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# Almost everywhere summability of eigenfunction expansions associated to elliptic operators

by

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**Abstract.** Let $A$ be a positive-definite operator, with spectral measure $E$, on $L^2(X)$, where $X$ is a metric space with measure. We give sufficient conditions on $A$, $X$, and a function $K$ on $\mathbf{R}_+$, which imply

$$\lim_{t\to 0} \int K(t\lambda)\, dE(\lambda) f = f \quad \text{a.e.}$$

for $f \in L^1(X)$. As a special case we obtain the a.e. convergence of Riesz means of index greater than $q/2$ for $A$ being either a positive-definite elliptic differential operator acting on sections of a vector bundle on a compact $q$-dimensional manifold or a Schrödinger operator on $\mathbf{R}^q$ with nonnegative potential in $L^p_{\text{loc}}$, $p > q/2$.

**1. Introduction.** Almost everywhere summability of Riesz means of eigenfunction expansions of $L^1$ functions associated to various elliptic or subelliptic operators have been treated by a number of authors. Let us mention the classical result of L. Hörmander [3], [4] who has shown that for an elliptic differential operator on functions on a compact manifold of dimension $q$ the Riesz means of index greater than $q-1$ of an $L^1$ function are a.e. convergent.

Recently C. Sogge [8] improved greatly Hörmander's result as far as norm convergence goes, to show that for the same operators the Riesz means up to the critical index $(q-1)/2$ of $L^1$ functions are norm convergent. Moreover, it seems that Sogge's method also gives a.e. results.

On the other hand, a.e. summability results for eigenfunction expansions of $L^1$ functions associated to Schrödinger operators with polynomial potentials have been obtained by A. Hulanicki and Joe Jenkins [7].

The aim of this paper is to present an abstract theorem, very much in the setting of E. M. Stein's *Topics in Harmonic Analysis*... [9], which yields the a.e. summability of the Riesz means of index greater than $q/2$ of $L^1$ functions associated both to an elliptic differential operator on vector bundles on a compact manifold of dimension $q$ and to Schrödinger operators on $\mathbf{R}^q$ with nonnegative potentials locally in $L^p$, $p > q/2$.

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The rapid decay of the kernels of semigroups generated by the operator in question which is basic for the functional calculus we apply, as well as some smoothness of these kernels essential for Zo's lemma are proved separately for the Schrödinger semigroups and the semigroups generated by elliptic operators on vector bundles on compact manifolds in Sections 7 and 8 (cf. (7.2)–(7.4), (8.11)–(8.13)).

Several proofs in the paper are modeled on arguments due to other people. For instance, for the idea to apply Zo's lemma rather than to compare with the Hardy–Littlewood maximal function we are indebted to E. M. Stein [10]. We rely heavily on a functional calculus whose idea goes back to J. Dixmier [1] and which, in the form adapted for convolution semigroups, is due to A. Hulanicki [6]. The use of submultiplicative functions together with hypoellipticity, which is essential for our proof of (8.11), was borrowed from J. Dziubański and A. Hulanicki [2].

**2. An abstract theorem.** Let $M$ be a metric space with metric $d$ and a Borel measure $\mu$. Let $B(x, r) = \{y \in M\colon d(x, y) < r\}$. We assume that there are constants $C_1$, $C_2$, $q$ such that

$$\mu(B(x, 2r)) \leqslant C_1 \mu(B(x, r)), \qquad \mu(B(x, r)) \leqslant C_2 r^q.$$

Let $A$ be a nonnegative selfadjoint densely defined operator on $L^2(M, \mathbf{C}^n, \mu)$. By the spectral theorem, we write

$$Af = \int \lambda\, dE(\lambda) f, \qquad e^{-tA} f = \int e^{-\lambda t}\, dE(\lambda) f.$$

We assume that

$$e^{-tA} f(x) = \int e^{-tA}(x, y) f(y)\, d\mu(y)$$

where the kernels $e^{-tA}(x, y)$ (with values in $M_n(\mathbf{C})$, the space of $n\times n$ matrices equipped e.g. with the Hilbert–Schmidt norm $|\cdot|$) satisfy the following estimates: there exist positive numbers $m$, $\alpha$ and $C$ such that

(2.1) $$\sup_x \int |e^{-tA}(x, y)| e^{sd(x,y)}\, d\mu(y) \leqslant C \quad \text{for all } s, t \text{ with } s^m t = 1,\ t < 1,$$

