Questions. 1) Can we drop (i) in the condition (M_p)?

2) What condition on the Radon–Nikodym derivative v_0^* is equivalent to (M_p)?

Janusz Woś is unfortunately no longer among us. This is one of the last papers on which he worked.

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Cross-sections of solution funnels in Banach spaces

by

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Abstract. The present paper applies negligibility theory (a part of infinite-dimensional topology) to study the geometry of the failure of Kneser's theorem in infinite-dimensional Banach spaces. In particular, it turns out that arbitrary compact subsets of the infinite-dimensional separable Hilbert space can be represented as cross-sections of solution funnels. For general infinite-dimensional Banach spaces, the existence of initial value problems with exactly two solutions is proved.

1. Introduction. Let X and Y be Banach spaces. If $U \subset X$ is open and $V \subset Y$, then $C^p(U, V)$ denotes the set of all mappings $f\colon U \to V$ (with domain U) having continuous pth Fréchet derivative, $p = 0, 1, 2, \ldots$ ($C^0(U, V)$ is simply the set of all continuous mappings.) We also let $C^\infty(U, V) = \bigcap \{C^p(U, V) \mid p \in \mathbf{N}\}$. The derivative of $f \in C^p(U, V)$ at $u \in U$ is denoted by $D_x f(u)$. The origin of X is denoted by 0_X.

For $F \in C^0(\mathbf{R} \times X, X)$, consider the ordinary differential equation (ODE)

$$D_t x = F(t, x). \tag{1}$$

For $(t_0, x_0) \in \mathbf{R} \times X$, a function $x \in C^1(I_x, X)$ is called a *solution of* (1) *through* (t_0, x_0) if I_x is an open interval in $\mathbf{R}$ containing t_0, $x(t_0) = x_0$ and $D_t x(u) = F(u, x(u))$ for all $u \in I_x$. Solutions with domain $\mathbf{R}$ are called *global*.

Let $\mathscr{F}(X)$ denote the class of functions $F \in C^0(\mathbf{R} \times X, X)$ satisfying the following conditions:

(2) for each $(t_0, x_0) \in \mathbf{R} \times X$, the ODE (1) has at least one solution through (t_0, x_0);

(3) all solutions of (1) extend to global solutions.

The well-known Peano theorem states that all $F \in C^0(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ satisfy (2).

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As is observed in [19], (3) is satisfied for all $F \in C^0(\mathbf{R}\times\mathbf{R}^n, \mathbf{R}^n)$ which have supports in sets of the form $\mathbf{R}\times$compact. Unfortunately, the Peano theorem does not remain valid in infinite dimensions. Given an arbitrary infinite-dimensional Banach space X, (2) is satisfied only for those $f \in C^0(\mathbf{R}\times X, X)$ which fulfil some additional hypotheses (usually compactness assumptions and/or assumptions of dissipative type [8]) [12]. In general, slightly stronger hypotheses imply (maybe a weaker form of) condition (3). For example, as a simple corollary of Gronwall's inequality, (2) implies (3) provided that $\|F(t, x)\| \leqslant K + L\|x\|$, $(t, x) \in \mathbf{R}\times X$, for some $K, L > 0$.

Given $F \in \mathscr{F}(X)$, $(t_0, x_0) \in \mathbf{R}\times X$, the *cross-section of the solution funnel* at time t is the set

$$S_t(F, (t_0, x_0)) = \{x(t) \in X \mid x \text{ is a solution of (1) through } (t_0, x_0)\}.$$

The *solution funnel* (integral funnel) is the set

$$S(F, (t_0, x_0)) = \{(t, x(t)) \in \mathbf{R}\times X \mid x(t) \in S_t(F, (t_0, x_0))\}.$$

It is also natural to consider the set of all solutions of (1) through (t_0, x_0) in the function space $C^1(\mathbf{R}, X)$ (endowed with the topology of uniform convergence on compact subintervals), not just their graphs in $\mathbf{R}\times X$ [1], [17].

The study of the topological properties of the solution funnel was initiated by Kneser [15].

KNESER'S THEOREM. *Let $F \in \mathscr{F}(\mathbf{R}^n)$. Then $S_t(F, (t_0, x_0))$ is a nonempty, compact and connected subset of $\mathbf{R}^n$, for all $(t_0, x_0) \in \mathbf{R}\times\mathbf{R}^n$, $t \in \mathbf{R}$.*

Arguing similarly to Section 2 of [19], it is easy to show that the assumption $F \in \mathscr{F}(\mathbf{R}^n)$ is effectively equivalent to the standard assumptions due to Kamke [14] (in his version of Kneser's [15] original theorem).

There is enormous literature on generalizations of Kneser's theorem (for certain classes of integral equations, functional differential equations, differential inclusions etc. on manifolds, Banach spaces, locally convex spaces etc.). With more or less effort, most of these generalizations can be derived from abstract fixed point theorems in nonlinear analysis [16, Chap. 48], [23]. The interested reader is referred to [10], [13], [16]–[18], [23] and the references therein.

Less is known about the problem of characterizing cross-sections of solution funnels. In spite of various necessary resp. sufficient conditions (formulated in terms of algebraic and/or differential topology) obtained by Pugh [19] and Rogers [20] in the finite-dimensional case, the problem as a whole is far from being settled. The best result is due to Pugh [19, Cor. (5.4)] who solved the problem under the additional hypothesis "funnel cobordant to a point". In an ingenious mixture of examples and counterexamples, theorems, conjectures and open problems, Pugh [19] has made an attempt at considering the classification problem of cross-sections of solution funnels in terms of cobordism theory in algebraic topology. The point $x_0 \in \mathbf{R}^n$ and the compact set $A \subset \mathbf{R}^n$ are called *funnel cobordant* in $\mathbf{R}^n$ if there exists an $F \in \mathscr{F}(\mathbf{R}^n)$ such that $S_1(F, (0, x_0)) = A$ and $S_0(F, (1, a)) = \{x_0\}$ for all $a \in A$. Funnel cobordism will be denoted by $\{x_0\} \vdash\!\dashv_F A$. (Arguing similarly to Section 2 of [19], it is easy to show that $\{x_0\} \vdash\!\dashv_F A$ implies $\{x_0\} \vdash\!\dashv_G A$ for some $G \in \mathscr{F}(\mathbf{R}^n)$ which has support in some $\mathbf{R}\times$compact: this is Pugh's original definition.)

