, $\omega \to m(\omega, A)$ is Σ-measurable,
(2) for all $\omega \in \Omega$, $A \to m(\omega, A)$ is a vector measure,
(3) for all $\omega \in \Omega$ and all $A \in \mathscr{T}$, $m(\omega, A) \in M(\omega, A)$.

The set of all transition selectors of $M(\cdot, \cdot)$ will be denoted by TS_M. Similarly we can define an $X^*_{w^*}$-valued transition multimeasure $M: \Omega \times T \to P_{kc}(X^*_{w^*})$ and its set of transition selectors.

Let T be a Polish space and X a Banach space. By $C_b(T)$ we will denote the space of bounded continuous functions on T and by $C_b(T) \otimes X$ the space of bounded continuous functions with values in a finite-dimensional subspace of X. Also by $M^b(T, X)$ we will denote the space of X-valued vector measures of bounded variation defined on $(T, B(T))$. Similarly we define $M^b(T, X^*_{w^*})$ and $C_b(T) \otimes X^*$ (see Sainte-Beuve [23]). Finally, if $m \in M^b(T, X)$ and $B \in B(T)$, $\chi_B m$ is the vector measure defined by $\chi_B m(A) = m(A \cap B)$, $A \in B(T)$.

3. Transition selectors. In this section we prove a theorem that establishes the existence of a transition selector for a transition multimeasure. Our result extends Theorem 2.3 of Hiai [14] to transition multimeasures and also it extends Theorem 5 of Godet-Thobie [13].

So assume that (Ω, Σ, μ) is a complete, finite measure space, T a Polish space with $B(T)$ denoting its Borel σ-field and X a separable Banach space.

THEOREM 3.1. *If $M: \Omega \times B(T) \to P_{kc}(X^*_{w^*})$ is a transition multimeasure of bounded variation and $h: \Omega \to X^*_{w^*}$ is a measurable map s.t. for some $A \in B(T)$, $h(\omega) \in M(\omega, A)$ for all $\omega \in \Omega$, then there exists $m \in TS_M$ s.t. $m(\omega, A) = h(\omega)$ for all $\omega \in \Omega$.*

Proof. Let $R_A: \Omega \to M^b(T, X^*_{w^*})$ be defined by

$$R_A(\omega) = \{m \in M^b(T, X^*_{w^*}): m \in S_{M(\omega,\cdot)}, \ m(A) = h(\omega)\}.$$

From Theorem 1 of Godet-Thobie [13] (see also Theorem 2.3 of Hiai [14]), we know that $R_A(\omega) \neq \emptyset$ for all $\omega \in \Omega$.

Next let $x \in X$ and consider the function $\phi_{A,x}: \Omega \times M^b(T, X^*_{w^*}) \to \mathbf{R}$ defined by

$$\phi_{A,x}(\omega, m) = (x, m(A) - h(\omega)).$$

Since by hypothesis $h(\cdot)$ is w^*-measurable, $\omega \to (x, h(\omega))$ is a measurable $\mathbf{R}$-valued function. On the other hand, recall that by the definition of $C_b(T) \otimes X$, the $w(M^b(T, X^*_{w^*}), C_b(T) \otimes X)$-topology is the weakest topology on $M^b(T, X^*_{w^*})$ for which $m \to x o m$ is continuous from $M^b(T, X^*_{w^*})$ into $M^b(T)$ with the weak (narrow) topology; here $x o m(\cdot)$ denotes the $\mathbf{R}$-valued measure $A \to (x, m(A))$. Also from the Dynkin system theorem, we deduce that for every $C \in B(T)$, the map $\lambda \to \lambda(C)$ on $M^b(T)$ with the weak topology is measurable. Hence we finally conclude that $m \to (x o m)(C) = (x, m(C))$ is measurable. Therefore we see that $(\omega, m) \to (x, m(A)) - (x, h(\omega)) = \phi_{A,x}(\omega, m)$ is jointly measurable on $\Omega \times M^b(T, X^*_{w^*})$ when $M^b(T, X^*_{w^*})$ is endowed with the $w(M^b(T, X^*_{w^*}), C_b(T) \otimes X)$-topology.

By definition m is a *measure selector* of $M(\omega, \cdot)$ (denoted by $m \in S_{M(\omega,\cdot)}$) if and only if $m(C) \in M(\omega, C)$ for all $C \in B(T)$. Since $M(\cdot,\cdot)$ is $P_{kc}(X^*_{w^*})$-valued we have $(x, m(C)) \leqslant \sigma(x, M(\omega, C))$ for all $x \in X$ and all $C \in B(T)$. Note that since $M(\cdot,\cdot)$ is a transition multimeasure, $\omega \to \sigma(x, M(\omega, C))$ is measurable while as above we can see that $m \to (x, m(C))$ is measurable from $M^b(T, X^*_{w^*})$ with the $w(M^b(T, X^*_{w^*}), C_b(T) \otimes X)$-topology into $\mathbf{R}$. Hence the map

$$\varphi_{C,x}(\omega, m) = \sigma(x, M(\omega, C)) - (x, m(C))$$

is jointly measurable.