(2.2) $$\sup_x \int |e^{-tA}(x, y)|^2\, d\mu(y) \leqslant Ct^{-q/m} \quad \text{for all } t < 1,$$

(2.3) $$|e^{-tA}(x, y) - e^{-tA}(x', y)| \leqslant Ct^{-(q+\alpha)/m} d(x, x')^\alpha$$

for all $t < 1$ and all $x, x', y \in M$, or for every ball $B$ there exists a constant $C_B$ such that

(2.3′) $$|e^{-tA}(x, y) - e^{-tA}(x', y)| \leqslant C_B t^{-(q+\alpha)/m} d(x, x')^\alpha$$

for all $t < 1$ and all $x, x', y \in B$.

Let $l$ be a nonnegative real number. We denote by $\mathscr{L}^l_p(\mathbf{R})$ the space of Bessel potentials of order $l$, i.e. $f \in \mathscr{L}^l_p(\mathbf{R})$ iff $(I-\Delta)^{l/2} f \in L^p(\mathbf{R})$.



(2.4) THEOREM. *Suppose* (2.1)–(2.3) *are satisfied. If* $F \in \mathscr{L}^l_{p,\mathrm{loc}}(\mathbf{R})$, $l > q/2+1$, *and for a nonzero* $\varphi \in C_c^\infty(\mathbf{R}_+)$ *and* $\varepsilon > 0$

$$\|F_t \varphi\|_{\mathscr{L}^l_p} = O\big((\log t)^{-1-\varepsilon}\big) \quad \text{as } t \to \infty$$

*where* $F_t(\lambda) = F(t\lambda)$, *then* $F_t(A) = \int F(t\lambda)\, dE(\lambda)$ *is bounded on* $L^p(M)$, $1 \leqslant p \leqslant \infty$, *uniformly in* $t < 1$, *and the maximal function*

$$F^* f(x) = \sup_{0<t<1} |F_t(A) f(x)|$$

*is of weak type* (1, 1).

Remark. Sup of a family of $\mu$-measurable functions on $M$ always means sup in the lattice of $\mu$-measurable functions on $M$.

(2.5) THEOREM. *Suppose* (2.1), (2.2) *and* (2.3′) *are satisfied. Then for a function* $F$ *which satisfies the conditions above and* $F(0) = 1$ *we have*

$$\lim_{t\to 0} F_t(A) f = f$$

*almost everywhere and in norm for* $f \in L^p(M)$, $1 \leqslant p < \infty$.

Remark. Let $f \in L^p$, $1 \leqslant p < \infty$. For a fixed $t$ we select a representative from the class of a.e. equal functions $F_t f$ in $L^p$ such that $\Phi\colon (t, x) \to F_t(A) f(x)$ is measurable. One can prove that (2.1), (2.2) and (2.3′) imply that this can be done in such a way that for almost all $x$, $\Phi$ is continuous with respect to $t$.

Remark. Let $(1-\lambda)_+^\alpha = R^\alpha(\lambda)$ be the Riesz kernel. One can check that for $\alpha > l-1$, $R^\alpha \in \mathscr{L}^l_{p,\mathrm{loc}}$ for some $p > 1$. Thus for $\alpha > l-1 > q/2$ the Riesz means of $L^1$ functions are a.e. convergent to $f$.

**3. Banach algebras**

DEFINITION. A continuous function $\varphi\colon M \times M \to \mathbf{R}$ is called *submultiplicative* if for all $x$, $y$, $z$

$$\varphi(x, y) \geqslant 1, \qquad \varphi(x, y)\varphi(y, z) \geqslant \varphi(x, z).$$

Of course $\omega_a = (1+d)^a$, $e^{bd}$, $\omega_a e^{bd}$ are submultiplicative.

Our functional calculus is based on Banach *-algebras whose elements are kernels $K$, $K(x, y)$ being an operator on $\mathbf{C}^n$ and $|K(x, y)|$ its Hilbert–Schmidt norm. For a submultiplicative function $\varphi$ we write

$$\|K\|_{B(\varphi)} = \max\Big\{\sup_x \int |K(x, y)|\varphi(x, y)\, dy,\ \sup_y \int |K(x, y)|\varphi(y, x)\, dx\Big\}$$

and we define a Banach *-algebra with unit element by

$$B(\varphi) = \{K\colon \|K\|_{B(\varphi)} < \infty\} + \mathbf{C}I.$$

The multiplication is defined by

$$K_1 \cdot K_2(x, y) = \int K_1(x, s) K_2(s, y)\, d\mu(s),$$

and $K^*(x, y) = \bar{K}(y, x)$.