PUGH'S THEOREM [19, Cor. (5.4)]. (a) *If $\{x_0\} \vdash\!\dashv_F A$, then there is a C^∞ diffeomorphism from $\mathbf{R}^n \setminus A$ onto $\mathbf{R}^n \setminus \{x_0\}$.*

(b) *If $A \subset \mathbf{R}^n$ is compact, $x_0 \in \mathbf{R}^n$ and there exists a C^∞ diffeomorphism* (in fact, if $n \neq 4$ then "C^∞ diffeomorphism" can be replaced [19, Remark 2] by "homeomorphism") *from $\mathbf{R}^n \setminus A$ onto $\mathbf{R}^n \setminus \{x_0\}$, then there exists an $F \in \mathscr{F}(\mathbf{R}^n)$ with $\{x_0\} \vdash\!\dashv_F A$.*

The construction of F in [19, pp. 287–292] is based on the construction [19, Cor. (5.10)] of a C^∞ diffeomorphism from $\mathbf{R}^n \setminus A$ onto $\mathbf{R}^n \setminus \{x_0\}$ which fixes all points in some neighbourhood of infinity. It is worth mentioning that $F \in \mathscr{F}(\mathbf{R}^n)$ constructed in [19, pp. 287–292] has the additional property that (in case A consists of at least two points) $(0, x_0) \in \mathbf{R}\times\mathbf{R}^n$ is the only point of nonuniqueness for (1).

The present paper is devoted to the problem of characterizing cross-sections of solution funnels in infinite dimensions. More precisely, the problem we investigate is to what extent Kneser's theorem fails in infinite-dimensional Banach spaces [19, Problem 4].

By generalizing Part (b) of Pugh's theorem, we provide a wide class of examples for cross-sections of solution funnels in infinite-dimensional Banach spaces. The method we apply is an extension of the one developed in [11] for constructing counterexamples to the Peano theorem in infinite-dimensional reflexive Banach spaces and is independent of techniques used in [19].

2. To what extent does Kneser's theorem fail in infinite-dimensional Banach spaces? Let X be an infinite-dimensional Banach space. As was observed by Binding [3, Section 2], it is easy to construct an $F \in \mathscr{F}(X)$ such that $S_1(F, (0, 0_X)) = \{x \in X \mid \|x\| \leqslant 1\}$. Thus, cross-sections of solution funnels need not be compact. To our best knowledge, the first example of this kind – in case $X = l_\infty$, the Banach space of real sequences $c = \{c_n\}$ with $\|c\| = \sup|c_n| < \infty$ – is due to Cellina [7, p. 135]. Further, by a modification of Godunov's general counterexample [12] for the Peano theorem in infinite-dimensional Banach spaces, Binding [3, Section 4] has constructed an $F \in C^0(\mathbf{R}\times X, X)$ such that $S_1(F, (0, 0_X))$ is not connected. Unfortunately, $F \notin \mathscr{F}(X)$. In fact, the disconnectedness of $S_1(F, (0, 0_X))$ in [3] is caused by a certain strong kind of violation of (2).

The aim of the present paper is to point out that the disconnectedness of cross-sections of solution funnels may be caused by a very complicated global behaviour of the trajectories as well and not only by the failure of local existence. For various types of Banach spaces, we construct ordinary differential equations $D_t x = F(t, x)$, $F \in \mathcal{F}(X)$, with disconnected cross-sections of a solution funnel.

THEOREM 1. *Let X be a Banach space and let A be a nonempty bounded closed subset of X. Let $\alpha = \sup\{\|a\| \mid a \in A\}$. Assume that there exists a C^1 diffeomorphism h mapping $X \setminus A$ onto $X \setminus \{0_X\}$ such that $h(x) = x$ whenever $\|x\| \geqslant \alpha + 1$. Then there is an $F \in \mathcal{F}(X)$ such that $S_1(F, (0, 0_X)) = A$. Moreover, if A consists of at least two points, then $F \in \mathcal{F}(X)$ can be chosen so that*

(4) $(0, 0_X) \in \mathbf{R} \times X$ *is the only point of nonuniqueness for* (1).

Given an infinite-dimensional Banach space X and a nonempty bounded closed set A in $\{x \in X \mid \|x\| \leqslant \alpha\}$, it is an extremely difficult task to decide whether there exists a homeomorphism/diffeomorphism h mapping $X \setminus A$ onto $X \setminus \{0_X\}$ (with or without the property that $h(x) = x$ whenever $\|x\| \geqslant \alpha + 1$). The topological case of this problem is thoroughly discussed in [2]. We do not know of such a survey in the differentiable case. (For various smoothness properties in Banach spaces, see [21].)

We now list several examples of pairs (X, A) satisfying the conditions of Theorem 1.

EXAMPLE 1 (Part (b) of Pugh's theorem). Let $X = \mathbf{R}^n$ and let A be a compact subset of X. Assume that there exists a C^1 diffeomorphism from $X \setminus A$ onto $X \setminus \{0_X\}$. By [19, Cor. (5.10)], the pair (X, A) satisfies the conditions of Theorem 1.

EXAMPLE 2. Let X be the separable infinite-dimensional Hilbert space and let A be a nonempty closed locally compact bounded subset of X. By a twofold application of [24, Theorem 1], the pair (X, A) satisfies the conditions of Theorem 1. In fact, there exists [24, Theorem 1] a C^∞ diffeomorphism h_1 from $X \setminus A$ onto X with $h_1(x) = x$ whenever $\|x\| \geqslant \alpha + 1$. Similarly, there exists a C^∞ diffeomorphism h_2 from X onto $X \setminus \{0_X\}$ with $h_2(x) = x$ whenever $\|x\| \geqslant \alpha + 1$. The desired diffeomorphism h can be chosen as $h_2 \circ h_1$.