Now let $\{x_k\}_{k \geqslant 1}$ be dense in X and let $\{C_n\}_{n \geqslant 1}$ be a field generating $B(T)$, i.e. $\sigma(\{C_n\}_{n \geqslant 1}) = B(T)$ (recall that since T is a Polish space, $B(T)$ is countably generated so such a countable field exists). Then by setting

$$\phi_{A,k}(\omega, m) = (x_k, m(A) - h(\omega)), \qquad \varphi_{n,k}(\omega, m) = \sigma(x_k, M(\omega, C_n)) - (x_k, m(C_n)),$$

we can write

$$\operatorname{Gr} R_A = \bigcap_{\substack{k \geqslant 1 \\ n \geqslant 1}} \{(\omega, m) \in \Omega \times M^b(T, X^*_{w^*}): \ \phi_{A,k}(\omega, m) = 0, \ \varphi_{n,k}(\omega, m) \geqslant 0\}$$

$$\in \Sigma \times B(M^b(T, X^*_{w^*})).$$

From Theorem 3, p. 337 of Sainte-Beuve [23], we know that $M^b(T, X^*_{w^*})$ equipped with the $w(M^b(T, X^*_{w^*}), C_b(T) \otimes X)$-topology is a Suslin space. Thus we can apply Aumann's selection theorem (see Sainte-Beuve [22], Theorem 3), to get $r: \Omega \to M^b(T, X^*_{w^*})$ measurable s.t. $r(\omega) \in R(\omega)$ for all $\omega \in \Omega$. Set $r(\omega)(C) = m(\omega, C)$ for all $(\omega, C) \in \Omega \times B(T)$. Then clearly $m(\cdot,\cdot)$ is a transition selector of $M(\cdot,\cdot)$ and $m(\omega, A) = h(\omega)$ for all $\omega \in \Omega$. ∎

4. Integration with respect to a transition multimeasure. Now we turn our attention to integration with respect to a transition multimeasure, extending the work of Costé [8].

Let $f: \Omega \times T \to \mathbf{R}_+$ be a measurable function s.t. $f(\omega, \cdot) \in L^1(T, \lambda)$ for all $\omega \in \Omega$. Motivated from the definition of the Aumann integral (see Section 2), we define the integral of $f(\cdot,\cdot)$ with respect to a multimeasure $M(\cdot,\cdot)$, $|M(\omega, \cdot)| \ll \lambda$ μ-a.e., as follows:

$$\int_C f(\omega, t) M(\omega, dt) = \{\int_C f(\omega, t)\, m(\omega, dt): \ m \in TS_M\}, \qquad C \in B(T),$$

Note that $\omega \to \int_C f(\omega, t)\, m(\omega, dt)$ is measurable for every $m \in TS_M$. To see this let $s_n: \Omega \times T \to \mathbf{R}_+$ be simple functions s.t. $|s_n(\omega, t)| \leqslant f(\omega, t)$ and $s_n(\omega, t) \to f(\omega, t)$ $\mu \times \lambda$-a.e. Clearly $\omega \to \int_C s_n(\omega, t)\, m(\omega, dt)$, $n \geqslant 1$, are measurable and by the dominated convergence theorem $\int_C s_n(\omega, t)\, m(\omega, dt) \to \int_C f(\omega, t)\, m(\omega, dt)$ μ-a.e. Hence $\omega \to \int_C f(\omega, t)\, m(\omega, dt)$ is measurable.

Assume that (Ω, Σ, μ) is a complete, finite measure space with $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$, T is a Polish space with $B(T)$ its Borel σ-field while $\lambda(\cdot)$ is a finite measure on $(T, B(T))$ and X is a separable, reflexive Banach space.

THEOREM 4.1. *If $M: \Omega \times B(T) \to P_{fc}(X)$ is a transition multimeasure s.t. $M(\omega, C) \subseteq \lambda(C) W(\omega)$ with $W(\omega) \in P_{wkc}(X)$ for all $\omega \in \Omega$ and*

$$N(\omega, C) = \int_C f(\omega, t) M(\omega, dt),$$

then $N(\cdot,\cdot)$ is a $P_{wkc}(X)$-valued transition multimeasure.

Proof. From Theorem 3.1 we know that $TS_M \neq \emptyset$ and so $N(\cdot,\cdot)$ has nonempty values. Also since $M(\cdot,\cdot)$ is convex-valued, TS is convex and so $N(\cdot,\cdot)$ is convex-valued too.

Now note that since by hypothesis $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$, we have

$$\{\int_C f(\omega, t)\, m(\omega, dt): \ m \in TS_M\} = \{\int_C f(\omega, t)\, \hat{m}(dt): \ \hat{m} \in S_{M(\omega,\cdot)}\}.$$

Fix $\omega \in \Omega$ and consider a net $\{x_\alpha\}_{\alpha \in I} \subseteq N(\omega, C)$ s.t. $x_\alpha \xrightarrow{w} x$ in X. Then by definition we have

$$x_\alpha = \int_C f(\omega, t)\, \hat{m}_\alpha(dt), \qquad \hat{m}_\alpha \in S_{M(\omega,\cdot)}.$$

But from Theorem 1 of Godet-Thobie [13], we know that $S_{M(\omega,\cdot)} \subseteq M^b(T, X)$ is compact for the topology of pointwise weak convergence, denoted by $\hat{w} = w(M^b(T, X), \Sigma \otimes X^*)$. So we can find a subnet $\{\hat{m}_\beta\}_{\beta \in I'}$ of $\{\hat{m}_\alpha\}_{\alpha \in I}$ s.t. $\hat{m}_\beta \xrightarrow{\hat{w}} \hat{m} \in S_{M(\omega,\cdot)}$. We now claim that for each $x^* \in X^*$ and each $C \in B(T)$, the map $\bar{m} \to (x^*, \int_C f(\omega, t)\, \bar{m}(dt))$ is continuous from $S_{M(\omega,\cdot)}$ with the $\hat{w}$-topology into $\mathbf{R}$. To see this let $f(\omega, \cdot)$ be the simple function $\sum_{k=1}^{n} a_k \chi_{B_k}(\cdot)$. Then we have

$$\int_C f(\omega, t)\, \bar{m}(dt) = \sum_{k=1}^{n} a_k \bar{m}(C \cap B_k)$$

$$\Rightarrow \ \bar{m} \to (x^*, \int_C f(\omega, t)\, \bar{m}(dt)) \text{ is continuous.}$$