For a given kernel $K$ we are going to estimate the norm of $Ke^{inK}$ in $B(\omega_a)$. We will use the abbreviation $\|K\|_a = \|K\|_{B(\omega_a)}$.

(3.1) THEOREM. *Assume that*

$$K = K^*, \quad \|K\|_{B(e^{bd})} \leqslant C_0, \quad \sup_x \int |K(x, y)|^2 \, d\mu(y) \leqslant C_0.$$

*Then for every $a \geqslant 0$ there is a constant $C$, $C = C(a, b, C_0)$, such that*

$$\|e^{inK} K\|_a \leqslant C(1+|n|)^{q/2+a}.$$

Proof. For $c < b$ and fixed $a$, taking $\varphi = e^{cd}\omega_a$ we have

$$\|K\|_{B(\varphi)} \leqslant C\|K\|_{B(e^{bd})} \leqslant CC_0$$

with some constant $C$. Let $l = c^{-1}|n|\,\|K\|_{B(\varphi)}$. Then, since $e^{inK}$ is a unitary operator,

$$\begin{aligned}
\int |e^{inK} K|(x, y)\omega_a(x, y)\, d\mu(x) &= \int_{d(x,y) \leqslant l} + \int_{d(x,y) > l} \\
&\leqslant (1+l)^a \mu(B(y, l))^{1/2} \|e^{inK} K(\cdot, y)\|_{L^2(B(y,l))} + e^{-cl} \|\omega_a e^{cd} e^{inK} K(\cdot, y)\|_{L^1} \\
&\leqslant (1+l)^a \mu(B(y, l))^{1/2} \|e^{inK} K(\cdot, y)\|_{L^2} + e^{-cl} \|e^{inK} K\|_{B(\varphi)} \\
&\leqslant (1+l)^{a+q/2} C_2^{1/2} \|K(\cdot, y)\|_{L^2} + e^{-cl} \|K\|_{B(\varphi)} e^{|n|\,\|K\|_{B(\varphi)}} \\
&\leqslant (1+|n|)^{a+q/2} C.
\end{aligned}$$

**4. Zo's lemma.** The following is an easy generalization of a result from [11].

LEMMA (Zo [11]). *Let $M$ be as above and let $\{K_\delta\}_{\delta \in I}$ be a family of kernels which satisfies:*

$$\sup_{x,\delta} \int |K_\delta(x, y)| \, d\mu(y) < \infty,$$

$$\sup_{z,y} \int_{d(x,y) > 2d(y,z)} \sup_\delta |K_\delta(x, y) - K_\delta(x, z)| \, d\mu(x) < \infty.$$

*Then the operator $K^* f(x) = \sup_\delta |K_\delta f(x)|$ is of weak type* (1, 1).

It seems convenient to define a norm on the space of families of kernels which satisfy the conditions of Zo's lemma.

DEFINITION.

$$\|\{K_\delta\}_{\delta \in I}\|_{\mathrm{Zo}} = \max\Big\{\sup_{x,\delta} \int |K_\delta(x, y)| \, d\mu(y), \ \sup_{z,y} \int_{d(x,y) > 2d(y,z)} \sup_\delta |K_\delta(x, y) - K_\delta(x, z)| \, d\mu(x)\Big\}.$$

For a fixed index set $I$, $\|\cdot\|_{\mathrm{Zo}}$ is a norm on the space of families of kernels indexed by $I$ for which $\|\cdot\|_{\mathrm{Zo}}$ is finite, and this is a Banach space. It is easy to see that, if $a_\delta$, $\delta \in I$, are complex numbers, then

$$\|\{a_\delta K_\delta\}_{\delta \in I}\|_{\mathrm{Zo}} \leqslant \sup_\delta |a_\delta| \cdot \|\{K_\delta\}_{\delta \in I}\|_{\mathrm{Zo}},$$

and if $f\colon I \to I'$, then

$$\|\{K_{f(\delta)}\}_{\delta \in I}\|_{\mathrm{Zo}} \leqslant \|\{K_\delta\}_{\delta \in I'}\|_{\mathrm{Zo}}.$$