EXAMPLE 3. Let $(X, \|\cdot\|)$ be a separable Banach space with Schauder basis and let A be a nonempty compact subset of X. Assume that there is an equivalent norm $\|\cdot\|_1$ on X with the property that the function $x \to \|x\|_1$, $x \neq 0_X$, is in $C^p(X \setminus \{0_X\}, \mathbf{R})$, $p = 1, 2, \ldots, \infty$. By a twofold application of [22, Cor. 19], the pair (X, A)—with h being a C^p diffeomorphism, $p = 1, 2, \ldots, \infty$ —satisfies the conditions of Theorem 1.

EXAMPLE 4. Let X be a reflexive Banach space and let A be a nonempty bounded discrete subset of X. By a twofold application of [11, Lemma 2] (cf. [9, p. 138]), the pair (X, A) satisfies the conditions of Theorem 1.

CONJECTURE. As usual, let l_1 denote the Banach space of real sequences $c = \{c_n\}$ with $\|c\| = \sum |c_n| < \infty$. Given $a \in l_1$, $a \neq 0_X \in X = l_1$, we conjecture that the pair $(l_1, \{a, 0_X\})$ does not satisfy the conditions of Theorem 1.

The conjecture is motivated by the following remark.

Remark 1. Let X be an infinite-dimensional Banach space and let K be a nonempty closed bounded subset of X. Assume that there is a C^1 diffeomorphism mapping $X \setminus K$ onto X. (This—and moreover, the existence of a C^∞ diffeomorphism—is known to be true if (a) X is separable and K is compact [9, Theorem 5.2] or if (b) X is the separable Hilbert space and $X \setminus K$ is homeomorphic to X [5, Theorem 9]). It is natural to ask if there is a C^1 diffeomorphism h from $X \setminus K$ onto X with the property that $\{x \in X \mid h(x) \neq x\}$ is bounded. The answer is, in general, negative. More precisely, (for arbitrary nonempty bounded K in X) the answer is negative [22, p. 589] whenever $\{\varphi \in C^1(X, \mathbf{R}) \mid \text{the support of } \varphi \text{ is nonempty and bounded}\} = \emptyset$. For a list of spaces satisfying this latter property, see [25]. In particular [4], the answer is negative in case $X = l_1$. Nevertheless, the problem as a whole seems to be very difficult and is far from being solved.

THEOREM 2. *Let X be an arbitrary infinite-dimensional Banach space. Then there is an $F \in \mathcal{F}(X)$ satisfying* (4) *such that the ODE* (1) *has exactly two global solutions through* $(0, 0_X) \in \mathbf{R} \times X$.

The proof of Theorem 1 is divided into the proof of two propositions. As is pointed out by Remark 2 below, Proposition 1 has its own interest. The $A = \emptyset$ case of Proposition 1 was proved in [11].

The definition of $\Phi(X, A)$ (see Section 3 below) makes sense for $A = \emptyset$ as well. (11) goes over into the requirement that (5) has no bounded global solutions. Given an arbitary infinite-dimensional Banach space X, (by using different notation) we proved that—with (9) replaced by (9)′—$\Phi(X, \emptyset) \neq \emptyset$ [11, Theorem 1] and then, by applying the $A = \emptyset$ case of Proposition 1, we constructed counterexamples to the Peano theorem in X [11]. Theorem 2 of the present paper will be obtained as a simple corollary of $\Phi(X, \emptyset) \neq \emptyset$.

3. A simple reduction principle. From now on, let X be an arbitrary Banach space and let A be a nonempty bounded closed subset of X. Let

$$\alpha = \sup\{\|a\| \mid a \in A\}.$$

For $f \in C^0(X, X)$, consider the ODE

$$D_\tau y = f(y). \tag{5}$$

Let $\Phi(X, A)$ denote the class of functions $f \in C^0(X, X)$ satisfying

(6) $\|f(y) - y\| \leqslant 2(\alpha + 2)$ for all $y \in X$,

(7) $f(y) = y$ whenever $\|y\| \geqslant \alpha + 2$,

(8) $f(y) = 0_X$ if and only if $y \in A$,

(9) for each $y_0 \in X$, the ODE (5) has at least one solution through $(0, y_0) \in \mathbf{R} \times X$,

(10) all solutions of (5) extend to global solutions,

(11) if $y \in C^1(\mathbf{R}, X)$ is a bounded global solution of (5), then there exists an $a \in A$ such that $y(t) = a$ for all $t \in \mathbf{R}$.

PROPOSITION 1. *Assume that $\Phi(X, A) \neq \emptyset$. Then there exists an $F \in \mathscr{F}(X)$ satisfying*

(12a) *for each $a \in A$, the function $x_a \in C^1(\mathbf{R}, X)$ defined by*

$$x_a(t) = \begin{cases} t^2 a & \text{if } t \geqslant 0, \\ 0_X & \text{if } t < 0, \end{cases}$$

is a solution of (1) *and*

(12b) *conversely, if $x \in C^1(\mathbf{R}, X)$ is a global solution of* (1) *through $(0, 0_X)$, then $x = x_a$ for some $a \in A$. Thus,*

(12c) $S_1(F, (0, 0_X)) = A$.

Moreover, if (9) *is replaced by*

(9)′ *for each $y_0 \in X$, the ODE* (5) *has exactly one* (*global*) *solution through $(0, y_0) \in \mathbf{R} \times X$*

and if A consists of at least two points, then $F \in \mathscr{F}(X)$ can be chosen to satisfy (4) *as well.*

Proof. Consider the transformation $J\colon \mathbf{R} \times X \to \mathbf{R} \times X$ defined by

$$(t, x) := J(\tau, y) := \big(\exp(-\tau/2),\ y\exp(-\tau)\big), \qquad (\tau, y) \in \mathbf{R} \times X.$$

Clearly J is one-to-one and maps $\mathbf{R} \times X$ onto $(0, \infty) \times X$.