Now let $s_n(\cdot)$ be simple functions on T s.t. $\|f(\omega, \cdot)-s_n(\cdot)\|_1 \to 0$ as $n \to \infty$. Note that for all $\bar{m} \in S_{M(\omega,\cdot)}$ we have

$$|(x^*, \bar{m}(C))| \leqslant \lambda(C) \cdot \|x^*\| \cdot |W(\omega)|, \quad x^* \in X^*.$$

Therefore

$$\lim_{n\to\infty} \int_C |f(\omega, t)-s_n(t)|\, d|x^* \circ \bar{m}|(dt) = 0 \quad \text{uniformly in } \bar{m} \in S_{M(\omega,\cdot)}$$

$$\Rightarrow \lim_{n\to\infty} \int_C s_n(t)\, d(x^* \circ \bar{m})(dt) = \int_C f(\omega, t)\, d(x^* \circ \bar{m})(dt) \quad \text{uniformly in } \bar{m} \in S_{M(\omega,\cdot)},$$

where $(x^* \circ m)(\cdot) = (x^*, m(\cdot))$.

Since the members of the uniformly convergent sequence are continuous in $\bar{m}$, we conclude that the limit is continuous in $\bar{m}$, i.e. $\bar{m} \to \int_C f(\omega, t)\, d(x^* \circ \bar{m})(dt) = (x^*, \int_C f(\omega, t)\, \bar{m}(dt))$ is continuous as claimed. So we have

$$\int_C f(\omega, t)\, \hat{m}_\beta(dt) \to \int_C f(\omega, t)\, \hat{m}(dt)$$

$$\Rightarrow x = \int_C f(\omega, t)\, \hat{m}(dt), \quad \hat{m} \in S_{M(\omega,\cdot)}$$

$$\Rightarrow N(\omega, C) \in P_{fc}(X) \quad \text{for all } (\omega, C) \in \Omega \times B(T).$$

Also note that

$$N(\omega, C) = \int_C f(\omega, t)\, M(\omega, dt) \subseteq \left(\int_C f(\omega, t)\, \lambda(dt)\right) W(\omega) \in P_{wkc}(X)$$

$$\Rightarrow N(\omega, C) \in P_{wkc}(X) \quad \text{for all } (\omega, C) \in \Omega \times B(T).$$

Next let $m \in TS_M$ and $x^* \in X^*$. We have (recall $(x^* \circ m)(\cdot) = (x^*, m(\cdot))$)

$$(x^*, \int_C f(\omega, t)\, m(\omega, dt)) = \int_C f(\omega, t)\, d(x^* \circ m)(\omega, dt) \leqslant \int_C f(\omega, t)\, \sigma(x^*, M(\omega, dt))$$

$$\Rightarrow \sigma(x^*, N(\omega, C)) \leqslant \int_C f(\omega, t)\, \sigma(x^*, M(\omega, dt)).$$

Fix $x^* \in X^*$ and consider the following multifunction:

$$H(\omega) = \{\hat{m} \in S_{M(\omega,\cdot)}:\ \sigma(x^*, M(\omega, C)) = (x^*, \hat{m}(C))\}.$$

Consider a well-ordering on X^* (it exists by the well-ordering principle) and give X the corresponding lexicographic ordering (see for example Bourbaki [7]). Since by hypothesis $M(\cdot,\cdot)$ is $P_{wkc}(X)$-valued, we can find a lexicographic maximum $\hat{m}(C)$ of $M(\omega, C)$. Then $(x^*, \hat{m}(C)) = \sigma(x^*, M(\omega, C))$. We will show that $\hat{m}(\cdot) \in S_{M(\omega,\cdot)}$. According to Proposition 2 of Godet-Thobie [13], it is enough to show that $\hat{m}(\cdot)$ is additive. So let B_1, B_2 be two disjoint elements of $B(T)$. Then, if by $<_L$ we denote the lexicographic order, then for all $b_1 \in B_1$ and all $b_2 \in B_2$ we have

$$b_1 <_L \hat{m}(B_1), \quad b_2 <_L \hat{m}(B_2).$$

Since $M(B_1 \cup B_2) = M(B_1)+M(B_2)$, every element $b \in M(B_1 \cup B_2)$ can be written as $b = b_1+b_2$ with $b_1 \in M(B_1)$ and $b_2 \in M(B_2)$. Because the lexicographic ordering is clearly compatible with vector addition, we have

$$b <_L \hat{m}(B_1)+\hat{m}(B_2)$$

$$\Rightarrow \hat{m}(B_1)+\hat{m}(B_2) \text{ is the lexicographic maximum of } M(B_1 \cup B_2)$$

$$\Rightarrow \hat{m}(B_1 \cup B_2) = \hat{m}(B_1)+\hat{m}(B_2)$$

$$\Rightarrow \hat{m}(\cdot) \text{ is additive, thus it belongs in } S_{M(\omega,\cdot)}.$$