(4.1) LEMMA. *If for some constants $R > 1$, $M$, $a > 0$, $\alpha > 0$, a family of kernels $\{K_n\}_{n=0}^\infty$ satisfies*

$$\|K_n\|_{(1+R^n d)^a} \leqslant M, \quad \int |K_n(x, y) - K_n(x, z)| \, d\mu(x) \leqslant MR^{n\alpha} d(y, z)^\alpha,$$

*then for some $C$ depending only on $R$, $a$, $\alpha$*

$$\|\{K_n\}_{n=0}^\infty\|_{\mathrm{Zo}} \leqslant MC.$$

Proof. Let $B = \{x\colon d(x, y) > 2d(y, z)\}$. Now

$$\int_B |K_n(x, y) - K_n(x, z)| \, d\mu(x) \leqslant \int_B |K_n(x, y)| \, d\mu(x) + \int_B |K_n(x, z)| \, d\mu(x) \leqslant 2MR^{-na} d(y, z)^{-a}.$$

Then

$$\begin{aligned}
\int_B \sup_n |K_n(x, y) - K_n(x, z)| \, d\mu(x) &\leqslant \sum_n \int_B |K_n(x, y) - K_n(x, z)| \, d\mu(x) \\
&\leqslant \sum_n \min\big(MR^{n\alpha} d(y, z)^\alpha, 2MR^{-na} d(y, z)^{-a}\big) \\
&\leqslant 2M(1-R^{-a})^{-1} + M(1-R^{-\alpha})^{-1} \leqslant CM.
\end{aligned}$$

**5. Elementary kernels.** Let $A$ be an operator which satisfies the conditions of Theorem (2.4) or Theorem (2.5). Theorem (3.1) will be applied to the operators $e^{-e^{-k}A}$ which we identify with their kernels.

Thus we shall estimate the kernels

$$e_{n,k} = \exp[ine^{-e^{-k}A}]\, e^{-2e^{-k}A}.$$

First we replace the metric $d$ and the measure $\mu$ by

$$d_t = t^{-1/m} d, \quad \mu_t = t^{-q/m} \mu,$$

respectively, and the kernel $e^{-tA}(x, y)$ by

$$e_t(x, y) = t^{q/m} e^{-tA}(x, y).$$

We notice that

$$e_t^* = e_t,$$

$$\sup_x \int |e_t(x, y)| e^{bd_t(x,y)} d\mu_t(y) \leqslant C_0,$$

$$\sup_x \int |e_t(x, y)|^2 d\mu_t(y) \leqslant C_0.$$

Also, for the operator $e^{-tA}$ we have

$$e^{-tA} f(x) = \int e_t(x, y) f(y) d\mu_t(y).$$

By an application of Theorem (3.1), with a constant $C_1$ independent of $k$ and $n$, we obtain

$$(5.1) \qquad \sup_x \int |e_{n,k}(x, y)| \big(1+e^{k/m} d(x, y)\big)^a d\mu(y) \leqslant C_1(1+|n|)^{q/2+a}.$$

Since we want to use results of this section in the proof of Theorems (2.4) and (2.5) we introduce kernels $e'_{n,k}$ writing

$$e'_{n,k}(x, y) = \int \exp\{ine^{-e^{-k}A}\} e^{-e^{-k}A}(x, s) v(s)\cdot e^{-e^{-k}A}(s, y) w(y) d\mu(s)$$

where $v$ and $w$ are Lipschitz functions equal to 1 on large balls and having bounded supports (we will specify this later).

By (2.1) and (2.3) we verify that for an $\varepsilon > 0$ and $\alpha' = \alpha - \varepsilon$

$$(5.2) \qquad \int |e_t(x, y) - e_t(x, z)| d\mu_t(x) \leqslant C d_t(y, z)^{\alpha'}.$$

In fact,

$$\int |e_t(x, y) - e_t(x, z)| d\mu_t(x) \leqslant \int_{d_t(x,y)<r_0} + \int_{d_t(x,y)\geqslant r_0}$$
$$\leqslant C_1 r_0^q d_t(y, z)^\alpha + 2\sup_x \int |e^{-tA}(x, y)| e^{d_t(x,y)} d\mu(y)\cdot \min(e^{-r_0+d_t(y,z)}, 1).$$

Thus with $r_0 = d_t^{-\varepsilon}(y, z)$ we obtain (5.2).