Given an $f \in \Phi(X, A)$, the ODE (5) is transformed into

(1R) $$D_t x = G(t, x) := 2t\big(t^{-2}x - f(t^{-2}x)\big), \qquad t > 0,\ x \in X.$$

In fact,

$$D_t x = (D_\tau t)^{-1}(D_\tau y - y)\exp(-\tau) = -2t^{-1}\big(f(y) - y\big)t^2 = G(t, x).$$

By the construction, if $y \in C^1(\mathbf{R}, X)$ is a solution of (5), then $x(t) = y\big(-2\ln(t)\big)t^2$, $t > 0$, is a solution of (1R) and conversely, if $x \in C^1(I_x, X)$ is a nonextendable solution of (1R), then $I_x = (0, \infty)$ and $y(\tau) = x\big(\exp(-\tau/2)\big)\exp(\tau)$ is a global solution of (5). By (7), $G(t, x) = 0_X$ whenever $\|x\| \geqslant (\alpha+2)t^2$, $t > 0$. Consequently, given a nonextendable solution x of (1R), there is an $\tilde{x} = \tilde{x}(x) \in X$ such that $x(t) \to \tilde{x}$ as $t \to 0^+$. Moreover, if $\tilde{x} \neq 0_X$, then $x(t) = \tilde{x}$ whenever $t \in \big(0, (\|\tilde{x}\|/(\alpha+2))^{1/2}\big)$. If $\tilde{x} = 0_X$, then, in virtue of (6), $\|x(t)\| \leqslant 2(\alpha+2)t^2$ for all $t > 0$.

Using (6) again, we obtain $\|G(t, x)\| \leqslant 4(\alpha+2)t$ for all $(t, x) \in (0, \infty) \times X$. It follows immediately that

$$F(t, x) = \begin{cases} G(t, x) & \text{if } t > 0,\ x \in X, \\ 0_X & \text{if } t \leqslant 0,\ x \in X, \end{cases}$$

defines a continuous extension of G. With F defined above, consider the ODE (1). Observe that (1R) is the restriction of (1) to $(0, \infty) \times X$ and that each nonextendable solution x of (1R) extends uniquely to a global solution of (1) by letting $x(t) = \tilde{x}$ for all $t \leqslant 0$.

The previous considerations imply immediately that $F \in C^0(\mathbf{R} \times X, X)$ and that F satisfies (2), (3) and (12a) (and also (4) provided that (9) is replaced by (9)′ and A consists of at least two points). Since (12c) is now a trivial consequence of (12a) and (12b), it remains to prove (12b).

In fact, if $x \in C^1(\mathbf{R}, X)$ is a global solution of (1) through $(0, 0_X)$, then, in virtue of (6), $\|x(t)\| \leqslant 2(\alpha+2)t^2$ for all $t > 0$. Since J maps $\mathbf{R} \times \{y \in X \mid \|y\| \leqslant k\}$, $k > 0$, onto $\{(t, x) \in \mathbf{R} \times X \mid t > 0,\ \|x\| \leqslant kt^2\}$, it follows that x corresponds to a bounded solution of (5). More precisely, $x(t) = y\big(-2\ln(t)\big)t^2$, $t > 0$, where $y \in C^1(\mathbf{R}, X)$ is a solution of (5) with $\|y(\tau)\| \leqslant 2(\alpha+2)$, $\tau \in \mathbf{R}$. In virtue of (11), there exists an $a \in A$ such that $y(\tau) = a$ for all $\tau \in \mathbf{R}$. Consequently, $x = x_a$ and this concludes the proof of (12b) as well as the proof of Proposition 1.

Remark 2. For $\varepsilon > 0$, replace J by $J_\varepsilon\colon \mathbf{R} \times X \to \mathbf{R} \times X$ defined by

$$(t, x) := J_\varepsilon(\tau, y) := \big(\exp(-\tau/2),\ \varepsilon y\exp(-\tau)\big), \qquad (\tau, y) \in \mathbf{R} \times X.$$

Observe that $J = J_1$. Repeating the proof of Proposition 1, we obtain

$$S_1(F_\varepsilon, (0, 0_X)) = \{\varepsilon a \in X \mid a \in A\}, \qquad \varepsilon > 0,$$

where $F_\varepsilon \in \mathscr{F}(X)$ is defined by

$$F_\varepsilon(t, x) = \begin{cases} 2\varepsilon t\big(\varepsilon^{-1}t^{-2}x - f(\varepsilon^{-1}t^{-2}x)\big) & \text{if } t > 0,\ x \in X, \\ 0_X & \text{if } t \leqslant 0,\ x \in X. \end{cases}$$

Observe that $F_\varepsilon(t, x) \to 0_X$ as $\varepsilon \to 0^+$, uniformly on bounded subsets of $\mathbf{R} \times X$. For $\varepsilon \leqslant 0$, define $F_\varepsilon(t, x) = 0_X$ for all $(t, x) \in \mathbf{R} \times X$. Clearly

$$S_1(F_\varepsilon, (0, 0_X)) = \{0_X\}, \qquad \varepsilon \leqslant 0.$$

The one-parameter family of the ordinary differential equations

$$D_t x = F_\varepsilon(t, x), \qquad \varepsilon \in \mathbf{R},$$

provides an example for a new type of bifurcation phenomena, for the birth of a nontrivial (if A consists of at least two points) solution funnel. This yields a positive answer to a question raised by Pugh [19, Problem 5].

Remark 3a. Let $W = \mathbf{R} \times X \setminus \{(0, 0_X)\}$. As usual $F|W$ denotes the restriction of F to W. Returning to the proof of Proposition 1, observe that $F|W \in C^p(W, X)$ provided that $f \in C^p(X, X)$, $p = 1, 2, \ldots, \infty$, and that $F|W$ is locally Lipschitzian provided that so is f.

4. The second part of the proof of Theorem 1. It is left to prove that with (9) replaced by (9)′, $\Phi(X, A) \neq \emptyset$ provided that the pair (X, A) satisfies the conditions of Theorem 1.

PROPOSITION 2. *Assume that the pair (X, A) satisfies the conditions of Theorem* 1. *Then with* (9) *replaced by* (9)′, $\Phi(X, A) \neq \emptyset$.