Hence $H(\omega) \neq \emptyset$ for all $\omega \in \Omega$. Then we have

$$H(\omega) = \{\hat{m} \in M^b(T, X):\ \sigma(x^*, M(\omega, B)) = (x^*, \hat{m}(B)),\ \hat{m}(B) \in M(\omega, B),\ B \in B(T)\}$$

$$\Rightarrow \operatorname{Gr} H = \{(\omega, \hat{m}) \in \Omega \times M^b(T, X):\ \sigma(x^*, M(\omega, B)) = (x^*, \hat{m}(B)),\ \hat{m}(B) \in M(\omega, B),\ B \in B(T)\}$$

$$\Rightarrow \operatorname{Gr} H = \bigcap_{\substack{k\geqslant 1\\ n\geqslant 1}} \{(\omega, \hat{m}) \in \Omega \times M^b(T, X):\ \sigma(x^*, M(\omega, B)) = (x^*, \hat{m}(B)),\ (z_k^*, \hat{m}(B_n)) \leqslant \sigma(z_k^*, M(\omega, B_n))\}$$

where $\{z_k^*\}_{k\geqslant 1}$ is dense in X^* and $\{B_n\}_{n\geqslant 1}$ is a field generating $B(T)$, i.e. $\sigma(B_n:\ n \geqslant 1) = B(T)$ (again since by hypothesis T is a Polish space, $B(T)$ is countably generated and so such a countable field exists). Then, as in the proof of Theorem 3.1, we find that $\operatorname{Gr} H \in \Sigma \times B(M^b(T, X))$ (recall that $M^b(T, X)$ is equipped with the $w(M^b(T, X), C_b(T)\otimes X^*)$-topology). But $M^b(T, X)$ with this topology is Suslin (see Sainte-Beuve [23]). So we can apply Aumann's selection theorem and get $m \in TS_M$ s.t.

$$\sigma(x^*, M(\omega, C)) = (x^*, m(\omega, C))$$

$$\Rightarrow \sigma(x^*, N(\omega, C)) = \int_C f(\omega, t)\, d(x^* \circ m)(\omega, dt)$$

$$\Rightarrow \omega \to \sigma(x^*, N(\omega, C)) \text{ is measurable.}$$

Observe that

$$\operatorname{Gr} N(\cdot, C) = \bigcap_{k\geqslant 1} \{(\omega, y) \in \Omega \times X:\ (z_k^*, y) \leqslant \sigma(z_k^*, N(\omega, C))\} \in \Sigma \times B(X)$$

and since (Ω, Σ, μ) is by hypothesis a complete, finite measure space, we conclude (see Section 2) that $N(\cdot, C)$ is measurable for every $C \in B(T)$. Clearly $C \to \sigma(x^*, N(\omega, C))$ is a signed measure, hence $N(\omega, \cdot)$ is a multimeasure. Therefore we conclude that $N(\cdot, \cdot)$ is a transition multimeasure with values in $P_{wkc}(X)$. ■

Remark. An interesting and useful byproduct of the proof of Theorem 4.1 is that under the hypotheses of that theorem, we have $\sigma(x^*, N(\omega, C)) = \int_C f(\omega, t)\sigma(x^*, M(\omega, dt))$, for all $(\omega, C, x^*) \in \Omega \times B(T) \times X^*$.

Next we will derive a useful characterization of the measure selectors of the multimeasure $N(\omega, \cdot)$, $\omega \in \Omega$.

THEOREM 4.2. *If the hypotheses of Theorem 4.1 hold, then for all $\omega \in \Omega$, we have*

$$S_{N(\omega,\cdot)} = \{\int f(\omega, t)\, m(\omega, dt):\ m \in TS_M\}.$$

Proof. Recall that

$$\{v(\cdot) = \int f(\omega, t)\, m(\omega, dt):\ m \in TS_M\}$$

$$= \{v(\cdot) = \int f(\omega, t)\, \hat{m}(dt):\ \hat{m} \in S_{M(\omega,\cdot)}\} = \Gamma(\omega).$$

Clearly $\Gamma(\omega)$ is convex for all $\omega \in \Omega$. Also in the proof of Theorem 4.1 we saw that $\hat{m} \to (x^*, \int_\Omega f(\omega, t)\hat{m}(dt))$ is continuous on $S_{M(\omega,\cdot)}$ with the topology of pointwise weak convergence (i.e. with the $\hat{w} = w(M^b(T, X), \Sigma \otimes X^*)$-topology). Furthermore, recall that $S_{M(\omega,\cdot)}$ is $\hat{w}$-compact. Combining those two facts, we can easily check that $\Gamma(\omega)$ is $\hat{w}$-closed in $M^b(T, X)$.

Next let $v_1, v_2 \in \Gamma(\omega)$. By definition we have

$$v_1(B) = \int_B f(\omega, t)\, \hat{m}_1(dt), \quad \hat{m}_1 \in S_{M(\omega,\cdot)},\ B \in B(T),$$

$$v_2(B) = \int_B f(\omega, t)\, \hat{m}_2(dt), \quad \hat{m}_2 \in S_{M(\omega,\cdot)},\ B \in B(T).$$

Then if (B_1, B_2) is a Borel partition of T, we have

$$(\chi_{B_1} v_1 + \chi_{B_2} v_2)(\cdot) = \int f(\omega, t)\, \hat{m}_0(dt)$$

where $\hat{m}_0 = \chi_{B_1}\hat{m}_1 + \chi_{B_2}\hat{m}_2$. Clearly $\hat{m}_0 \in S_{M(\omega,\cdot)}$ and so $\chi_{B_1} v_1 + \chi_{B_2} v_2 \in \Gamma(\omega)$. Here for every $\omega \in \Omega$, $\Gamma(\omega)$ is a nonempty, $\hat{w}$-closed, convex and decomposable subset of $M^b(T, X)$. Thus Theorem 2 of Pallu de la Barrière [17] tells us that $\Gamma(\omega) = S_{N_1(\omega)(\cdot)}$, where $N_1(\omega)(\cdot)$: $B(T) \to P_{fc}(X)$ is a multimeasure. But clearly $\Gamma(\omega) \subseteq S_{N(\omega,\cdot)} \Rightarrow N_1(\omega)(\cdot) \subseteq N(\omega, \cdot) \Rightarrow N_1(\omega)(\cdot)$ is $P_{wkc}(X)$-valued. Also from