Now we write

$$\int |e_{n,k}(x, y) - e_{n,k}(x, z)| d\mu(x)$$
$$\leqslant \sup_s \int |\exp\{ine^{-e^{-k}A}\} e^{-e^{-k}A}(x, s)| d\mu(x)\cdot \int |e^{-e^{-k}A}(s, y) - e^{-e^{-k}A}(s, z)| d\mu(s),$$

whence by another application of Theorem (3.1) and (5.2) (with $t = e^{-k/m}$),

$$\int |e_{n,k}(x, y) - e_{n,k}(x, z)| d\mu(x) \leqslant C_2(1+|n|)^{q/2} e^{\alpha' k/m} d(y, z)^{\alpha'}.$$

Thus the family $\{e_{n,k}\}_{k\in\mathbf{N}}$ satisfies the assumptions of Lemma (4.1) and so we arrive at the main formula of this section:

$$(5.3) \qquad \|\{e_{n,k}\}_{k\in\mathbf{N}}\|_{\mathrm{Zo}} \leqslant C_a(1+|n|)^{q/2+a}$$

for every $a > 0$.

The same argument also yields

$$(5.3') \qquad \|\{e'_{n,k}\}_{k\in\mathbf{N}}\|_{\mathrm{Zo}} \leqslant C_{a,v,w}(1+|n|)^{q/2+a}.$$



**6. Functional calculus.** In this section a functional calculus based on estimates of Section 5 is described. We operate with functions $F$ on an operator $A$ which satisfies (2.1)–(2.3) or (2.1), (2.2) and (2.3′), and we place $F$ in a Sobolev space $\mathscr{L}^l_p(\mathbf{R})$ with $p > 1$ and $l > q/2+1$, which seems to be the most convenient space, being invariant under diffeomorphisms of $\mathbf{R}$, good enough to estimate $\max_n|\hat{F}(n)|$ and including the Riesz means $R^\alpha$. First we assume that

$$(6.1) \qquad F \in \mathscr{L}^l_p(\mathbf{R}), \quad \operatorname{supp} F \subset [-e^{-1}, e],$$

and we write

$$F_t(\lambda) = F(t\lambda),$$
$$G_t(\lambda) = \begin{cases} \lambda^{-2} F_t(-\log\lambda) & \text{for } \lambda > 0,\\ 0 & \text{otherwise,}\end{cases}$$

One can prove that

$$(6.2) \qquad \|G_t\|_{\mathscr{L}^l_p} \leqslant C\|F\|_{\mathscr{L}^l_p} \quad \text{for } t \text{ in } [a, b],\ a > 0,$$

where $C$ is independent of $F$. We write $F(A)f = \int F(\lambda) dE(\lambda) f$ where $E$ is the spectral measure of $A$. We also note

$$\operatorname{supp} G_t \subset [0, e] \quad \text{if } t \geqslant e^{-1},$$
$$F_t(A) = F(tA) = G_t(e^{-A}) e^{-2A},$$
$$F_{t_1 \cdot t_2}(A) = F_{t_1}(t_2 A) = G_{t_1}(e^{-t_2 A}) e^{-2t_2 A}.$$

Let $G_t(\lambda) = \sum_n \hat{G}_t(n) e^{in\lambda}$. Then

$$F_{te^{-k}}(A) = G_t(e^{-e^{-k}A}) e^{-2e^{-k}A} = \sum \hat{G}_t(n) e_{n,k}$$

and, of course, we restrict $t$ to the interval $[e^{-1}, 1] = I$. We estimate

$$\|\{F_t(A)\}_{t\in(0,1)}\|_{\mathrm{Zo}} = \|\{F_{se^{-k}}(A)\}_{s\in I, k\in\mathbf{N}}\|_{\mathrm{Zo}}$$
$$= \Big\|\{\sum_n \hat{G}_s(n) e_{n,k}\}_{s\in I, k\in\mathbf{N}}\Big\|_{\mathrm{Zo}}$$
$$\leqslant \sum_n \sup_{s\in I} |\hat{G}_s(n)|\cdot \|\{e_{n,k}\}_{k\in\mathbf{N}}\|_{\mathrm{Zo}}.$$