Proof. Consider a C^1 diffeomorphism h: $X \setminus A \to X \setminus \{0_X\}$, $x \to w = h(x)$, such that $h(x) = x$ whenever $\|x\| \geqslant \alpha+1$. The ODE

$$D_t w = w, \qquad w \neq 0_X,$$

is transformed into

(13) $$D_t x = e(x) := [D_w h^{-1}(h(x))]\, h(x), \qquad x \in X \setminus A.$$

Note that $e \in C^0(X \setminus A, X)$ and $e(x) = x$ whenever $\|x\| \geqslant \alpha+1$. Formula (13) defines an ODE on $X \setminus A$. For each $z_0 \in X \setminus A$, define a function $x(\cdot, z_0)$: $\mathbf{R} \to X$ by $x(t, z_0) = h^{-1}(h(z_0)\exp(t))$. Obviously $x(\cdot, z_0)$ is the unique (nonextendable) solution of (13) through $(0, z_0) \in \mathbf{R} \times (X \setminus A)$. It follows that (13) has no bounded global solutions.

Let M: $(\alpha+1, \infty) \to [1, \alpha+2]$ be a real function with $M(k) = k$ whenever $k \in (\alpha+1, \alpha+3/2)$ and $M(k) = 1$ whenever $k \geqslant \alpha+2$. For $y \in X \setminus A$, define

$$\lambda(y) = \begin{cases} \|e(y)\| & \text{if } \|y\| \leqslant \alpha+1, \\ M(\|y\|) & \text{if } \|y\| > \alpha+1. \end{cases}$$

Note that $\lambda \in C^0(X \setminus A, \mathbf{R})$ and $\lambda(y) > 0$ for all $y \in X \setminus A$.

Let κ: $X \to [0, 1]$ be an Urysohn function for the pair $(A, \{x \in X \mid \|x\| \geqslant \alpha+2\})$, i.e. $\kappa \in C^0(X, \mathbf{R})$, $\kappa(x) \in [0, 1]$ for all $x \in X$, $\kappa^{-1}(0) = A$ and $\kappa^{-1}(1) = \{x \in X \mid \|x\| \geqslant \alpha+2\}$. Since $\lambda(y) = 1$ whenever $\|y\| \geqslant \alpha+2$, there is no loss of generality in assuming that

(14) $$\kappa(y) \leqslant \lambda(y) \quad \text{for all } y \in X \setminus A.$$

Finally, for $y \in X$, define

(15) $$f(y) = \begin{cases} \kappa(y)(\lambda(y))^{-1} e(y) & \text{if } y \in X \setminus A, \\ 0_X & \text{if } y \in A. \end{cases}$$

We claim that $f \in \Phi(X, A)$.

Since $A \subset \{y \in X \mid \|y\| \leqslant \alpha\}$ and $\|(\lambda(y))^{-1} e(y)\| = 1$ whenever $\|y\| \leqslant \alpha+1$, it follows immediately that $f \in C^0(X, X)$ and that f satisfies (8). Recall that $e(y) = y$ whenever $\|y\| \geqslant \alpha+1$. Consequently, by the properties of M and κ, we see that $f(y) = y$ whenever $\|y\| \geqslant \alpha+2$. In particular, f satisfies condition (7) and, in order to prove (6), we may assume that $\|y\| < \alpha+2$. If $\|y\| \leqslant \alpha+1$, then $\|f(y)\| \leqslant 1 < \alpha+2$. Similarly, if $\alpha+1 < \|y\| < \alpha+2$, then $\|f(y)\| \leqslant \|y\| < \alpha+2$. Hence $\|f(y) - y\| \leqslant \|f(y)\| + \|y\| < 2(\alpha+2)$ whenever $\|y\| < \alpha+2$.

It is left to prove that f satisfies (9)–(11).

For $y \in X \setminus A$, define $\mu(y) = \lambda(y)(\kappa(y))^{-1}$ and consider the ODE

(16) $$D_\tau y = (\mu(y))^{-1} e(y), \qquad y \in X \setminus A.$$

Observe that (16) is the restriction of (5) (with f defined by (15)) to $X \setminus A$. The

solutions of (16) are related to those of (13) by (an abstract formulation [6] of) Vinograd's reparametrization principle. For $z_0 \in X \setminus A$, define

$$\tau^-(z_0) = \int_0^{-\infty} \mu(x(s, z_0))\, ds, \qquad \tau^+(z_0) = \int_0^{\infty} \mu(x(s, z_0))\, ds,$$

where $x(\cdot, z_0)$ is the unique (nonextendable) solution of (13) through $(0, z_0)$. For each $z_0 \in X \setminus A$, it is well known [6] that (16) has a unique (nonextendable) solution $y(\cdot, z_0)$ through $(0, z_0)$ which is defined for $\tau \in (\tau^-(z_0), \tau^+(z_0))$. Further [6], for all $t \in \mathbf{R}$,

$$h^{-1}(h(z_0)\exp(t)) = x(t, z_0) = y(\tau, z_0) \quad \text{where} \quad \tau = \int_0^t \mu(x(s, z_0))\, ds.$$

It is easy to see that $\tau^-(z_0) = -\infty$, $\tau^+(z_0) = \infty$ for all $z_0 \in X \setminus A$. In fact, there is no loss of generality in assuming that $\|z_0\| = \alpha+2$. Then $\|x(s, z_0)\| \leqslant \alpha+2$ whenever $s \leqslant 0$. Consequently, in virtue of (14),

$$\tau^-(z_0) = \int_0^{-\infty} \lambda(x(s, z_0))(\kappa(x(s, z_0)))^{-1}\, ds \leqslant \int_0^{-\infty} 1\, ds = -\infty.$$

Similarly, by the properties of λ and κ,

$$\tau^+(z_0) = \int_0^{\infty} \lambda(z_0 \exp(s))(\kappa(z_0 \exp(s)))^{-1}\, ds = \int_0^{\infty} 1 \cdot 1\, ds = \infty.$$

For each $z_0 \in X \setminus A$, we arrived at the conclusion that (5) (with f defined by (15)) has a unique (nonextendable) solution $y(\cdot, z_0) \in C^1(\mathbf{R}, X)$. Further, $y(\tau, z_0) \in X \setminus A$ for all $\tau \in \mathbf{R}$ and $\|y(\tau, z_0)\| \to \infty$ as $\tau \to \infty$.