Theorem 1 of Godet-Thobie [13] we know that for all $C \in B(T)$, we have

$$\{\int_C f(\omega, t)\, \hat{m}(dt):\ \hat{m} \in S_{M(\omega,\cdot)}\} = \{\int_C f(\omega, t)\, m(\omega, dt):\ m \in TS_M\}$$

$$\Rightarrow N_1(\omega)(C) = \int_C f(\omega, t)\, M(\omega, dt)$$

$$\Rightarrow N_1(\omega)(C) = N(\omega, C) \quad \text{for all } (\omega, C) \in \Omega \times B(T)$$

$$\Rightarrow \Gamma(\omega) = S_{N(\omega,\cdot)} \quad \text{for all } \omega \in \Omega, \text{ as claimed by the theorem. } \blacksquare$$

An immediate interesting consequence of Theorem 4.2 is the following fact:

COROLLARY. *If the hypotheses of Theorem 4.2 hold, h: $\Omega \to X$ is measurable and for some $C \in B(T)$, $h(\omega) \in N(\omega, C)$ for all $\omega \in \Omega$, then there exists $m \in TS_M$ s.t. $h(\omega) = \int_C f(\omega, t)\, m(\omega, dt)$ for all $\omega \in \Omega$.*

Proof. From Theorem 3.1 we know that there exists $n \in TS_N$ s.t. $n(\omega, C) = h(\omega)$ for all $\omega \in \Omega$. Then applying Theorem 4.2, we see that for some $m \in TS_M$ and for all $B \in B(T)$, we have

$$n(\omega, B) = \int_B f(\omega, t)\, m(\omega, dt) \Rightarrow h(\omega) = \int_C f(\omega, t)\, m(\omega, dt) \text{ with } m \in TS_M. \ \blacksquare$$

5. The multivalued Feller property. In this section we turn our attention to transition multimeasures for which the parameter varies over a topological space. Hence instead of simple measurability with respect to that parameter, we can require a continuity type property. Recall (see Klein–Thompson [16]) that if Y, Z are Hausdorff topological spaces, then a multifunction G: $Y \to 2^Z \setminus \{\emptyset\}$ is said to be *upper semicontinuous* (u.s.c.) if and only if for every $U \subseteq Z$ nonempty, open, $G^+(U) = \{y \in Y:\ G(y) \subseteq U\}$ is open in Y. So if Z is a Polish space, Theorem 4.2 of Wagner [26] tells us that an u.s.c. multifunction G: $Y \to P_f(Z)$ is automatically $B(Y)$-measurable.

If Y, Z are separable metric spaces, $m(y, dz)$ is a continuous stochastic kernel (i.e. a continuous transition measure) and $f \in C(Y \times Z)$, then according to Feller's property $y \to n(y) = \int_Z f(y, z)\, m(y, dz)$ is continuous. Feller's property is crucial in establishing the existence of invariant probability measures for transition probabilities.

Our next theorem derives a multivalued version of Feller's property. So assume that: (i) S is a Polish space with a Radon measure $\mu(\cdot)$ and $\hat{B}(S)$ denotes the completion of the Borel σ-field $B(S)$ with respect to $\mu(\cdot)$, (ii) T is another Polish space, with Borel σ-field $B(T)$ and $\lambda(\cdot)$ a Radon measure on $(T, B(T))$, and (iii) X is a separable reflexive Banach space. Also a transition multimeasure M: $S \times B(T) \to P_f(X)$ which is u.s.c. in the s variable from S into X_w will be called an *u.s.c. transition multimeasure*. Finally, we will say that $M(\cdot, T)$ is *scalarly continuous* if $s \to \sigma(x^*, M(s, T))$ is continuous for all $x^* \in X^*$. This is trivially satisfied if for instance $M(s, T)$ is independent of s.

THEOREM 5.1. *If* $M\colon S\times B(T)\to P_{fc}(X)$ *is an u.s.c. transition multimeasure s.t.* $M(s, A)\subseteq \lambda(A)W(s)$ *for all* $(s, A)\in S\times B(T)$, *with* $W(s)\in P_{wkc}(X)$, *and* $M(\cdot, T)$ *is scalarly continuous,* $f\colon S\times T\to \mathbf{R}_+$ *is an u.s.c., bounded above function s.t.* $f(s,\cdot)\in L^1(T)$ *for all* $s\in S$ *and*

$$N(s, C)=\int_C f(s, t)M(s, dt) \quad \text{for all } (s, C)\in S\times B(T),$$

then $N(\cdot,\cdot)$ *is an u.s.c.,* $P_{wkc}(X)$*-valued transition multimeasure.*

Proof. That $N(\cdot,\cdot)$ is a P_{wkc}-valued transition multimeasure follows immediately from Theorem 4.1. Also from the same theorem (see the remark following the proof), for any $x^*\in X^*$ we have

$$\sigma(x^*, N(s, C))=\int_S f(s, t)\sigma(x^*, M(s, dt)).$$

Let $\phi_1\colon S\to M^b(S)$ be defined by $\phi_1(s)=\delta_s$, where $\delta_s(\cdot)$ is the Dirac point mass measure at $s\in S$. It is clear that $\phi_1(\cdot)$ is continuous from S into $M^b(S)$ with the weak topology. Also let $\phi_2\colon S\to M^b(T)$ be defined by $\phi_2(s)=\sigma(x^*, M(s,\cdot))$. If $s_n\to s$ in S and K is a closed subset of T, from the upper semicontinuity of $M(\cdot, K)$ we have

$$\limsup \sigma(x^*, M(s_n, K))\leqslant \sigma(x^*, M(s, K))$$

(see for example Proposition 2, p. 122 of Aubin–Ekeland [3]). Since K was any closed subset of T and $\sigma(x^*, M(s_n, T))\to\sigma(x^*, M(s, T))$ ($M(\cdot, T)$ being by hypothesis scalarly continuous), we deduce that

$$\sigma(x^*, M(s_n,\cdot))\xrightarrow{w}\sigma(x^*, M(s,\cdot)) \quad \text{in } M^b(T)$$

$$\Rightarrow \phi_2(\cdot) \text{ is continuous into } M^b(T) \text{ with the weak topology.}$$

Therefore the map $\phi\colon S\to M^b(S)\times M^b(T)$ defined by $\phi(s)=(\phi_1(s), \phi_2(s))$ is continuous into $M^b(S)\times M^b(T)$ with the product weak topology.