Since by (6.2)

$$|\hat{G}_s(n)| \leqslant c(1+|n|)^{-l} \|G_s\|_{\mathscr{L}^l_p} \leqslant C(1+|n|)^{-l} \|F\|_{\mathscr{L}^l_p}$$

and by (5.3)

$$\|\{e_{n,k}\}_{k\in\mathbf{N}}\|_{\mathrm{Zo}} \leqslant C(1+|n|)^{q/2+a'},$$

choosing $a'$ small enough, we obtain

$$(6.3) \qquad \|\{F_t(A)\}_{t\in(0,1)}\|_{\mathrm{Zo}} \leqslant C\|F\|_{\mathscr{L}^l_p}.$$

Now we assume that $F \in \mathscr{L}^l_{p,\mathrm{loc}}(\mathbf{R})$ and satisfies the conditions of Theorem

(2.4). We write

$$F = \varphi_1 F + \sum_{k=1}^{\infty} (\varphi_{k+1} - \varphi_k) F$$

where $\varphi \in C_c^\infty(\mathbf{R})$, $\varphi(\lambda) = 1$ for $\lambda \in [0, 1]$, $\varphi(\lambda) = 0$ outside $(-1/10, 3/2)$, $0 \leqslant \varphi \leqslant 1$ and $\varphi_k(\lambda) = \varphi(2^{-k}\lambda)$. Let

$$H_0(\lambda) = \varphi_1(\lambda) F(\lambda), \qquad H_k(\lambda) = [(\varphi_{k+1} - \varphi_k) F](2^k \lambda).$$

Then $\operatorname{supp} H_k \subset [-e^{-1}, e]$ and $F(\lambda) = \sum H_k(2^{-k}\lambda)$ with $\sum \|H_k\|_{\mathscr{L}^1_p} < \infty$. Consequently,

$$\|\{F(tA)\}_{t \in (0,1)}\|_{Z_0} \leqslant \sum \|\{H_k(2^{-k} tA)\}_{t \in (0,1)}\|_{Z_0}. \tag{6.4}$$

Using (5.1) instead of (5.3) one can prove that $\sup_{t<1} \|F_t(A)\|_{L^p, L^p} < \infty$, which ends the proof of (2.4).

We omit an easy but tedious proof that norm convergence takes place on a dense subset of $L^p$, $1 \leqslant p < \infty$.

By the spectral theorem $\lim_{t \to 0} F_t(A) f = f$ in $L^2$ norm. By (2.2) each $e^{tA}$ is bounded from $L^2$ into $L^\infty$ so $e^{sA} F_t(A) f$ converges to $e^{sA} f$ in $L^\infty$ and consequently a.e. Since $e^{-A} L^2$ is dense in $L^2$ and $L^2 \cap L^p$, $1 \leqslant p < \infty$, is dense in $L^p$ we need only the continuity of $F^*$.

Now, we are going to prove that $F^*$ is continuous from $L^p$, $p \in [1, \infty]$, into the space of $\mu$-measurable functions on $M$. To do this, it is enough to estimate all operators of the form $uF^*$, where $u$ is a Lipschitz function with bounded support. By the Marcinkiewicz interpolation theorem and $L^\infty$ estimate we only have to prove that each of the $uF^*$ is of weak type (1,1).

We write

$$F_{te^{-k}}(A) = \sum \hat{G}_t(n) e'_{n,k} + \sum \hat{G}_t(n)(e_{n,k} - e'_{n,k}) = I(t, k) + J(t, k)$$

and, by (6.4), we see that $\|\{I(t, k)\}\|_{Z_0} < \infty$. On the other hand, we are going to show that for every bounded function $u$ with bounded support there are $v$ and $w$ (cf. Section 5) such that

$$M_u f(x) = \sup\{|u(x)[J(t, k) f](x)|: t \in (e^{-1}, 1),\ k \in \mathbf{N}\}$$

is of strong type (1, 1).