For $\tau \in \mathbf{R}$, $z_0 \in A$, define $y(\tau, z_0) = z_0$. Obviously $y(\cdot, z_0) \in C^1(\mathbf{R}, X)$ is a solution of (5) (with f defined by (15)) through $(0, z_0) \in \mathbf{R} \times A$ and if $y \in C^1(\mathbf{R}, X)$ is a solution with $y(\tau) \in A$ for all $\tau \in \mathbf{R}$, then $y = y(\cdot, z_0)$ for some $z_0 \in A$.

Summarizing the properties of the solutions of (5) (with f defined by (15)) established above, it follows immediately that f satisfies (9)′ (a stronger form of (9)), (10) and (11) and this concludes the proof of Proposition 2.

Remark 3b. The smoothness properties of $f \in \Phi(X, A)$ (defined by (15)) depend crucially on those of κ constructed in the proof of Proposition 2. For simplicity, let X be the separable infinite-dimensional Hilbert space and let A be a nonempty bounded closed subset of X. Let $\alpha = \sup\{\|a\| \mid a \in A\}$. Assume that there exists a C^∞ diffeomorphism h mapping $X \setminus A$ onto $X \setminus \{0_X\}$ such that $h(x) = x$ whenever $\|x\| \geqslant \alpha+1$. For $p = 1, 2, \ldots$, by choosing κ to be an appropriate Whitney–Urysohn function (this can be done by applying the methods used e.g. in [9, pp. 118–119]), it is not hard to prove that f (defined by (15)) is in $C^p(X, X) \cap \Phi(X, A)$. The core of the construction of such a function κ is to ensure that f be p times continuously differentiable at all $a \in A$.

Remark 4. Consider an $f \in C^1(X, X) \cap \Phi(X, A)$ where X is the separable infinite-dimensional Hilbert space and $A = \{a_1, a_2, \ldots\}$ is a discrete subset in $\{x \in X \mid \|x\| = 1\}$. Modifying F (obtained from f in proving Proposition 1) on $[1, \infty) \times X$, it is easy to construct an $\tilde{F} \in \mathcal{F}(X)$ such that $S_1(\tilde{F}, (0, 0_X)) = A$ but $S_2(\tilde{F}, (0, 0_X)) = \{na_n \mid a_n \in A,\ n = 1, 2, \ldots\}$. In other words, cross-sections of solution funnels need not be bounded.

5. The proof of Theorem 2. Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space. Choose an $x_0 \in X$ so that $\|x_0\| = 1$. Consider a continuous linear functional $L\colon X \to \mathbf{R}$ for which $L(x_0) = 1$. Set $Z = \{y \in X \mid L(y) = 0\}$. Since $y - L(y)x_0 \in Z$ and $y = (y - L(y)x_0) + L(y)x_0$ for all $y \in X$, the space X can be represented as $X = Z \times \mathbf{R}$ and $y \in X$ can be written as $y = (z, \lambda) \in Z \times \mathbf{R}$ where $z = y - L(y)x_0$ and $\lambda = L(y)$. In particular, $0_X = (0_Z, 0)$, $x_0 = (0_Z, 1)$, $-x_0 = (0_Z, -1)$. By passing to an equivalent norm, there is no loss of generality in assuming that $\|y\| = \max\{\|z\|, |\lambda|\}$ for all $y = (z, \lambda) \in Z \times \mathbf{R}$.

In virtue of Proposition 1, it is enough to prove that — with (9) replaced by (9)′ — $\Phi(X, \{x_0, -x_0\}) \neq \emptyset$.

Observe that $Z \subset X$ is an infinite-dimensional Banach space. By [11, Theorem 1], there exists a $g \in C^0(Z, Z)$ satisfying

(17) $\|g(z) - z\| \leqslant 4$ for all $z \in Z$,

(18) $g(z) = z$ whenever $\|z\| \geqslant 2$,

(19) for each $z_0 \in Z$, the ODE $D_s z = g(z)$ has exactly one solution through $(0, z_0) \in \mathbf{R} \times Z$,

(20) all solutions of the ODE $D_s z = g(z)$ extend to global solutions,

(21) the ODE $D_s z = g(z)$ has no bounded global solutions.

The similarity between properties (17)–(21) and (6)–(11) is conspicuous. Using the terminology adopted in Section 3, the existence of a $g \in C^0(Z, Z)$ with properties (17)–(21) means that $\Phi(Z, \emptyset) \neq \emptyset$. In proving $\Phi(Z, \emptyset) \neq \emptyset$, i.e. in proving [11, Theorem 1], we have distinguished two cases. For infinite-dimensional reflexive Banach spaces, the proof is parallel to the one of Proposition 2 with an important result of Dobrowolski [9] as its starting point: Given an infinite-dimensional reflexive Banach space Z, there is a C^1 diffeomorphism h mapping Z onto $Z \setminus \{0_Z\}$ such that $h(z) = z$ whenever $\|z\| \geqslant 1$ (see Example 4 of the present paper; cf. also Remark 1). For the nonreflexive case, the proof is completely different. It is a direct construction based on the existence of a nested system of nonempty convex closed subsets $\{Q_c \mid 0 < c \leqslant 1\}$ of $Q_1 = \{z \in Z \mid \|z\| \leqslant 2\}$ satisfying $\bigcap \{Q_c \mid 0 < c \leqslant 1\} = \emptyset$ and $Q_c \subset Q_d$, $0 < c \leqslant d \leqslant 1$. The function $g\colon Z \to Z$ constructed in [11] is locally Lipschitzian and satisfies $\|g(z)\| \leqslant 2 + \|z\|$, $z \in Z$. Further, g has properties (17), (18) and, last but not least, in a technical sense, it is transversal to the boundary of Q_c, $0 < c \leqslant 1$. This latter property "kills" the bounded solutions and ensures the "repulsivity" of the empty set $\emptyset = \bigcap \{Q_c \mid 0 < c \leqslant 1\}$.

Let Γ be a closed subset of the interval $[-1, 1] \subset \mathbf{R}$. Assume that $\{1, -1\} \subset \Gamma$. In what follows we construct an $f \in \Phi(X, \{(0_Z, \gamma) \in Z \times \mathbf{R} = X \mid \gamma \in \Gamma\})$ satisfying (9)′. The desired result $\Phi(X, \{x_0, -x_0\}) \neq \emptyset$ corresponds to the special case $\Gamma = \{1, -1\}$.