Now let $\varphi\colon M^b(S)\times M^b(T)\to M^b(S\times T)$ be defined by

$$\varphi(m, n)=m\otimes n.$$

From Theorem 3.2, p. 21 of Billingsley [5], we know that φ is continuous for the weak topology. So $h=\varphi\circ\phi\colon S\to M^b(S\times T)$ is continuous. Also let $p_f^C\colon M^b(S\times T)\to\mathbf{R}$ be defined by

$$p_f^C(v)=\int_{S\times C} f(s, t)\,dv.$$

Recall that the upper semicontinuity and boundedness from above of $f(\cdot,\cdot)$ is equivalent to the existence of $f_n(\cdot,\cdot)\in C_b(S\times T)$ s.t. $f_n\downarrow f$ (consider for example the "Weierstrass needle functions" $f_n(s, t)=\sup_{(s',t')\in S\times T}[f(s', t')-nd_S(s, s')-nd_T(t, t')]$). Then let

$$p_{f,n}^C(v)=\int_{S\times C} f_n(s, t)\,dv.$$

Clearly from the definition of the weak topology on $M^b(S\times T)$, $p_{f,n}^C(\cdot)$ is continuous for all $n\geqslant 1$. Then by the monotone convergence theorem we have $p_{f,n}^C\downarrow p_f^C$ and so conclude that $p_f^C(\cdot)$ is u.s.c. Hence the composite map

$$p_f^C(h(s))=\int_C f(s, t)\sigma(x^*, M(s, dt))=\sigma(x^*, N(s, C))$$

is u.s.c. in s. Since $N(\cdot,\cdot)$ is $P_{wkc}(X)$-valued, Theorem 10, p. 128 of Aubin–Ekeland [3], tells us that $N(\cdot, C)$ is u.s.c. from S into X_w. ∎

Remarks. (1) If $f\in C_b(S\times T)$, $\dim X<\infty$ and $M(\cdot,\cdot)$ is as in Theorem 5.1, then $N(\cdot, C)$ is continuous in the Hausdorff metric. This follows from Corollary 3A of Salinetti–Wets [24].

(2) This result can be useful in establishing the existence of stochastic equilibria in dynamic economies.

6. Integration with respect to the parameter. As stochastic kernels act upon probabilities on the parameter space, by integration with respect to the parameter, a similar action can be defined for transition multimeasures. So for a transition multimeasure $M\colon S\times\mathscr{T}\to P_f(X)$ and for $C\in B(S)\times\mathscr{T}$ we consider the Aumann integral $\int_S M(s, C(s))\mu(ds)$, where $C(s)$ is the section of C by s and $\mu(\cdot)$ is a measure on $(S, B(S))$. To guarantee that the above set-valued integral will be nonempty, we need to know that the multifunction $s\to M(s, C(s))$ is measurable.

So assume that: (i) S is a Polish space, (ii) $(T,\mathscr{T})$ is a measurable space, and (iii) X is a separable Banach space.

THEOREM 6.1. *If* $M\colon S\times\mathscr{T}\to P_{wkc}(X)$ *is a transition multimeasure,* $C\in B(S)\times\mathscr{T}$ *and* $C(s)=\{t\in T\colon (s, t)\in C\}$, *then* $s\to F_C(s)=M(s, C(s))$ *is a measurable multifunction.*

Proof. From Fubini's theorem, we know that $C(s)\in\mathscr{T}$ for all $s\in T$. So $M(s, C(s))=F_C(s)$ is well defined.

Next consider the family

$$\mathscr{L}=\{C\in B(S)\times\mathscr{T}\colon F_C(\cdot) \text{ is a measurable multifunction}\}.$$

Clearly $C=S\times T\in\mathscr{L}$. Also assume that $C_1, C_2\in\mathscr{L}$ and $C_2\subseteq C_1$. Then for $x^*\in X^*$, we have

$$\sigma(x^*, M(s, C_1(s)))=\sigma(x^*, M(s, C_1(s)\backslash C_2(s)))+\sigma(x^*, M(s, C_2(s)))$$

$$\Rightarrow \sigma(x^*, M(s, C_1(s)\backslash C_2(s)))=\sigma(x^*, M(s, C_1(s)))-\sigma(x^*, M(s, C_2(s)))$$

$$\Rightarrow s\to\sigma(x^*, M(s, C_1(s))) \text{ is measurable.}$$

Since $M(\cdot,\cdot)$ is $P_{wkc}(X)$-valued, as in the proof of Theorem 4.1, we see that $s\to M(s, C_1(s)\backslash C_2(s))$ is measurable.