We select $v$ and $w$ in such a way that $w(x) = 1$ if $d(x, \operatorname{supp} u) < 1$ and $v(x) = 1$ if $d(x, \operatorname{supp} w) < 1$. As in the proof of (6.3), it is sufficient to show

$$\sum_k \sup_y \int |u(x)[e_{n,k} - e'_{n,k}](x, y)|\, d\mu(x) \leqslant (1 + |n|)^{q/2 + a}.$$

We write



$$e_{n,k}(x, y) w(y) - e'_{n,k}(x, y)$$
$$= \int \exp\{ine^{-e^{-k}A}\} e^{-e^{-k}A}(x, s)(1 - v)(s) e^{-e^{-k}A}(s, y) w(y)\, d\mu(s).$$

Hence

$$\|e_{n,k} w - e'_{n,k}\|_0 \leqslant \|\exp\{ine^{-e^{-k}A}\} e^{-e^{-k}A}\|_0 \|(1 - v) e^{-e^{-k}A} w\|_0.$$

The first factor on the right is bounded by $C(1 + |n|)^{q/2 + a}$ independently of $k$. Since $(1 - v) e^{-e^{-k}A}(x, y) w(y)$ is zero if $d(x, y) < 1$, (2.1) implies that

$$\|(1 - v) e^{-e^{-k}A} w\|_0 \leqslant Ce^{-ce^k}.$$

To estimate

$$u e_{n,k}(1 - w) \tag{6.5}$$

we recall (5.1) and again for $d(x, y) < 1$, (6.5) is equal to zero.

**7. Applications. Schrödinger operators.** Let $A$ be a Schrödinger operator on $\mathbf{R}^q$:

$$A = -\Delta + V,$$

where the potential $V$ is nonnegative and $V \in L^p_{\text{loc}}$, $p > q/2$. Formulas (7.2)–(7.4) show that our abstract theorem (2.5) is applicable to the eigenexpansions of such Schrödinger operators.

By the Feynman–Kac formula we see that

$$0 \leqslant e^{-At}(x, y) \leqslant p_t(x, y), \tag{7.1}$$

where $p_t(x, y) = (4\pi t)^{-q/2} \exp(-|x - y|^2/(4t))$. Formula (7.1) implies

$$\int e^{-At}(x, y) e^{s|x-y|}\, dx \leqslant Ce^{cs^2 t}, \tag{7.2}$$

$$\int |e^{-At}(x, y)|^2\, dx \leqslant ct^{-q/2}, \qquad \sup_{x,y} e^{-At}(x, y) \leqslant ct^{-q/2} \tag{7.3}$$

for some universal constants $C$ and $c$.

Let $A_t = \Delta - V_t$, where $V_t(x) = tV(t^{1/2} x)$. For every $t > 0$ we have

$$\|e^{zA_t}\|_{L^2, L^2} \leqslant 1 \quad \text{for } \operatorname{Re} z > 0, \qquad \|e^{sA_t}\|_{L^\infty, L^\infty} \leqslant 1,$$

so with some $\beta = \beta_p$ (see [9], p. 67, Theorem 1 and its proof)

$$\|e^{zA_t}\|_{L^p, L^p} \leqslant 1 \quad \text{for } \operatorname{Arg} z < \beta.$$

Then the Cauchy integral formula gives

$$\|A_t e^{A_t}\|_{L^p, L^p} \leqslant C_p.$$

Since $\|e^{A_t}(\cdot, y)\|_{L^p} \leqslant C$ we have

$$\|A_t e^{2A_t}(\cdot, y)\|_{L^p} \leqslant C_p$$

independently of $t$ and $y$.

For a fixed ball $B = B(0, r)$ with $B_t = B(0, rt^{1/2})$ and for $t < 1$ and $p > q/2$ we have

$$\begin{aligned}\|\Delta e^{A_t}(\cdot, y)\|_{L^p(B)} &\leqslant \|A_t e^{A_t}(\cdot, y) + V_t e^{A_t}(\cdot, y)\|_{L^p(B)}\\ &\leqslant \|A_t e^{A_t}(\cdot, y)\|_{L^p(B)} + \|V_t e^{A_t}(\cdot, y)\|_{L^p(B)}\\ &\leqslant \|A_t e^{A_t}(\cdot, y)\|_{L^p(B)} + \|V_t\|_{L^p(B)} \sup e^{A_t}(x, y) \leqslant C_p.\end{aligned}$$

Also for $t < 1$ and $p > q/2$

$$\|V_t\|_{L^p(B)} = t(t^{-q/2})^{1/p} \|V\|_{L^p(B_t)} \leqslant \|V\|_{L^p(B)}.$$