Define

$$X_1 = \{(z, \lambda) \in Z \times \mathbf{R} \mid |\lambda| \leqslant 1,\ \|z\| \leqslant 3 - |\lambda|\},$$
$$X_2 = \{(z, \lambda) \in Z \times \mathbf{R} \mid 1 \leqslant \lambda \leqslant 3,\ 2\|z\| \leqslant \lambda + 3\},$$
$$X_3 = \{(z, \lambda) \in Z \times \mathbf{R} \mid 2 \leqslant \|z\| \leqslant 3,\ 3 - \|z\| \leqslant \lambda \leqslant 2\|z\| - 3\},$$
$$X_4 = \{(z, \lambda) \in Z \times \mathbf{R} \mid 2 \leqslant \|z\| \leqslant 3,\ 3 - 2\|z\| \leqslant \lambda \leqslant 3 + \|z\|\},$$
$$X_5 = \{(z, \lambda) \in Z \times \mathbf{R} \mid -3 \leqslant \lambda \leqslant -1,\ 2\|z\| \leqslant 3 - \lambda\},$$
$$X_6 = \{(z, \lambda) \in Z \times \mathbf{R} \mid \max\{\|z\|, |\lambda|\} \geqslant 3\}.$$

Observe that $\bigcup \{X_i \mid i = 1, \ldots, 6\} = Z \times \mathbf{R} = X$ and $\operatorname{int}(X_i) \cap \operatorname{int}(X_j) = \emptyset$ for $i \neq j$. Here of course, $\operatorname{int}(X_i)$ denotes the interior of X_i in X.

For $(z, \lambda) \in Z \times \mathbf{R} = X$, let

$$p(z, \lambda) = \begin{cases} 0 & \text{if } (z, \lambda) \in X_1, \\ (3\lambda - 3)/2 & \text{if } (z, \lambda) \in X_2, \\ \|z\| + \lambda - 3 & \text{if } (z, \lambda) \in X_3, \\ 3 - \|z\| + \lambda & \text{if } (z, \lambda) \in X_4, \\ (3\lambda + 3)/2 & \text{if } (z, \lambda) \in X_5, \\ \lambda & \text{if } (z, \lambda) \in X_6. \end{cases}$$

It is easy to see that $p(z, \lambda)$ is well defined, $|p(z, \lambda) - \lambda| \leqslant 1$ for all $(z, \lambda) \in Z \times \mathbf{R}$, and $p\colon X \to \mathbf{R}$ is a continuous function.

For $y = (z, \lambda) \in Z \times \mathbf{R} = X$, let

$$f(y) = \begin{cases} (z, p(z, \lambda)) & \text{if } z \in Z,\ \lambda \in \tilde{\Gamma}, \\ (r(\lambda)\, g(z/r(\lambda)),\ p(z, \lambda)) & \text{if } z \in Z,\ \lambda \notin \tilde{\Gamma}, \end{cases}$$

where

$$\tilde{\Gamma} = \Gamma \cup \{\gamma \in \mathbf{R} \mid |\gamma| \geqslant 1\}, \qquad r(\lambda) = \inf\{|\lambda - \gamma| \mid \gamma \in \tilde{\Gamma}\}, \quad \lambda \in \mathbf{R}.$$

Observe that $0 \leqslant r(\lambda) \leqslant 1$ for all $\lambda \in \mathbf{R}$. In virtue of (17),

$$\|r(\lambda)\, g(z/r(\lambda)) - z\| = r(\lambda)\, \|g(z/r(\lambda)) - z/r(\lambda)\| \leqslant 4r(\lambda)$$

whenever $z \in Z$ and $r(\lambda) \neq 0$. Since $r\colon \mathbf{R} \to \mathbf{R}$ is Lipschitzian and $r(\lambda) = 0$ if and only if $\lambda \in \tilde{\Gamma}$, it follows immediately that $f\colon X \to X$ is continuous and

$$\|f(y)-y\| \leqslant \max\{4\tilde{r}(\lambda), |p(z,\lambda)-\lambda|\} \leqslant \max\{4, 1\} < 6$$

for all $y = (z, \lambda) \in Z \times \mathbf{R} = X$. Similarly, in virtue of (18), $f(y) = (z, p(z, \lambda))$ whenever $\|z\| \geqslant 2$. On the other hand, since $\tilde{\Gamma} \supset \{\gamma \in \mathbf{R} \mid |\gamma| \geqslant 1\}$, we have $f(y) = (z, p(z, \lambda))$ whenever $|\lambda| \geqslant 1$. In particular,

$$f(y) = y \quad \text{whenever} \quad \|y\| = \max\{\|z\|, |\lambda|\} \geqslant 3.$$

Further, since $g(z) \neq 0_Z$ for all $z \in Z$,

$$f(y) = 0_X \quad \text{if and only if} \quad y \in \{(0_Z, \gamma) \in Z \times \mathbf{R} = X \mid \gamma \in \Gamma\}.$$

Thus, with $A = \{(0_Z, \gamma) \in Z \times \mathbf{R} = X \mid \gamma \in \Gamma\}$ and $\alpha = 1$, it remains to prove that f satisfies (9)′, (10) and (11).

Consider the ODE

$$(D_\tau z, D_\tau \lambda) = D_\tau y = f(y), \quad y \in X. \tag{22}$$

Parallel to (22), consider the ODE

$$D_\tau y = \begin{cases} (z, 0) & \text{if } y = (z, \lambda) \in X_1,\ \lambda \in \Gamma, \\ \big(r(\lambda) g(z/r(\lambda)), 0\big) & \text{if } y = (z, \lambda) \in X_1,\ \lambda \notin \Gamma. \end{cases} \tag{22R}$$

It is obvious that (22R) is the restriction of (22) to X_1. Using (19)–(21), it is easy to see that given a $y_0 = (z_0, \lambda_0) \in X_1 \setminus A$, there is a unique nonextendable solution $y_{\mathbf{R}}(\cdot, y_0) = (z_{\mathbf{R}}(\cdot, y_0), \lambda_0)$ of (22R) through $(0, y_0)$ and this solution is defined on some interval $(-\infty, \tau(y_0)] \subset \mathbf{R}$ where $y_{\mathbf{R}}(\tau(y_0), y_0) \in X_1 \cap (X_3 \cup X_4)$.