Finally, let $\{C_n\}_{n\geq 1} \subseteq \mathscr{L}$, $C_1 \subseteq C_2 \subseteq \ldots$ Then $s \to \sigma(x^*, M(s, C_n(s)))$ is measurable for each $n \geq 1$ and each $x^* \in X^*$. Let $D_1(s) = C_1(s)$ and $D_n(s) = C_n(s) \setminus C_{n-1}(s)$, $n \geq 2$. Then we have

$$\sigma\big(x^*, M(s, \bigcup_{n\geq 1} C_n(s))\big) = \sigma\big(x^*, M(s, \bigcup_{n\geq 1} D_n(s))\big) = \sum_{n\geq 1} \sigma\big(x^*, M(s, D_n(s))\big)$$

$$= \sigma\big(x^*, M(s, C_1(s))\big) + \sum_{n\geq 2} \Big(\sigma\big(x^*, M(s, C_n(s))\big) - \sigma\big(x^*, M(s, C_{n-1}(s))\big)\Big)$$

$$\Rightarrow s \to \sigma\big(x^*, M(s, \bigcup_{n\geq 1} C_n(s))\big) \text{ is measurable,}$$

$$\Rightarrow s \to M(s, \bigcup_{n\geq 1} C_n(s)) \text{ is measurable,}$$

$$\Rightarrow \bigcup_{n\geq 1} C_n \in \mathscr{L}.$$

Thus we conclude that $\mathscr{L}$ is a Dynkin system (see Ash [2]). Clearly $\mathscr{L} \supseteq R = \{E_1 \times E_2: E_1 \in B(S), E_2 \in \mathscr{T}\}$. Therefore invoking the Dynkin system theorem (see Ash [2], Theorem 4.1.2, p. 169), we conclude that

$$\sigma(R) = B(S) \times \mathscr{T} \subseteq \mathscr{L}$$

$$\Rightarrow s \to F_C(s) \text{ is measurable for all } C \in B(S) \times \mathscr{T}. \ \blacksquare$$

Now we can integrate with respect to the parameter $s \in S$.

THEOREM 6.2. *If the hypotheses of Theorem 6.1 hold and in addition $\mu(\cdot)$ is a measure $(S, B(S))$, $\lambda(\cdot)$ is a measure on $(T, \mathscr{T})$ and for all $C \in \mathscr{T}$, $M(s, C) \subseteq \lambda(C) W(s)$ with $W: S \to P_{wkc}(X)$ integrably bounded, then*

$$N(C) = \int_S M(s, C(s)) \mu(ds)$$

is a multimeasure with values in $P_{wkc}(X)$.

Proof. From Theorem 6.1 and our boundedness hypothesis on $M(\cdot, \cdot)$, we deduce that $s \to M(s, C(s))$ is an integrably bounded multifunction. So the corollary to Proposition 3.1 of [18] tells us that $N(C) = \int_C M(s, C(s)) \mu(ds) \in P_{wkc}(X)$. Then for $x^* \in X^*$ we have

$$\sigma(x^*, N(C)) = \int_S \sigma\big(x^*, M(s, C(s))\big) \mu(ds)$$

(see Proposition 2.1 of [20]), from which we deduce that $\sigma(x^*, N(\cdot))$ is a signed measure, hence $N(\cdot)$ is a multimeasure. ■

We can characterize the measure selectors of $N(\cdot)$ using the elements of TS_M. So assume the following: (i) S is a Polish space with Borel σ-field $B(S)$ and a Radon measure $\mu(\cdot)$ on $(S, B(S))$, (ii) T is a Polish space with Borel σ-field $B(T)$ and a Radon measure $\lambda(\cdot)$ on $(T, B(T))$, and (iii) X is a separable, reflexive Banach space.

THEOREM 6.3. *If $M: S \times B(T) \to P_{wkc}(X)$ is a transition multimeasure s.t. for all $C \in B(T)$, $M(s, C) \subseteq \lambda(C) W(s)$ with $W(s) \in P_{wkc}(X)$ and*

$$x \in N(A \times B) = \int_A M(s, B) \mu(ds) \quad \textit{for some } (A, B) \in B(S) \times B(T),$$

then there exists $m \in TS_M$ s.t. $x = \int_A m(s, B) \mu(ds)$.

Proof. From the definition of the Aumann integral, we have $x = \int_A f(s) \mu(ds)$, $f \in S^1_{M(\cdot, B)}$. Applying Theorem 3.1 we can find $m \in TS_M$ s.t. $m(s, B) = f(s)$. Hence $x = \int_A m(s, B) \mu(ds)$. ■

7. Radon–Nikodym theorem for transition multimeasures. The Radon–Nikodym theorem for transition multimeasures is an interesting problem and can have useful applications, like the corresponding result for regular multimeasures (see Hildenbrand [15], the core of economies with production and with a continuum of agents).

So assume that (i) (Ω, Σ, μ) is a complete σ-finite measure space, (ii) T is a Polish space with a σ-finite measure $\lambda(\cdot)$ on $B(T)$, and (iii) X is a separable, reflexive Banach space. We start with a proposition that we will need in the proof of the main theorem.

PROPOSITION 7.1. *If $m: \Omega \times B(T) \to X$ is a transition measure of bounded variation s.t. $m(\omega, \cdot) \ll \lambda$ μ-a.e., $|m(\omega, \cdot)| \leq a(\omega)$ μ-a.e., $a(\cdot) \in L^1_+$, then there exists a measurable function $f: \Omega \times T \to X$ and $N \in \Sigma$ with $\mu(N) = 0$ s.t. $f(\omega, \cdot) \in L^1(T, \lambda, X)$ for every $\omega \in \Omega$ and*

$$m(\omega, C) = \int_C f(\omega, t) \lambda(dt)$$

for all $\omega \in \Omega \setminus N$ and all $C \in B(T)$.

Proof. Since by hypothesis $m(\omega, \cdot)$ is of bounded variation, $m(\omega, \cdot) \ll \lambda$ for all $\omega \in \Omega \setminus N$, $\mu(N) = 0$ and X is reflexive (hence has the Radon–Nikodym property (RNP)), for $\omega \in \Omega \setminus N$ there exists $f(\omega, \cdot) \in L^1(T, \lambda, X)$ s.t. $m(\omega, C) = \int_C f(\omega, t) \lambda(dt)$. By redefining $\omega \to f(\omega, \cdot)$ on N, we may assume that $f(\omega, \cdot) \in L^1(X)$ for all $\omega \in \Omega$. Then for every $x^* \in X^*$ and $C \in B(T)$ we have

$$\langle f(\omega, \cdot), \chi_C x^* \rangle = (x^*, m(\omega, C))$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $(L^1(T, \lambda, X), L^\infty(T, \lambda, X^*) = [L^1(T, \lambda, X)]^*)$. Hence $\omega \to \langle f(\omega, \cdot), \chi_C x^* \rangle$ is measurable. Since countably valued functions are dense in $L^\infty(T, \lambda, X^*)$ (see Corollary 3, p. 42 of Diestel–Uhl [11]), we deduce that $\omega \to \langle f(\omega, \cdot), u \rangle$ is measurable for all $u \in L^\infty(T, \lambda, X^*)$ $\Rightarrow$ $\omega \to f(\omega, \cdot)$ is weakly measurable from Ω into $L^1(T, \lambda, X)$ and since $L^1(T, \lambda, X)$ is separable, by the Pettis measurability theorem (see Diestel–Uhl [11], p. 42), we find that $\omega \to f(\omega, \cdot)$ is measurable from Ω into $L^1(T, \lambda, X)$, hence $f(\cdot, \cdot)$ is measurable from $\Omega \times T$ into X. ■

Now we can state the Radon–Nikodym theorem for transition multimeasures. The hypotheses on the spaces remain the same as in Proposition 7.1.

THEOREM 7.1. *If* $M: \Omega\times B(T)\to P_{wkc}(X)$ *is a transition multimeasure of bounded variation s.t.* $|M|(\omega,\cdot)\ll\lambda$ μ*-a.e.,* $|M|(\omega,\cdot)\leqslant a(\omega)$ μ*-a.e.,* $a(\cdot)\in L^1_+$, *then there exists a measurable multifunction* $F: \Omega\times T\to P_{wkc}(X)$ *and* $N\in\Sigma$ *with* $\mu(N)=0$ *s.t.* $F(\omega,\cdot)$ *is integrably bounded for every* $\omega\in\Omega$ *and*

$$M(\omega, C)=\int_C F(\omega, t)\lambda(dt),\qquad \omega\in\Omega\backslash N,\ C\in B(T).$$

Proof. Let $h_n: \Omega\to X$ be measurable functions s.t. $M(\omega, T)=\mathrm{cl}\{h_n(\omega)\}$ for all $\omega\in\Omega$. Invoking Theorem 3.1 of this paper, we know that we can find $m_n\in TS_M$ s.t. $h_n(\omega)=m_n(\omega, T)$, $n\geqslant 1$ for all $\omega\in\Omega$. Then for every $C\in B(T)$ we have

$$\overline{\{m_n(\omega, C)+m_n(\omega, C^c)\}}_{n\geqslant 1}=\overline{\{h_n(\omega)\}}_{n\geqslant 1}=M(\omega, T)=M(\omega, C)+M(\omega, C^c),$$

$$\overline{\{m_n(\omega, C)+m_n(\omega, C^c)\}}_{n\geqslant 1}\subseteq\overline{\mathrm{conv}}\{m_n(\omega, C)\}_{n\geqslant 1}+\overline{\mathrm{conv}}\{m_n(\omega, C^c)\}_{n\geqslant 1}.$$

Since $m_n\in TS_M$, $n\geqslant 1$, we deduce that

$$\overline{\mathrm{conv}}\{m_n(\omega, C)\}_{n\geqslant 1}=M(\omega, C).$$

Applying Proposition 7.1 above, we see that there exist $N\in\Sigma$ with $\mu(N)=0$ and $f_n: \Omega\times T\to X$ measurable s.t. for all $n\geqslant 1$, $f_n(\omega,\cdot)\in L^1(T,\lambda, X)$ for all $\omega\in\Omega$ and $m_n(\omega, C)=\int_C f_n(\omega, t)\lambda(dt)$ for every $\omega\in\Omega\backslash N$ and every $C\in B(T)$. Set $F(\omega, t)=\overline{\mathrm{conv}}\{f_n(\omega, t)\}_{n\geqslant 1}$. Clearly then $F: \Omega\times T\to P_{wkc}(X)$ is measurable and

$$|F(\omega, t)|\leqslant\frac{d|M|(\omega, t)}{d\lambda}\qquad \mu\times\lambda\text{-a.e.}$$

Since $d|M|(\omega,\cdot)/d\lambda\in L^1_+(T)$, we deduce that $F(\omega,\cdot)$ is integrably bounded μ-a.e. and by redefining it on the μ-null set we can have $F(\omega,\cdot)$ integrably bounded for all $\omega\in\Omega$. Finally, using Proposition 2.3 of [20], we have

$$\overline{\mathrm{conv}}\{m_n(\omega, C)\}_{n\geqslant 1}=\overline{\mathrm{conv}}\{\int_C f_n(\omega, t)\lambda(dt)\}_{n\geqslant 1}=\int_C\overline{\mathrm{conv}}\{f_n(\omega, t)\}_{n\geqslant 1}\lambda(dt)$$

$$=\int_C F(\omega, t)\lambda(dt),\qquad \omega\in\Omega\backslash N,\ C\in B(T).\ \blacksquare$$

Remark. If $f: \Omega\times T\to\mathbf{R}$ is a bounded measurable function, then we have

$$\int_C f(\omega, t)M(\omega, dt)=\int_C f(\omega, t)F(\omega, t)\lambda(dt)\qquad\text{for all } \omega\in\Omega\backslash N,\ C\in B(T).$$

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