Since

$$f(y) = (z, p(z, \lambda)) \quad \text{whenever} \quad y = (z, \lambda) \in X \setminus X_1, \quad \text{and} \tag{23}$$

$$\begin{cases} p(z, \lambda) > 0 \text{ if } (\lambda, z) \in (X_2 \cup X_3) \setminus X_1, \\ p(z, \lambda) < 0 \text{ if } (\lambda, z) \in (X_4 \cup X_5) \setminus X_1, \\ p(z, \lambda) = \lambda \text{ if } (\lambda, z) \in X_6, \\ p\colon X \to \mathbf{R} \text{ is Lipschitzian (with global Lipschitz constant 2),} \end{cases} \tag{24}$$

$y_{\mathbf{R}}(\cdot, y_0)$ extends uniquely to a global solution $y(\cdot, y_0) = (z(\cdot, y_0), \lambda(\cdot, y_0))$ of (22) through $(0, y_0) \in \mathbf{R} \times (X_1 \setminus A)$. It is clear that

$$\|z(\tau, y_0)\| \to \infty \quad \text{as } \tau \to \infty, \quad \text{and} \tag{25}$$

$$\begin{cases} \text{if } \lambda_0 > 0, \text{ then } \lambda(\tau, \lambda_0) \to \infty \text{ as } \tau \to \infty, \\ \text{if } \lambda_0 = 0, \text{ then } \lambda(\tau, \lambda_0) = 0 \text{ for all } \tau \in \mathbf{R}, \\ \text{if } \lambda_0 < 0, \text{ then } \lambda(\tau, \lambda_0) \to -\infty \text{ as } \tau \to \infty. \end{cases} \tag{26}$$

Consider now an arbitrary $y_0 = (z_0, \lambda_0) \in X \setminus X_1$. By the previous considerations, X_1 is negatively invariant (i.e. solutions starting in X_1 remain in X_1 for all negative time). Hence, as an easy corollary of (23) and (24), there exists a unique nonextendable solution $y(\cdot, y_0) = (z(\cdot, y_0), \lambda(\cdot, y_0))$ to (22) through $(0, y_0) \in \mathbf{R} \times (X \setminus X_1)$ and this solution is defined for all real τ and satisfies (25) and (26). (It is worth mentioning that $y(\cdot, y_0)$ can be computed explicitly provided that it remains in $X \setminus X_1$. In fact, $z(\tau, y_0) = z_0 \exp(\tau)$ and the differential equation for $\lambda(\cdot, y_0)$ is piecewise linear.)

Summarizing the properties of (22) established above, we see that f satisfies (9)′, (10) and (11) as well and this concludes the proof of Theorem 2.

Remark 5. The method we used for proving Theorem 2 can be applied to general product spaces. Let $(Z, \|\cdot\|_1)$ and $(\Lambda, \|\cdot\|_2)$ be Banach spaces. Assume that Z is infinite-dimensional. The product Banach space $(X, \|\cdot\|)$ is defined by letting $\|y\| = \max\{\|z\|_1, \|\lambda\|_2\}$ for all $y = (z, \lambda) \in Z \times \Lambda = X$. A simple modification of the proof of Theorem 2 shows that given an arbitrary closed subset Γ of $\{\lambda \in \Lambda \mid \|\lambda\|_2 \leqslant 1\}$ satisfying $\{\lambda \in \Lambda \mid \|\lambda\|_2 = 1\} \subset \Gamma$, we have

$$\Phi(X, \{(0_Z, \gamma) \in Z \times \Lambda = X \mid \gamma \in \Gamma\}) \neq \emptyset.$$

The condition $\{\lambda \in \Lambda \mid \|\lambda\|_2 = 1\} \subset \Gamma$ seems to be only technical. Nevertheless, we have no idea how to get rid of it: without it, there are too many technical difficulties (the main ones being connected with replacing (23) and (24) by their appropriate versions) and the method we used in proving Theorem 2 breaks down.

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On the uniqueness of equilibrium states for piecewise monotone mappings

by

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Abstract. Our main result is: Given a piecewise monotone interval map T and a continuous function φ with $P(T, \varphi) > \sup\varphi$ satisfying an additional regularity condition, there is at most one φ-equilibrium state for T on each topologically transitive component L_n of T, and only the finitely many L_n with $h(T|_{L_n}) \geqslant P(T, \varphi) - \sup\varphi$ can support such an equilibrium state. The additional regularity assumption is: φ is of bounded variation or φ has bounded distortion under T.

1. Introduction. For a continuous transformation T of a compact metric space X and a continuous function $\varphi\colon X \to \mathbf{R}$, the *pressure* is defined as

$$P(T, \varphi) = \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}}\, n^{-1} \log \sup_E \sum_{x\in E} \exp\left(\varphi(x) + \ldots + \varphi(T^{n-1}x)\right)$$

where the supremum extends over all (n, ε)-separated subsets of X (recall that E is (n, ε)-*separated* if for all $x, y \in E$ with $x \neq y$ we have $d(T^i x, T^i y) > \varepsilon$ for some $i \in \{0, \ldots, n-1\}$). Walters [W1] proved the variational principle

$$P(T, \varphi) = \sup\{h_\mu(T) + \int \varphi\, d\mu\}$$

where the supremum extends over all ergodic T-invariant measures μ. If the supremum is attained for some μ, then μ is called an *equilibrium state* for φ. For some classes of transformations such as expansive maps or piecewise monotone interval maps [MS] it is known that equilibrium states exist for all continuous φ.

The uniqueness problem is more difficult. Bowen proved uniqueness for irreducible subshifts of finite type and Hölder-continuous φ [B1] and also for general expansive systems with specification property if φ satisfies a condition similar to our (2.3) below [B2]. Walters [W2] proved uniqueness for β-transformations and Lipschitz-continuous φ, and Hofbauer [H1, H2] showed that for general piecewise monotone interval maps of positive entropy and $\varphi \equiv 0$ there is a unique equilibrium state (i.e. a measure of maximal

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