Thus

$$\|e^{A_t}(\cdot, y)\|_{\mathrm{Lip}_\alpha(B_{1/2})} \leqslant C_\alpha, \qquad \alpha = \alpha(p),$$

because by e.g. [5], pp. 123 and 246,

$$\|f\|_{\mathrm{Lip}_\alpha(B_{1/2})} \leqslant C_p(\|\Delta f\|_{L^p(B)} + \|f\|_{L^p(B)}).$$

Using dilations and translations we obtain

(7.4) $$|e^{tA}(x, y) - e^{tA}(x', y)| \leqslant C_B t^{-(q+\alpha)/2} d(x, x')^\alpha.$$

for all $x, x' \in B_{1/2}$ and $y \in \mathbf{R}^q$.

**8. Applications. Elliptic operators on vector bundles.** We consider an elliptic differential operator of degree $m$ acting on a hermitian vector bundle $E$ on a compact $q$-dimensional riemannian manifold $M$ with riemannian metric $d$ and riemannian measure $\mu$. We fix a sufficiently fine finite covering $\{U_i\}$ and coordinate systems $\{U_i, \varphi_i\}$ on $M$, and we assume that $\varphi_i$ are bounded together with all the derivatives of $\varphi_i$ and $\varphi_i^{-1}$. We define the Sobolev norms on sections:

$$\|f\|_k^2 = \sum \|(\psi_i f) \circ \varphi_i^{-1}\|^2_{H^k(\mathbf{R}^q, \mathbf{C}^r)},$$

where $\{\psi_i\}$ is a fixed partition of unity subordinate to $\{U_i\}$.

(8.1) THEOREM. *If $A$ is a nonnegative elliptic operator on $E$ and $e^{-tA}$ is the semigroup generated by $A$, then the kernels $e^{-tA}(x, y)$ satisfy* (2.1)–(2.3).

We begin with two simple observations. By passing to coordinates and dilating for every $k > q/2$ we have

(8.2) $$\sup|f| \leqslant C_k \lambda^{-q/2}(\lambda^k \|f\|_k + \|f\|_0).$$

Also we note that for $s > 1$

(8.3) $$\int_M e^{-sd(x,y)}\, d\mu(x) \leqslant cs^{-q}.$$

(8.4) LEMMA. *For every $m \in \mathbf{N}$ and $x_0 \in M$, $s > 1$ there is a function $\varrho\colon M \to \mathbf{R}$ (which depends on $x_0$ and $s$) such that*

$$|\varrho(y) - sd(y, x_0)| \leqslant 1 \quad \textit{for all } y \in M,$$
$$\sup|\partial^\alpha(\varrho \circ \varphi_i)| \leqslant C_m s^{|\alpha|} \quad \textit{for all } i \textit{ and } \alpha,\ 1 \leqslant |\alpha| \leqslant m.$$



Proof. For every $i$

$$v = d(\cdot, x_0) \circ \varphi_i^{-1}$$

is a Lipschitz function on $\mathbf{R}^q$ and so for a $C_c^\infty$ function $u$, with $u_\varepsilon(x) = \varepsilon^{-q} u(x/\varepsilon)$, $v * u_\varepsilon$ is a $C^\infty$ function for which we have

$$|\partial^\alpha(v * u_\varepsilon)| = |\partial_j v * \partial^{\alpha'} u_\varepsilon| \leqslant c\varepsilon^{-|\alpha'|} \int |(\partial^{\alpha'} u)_\varepsilon|.$$

Now taking $\varepsilon$ proportional to $1/s$ and small enough for $v * u_\varepsilon \circ \varphi_i$ to be defined on $\operatorname{supp}\psi_i$, we put

$$\varrho = s \sum \psi_i (v * u_\varepsilon \circ \varphi_i).$$

(8.5) LEMMA. *Let $A$ be a differential operator of order $m$ on $E$, let $\varrho$ be the function of* (8.4) *and let $\eta = e^{-\varrho}$. For every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ independent of $\varrho$ such that for all $s > 1$*

$$|(A(\eta f) - \eta(Af), \eta f)| \leqslant \varepsilon \|\eta f\|^2_{m/2} + C(\varepsilon) s^m \|\eta f\|_0^2.$$

Proof. We write $A = \sum \psi_i A$ and we reduce the lemma to a compactly supported scalar operator. A multiple application of the formula $\eta \partial_i f = \partial_i(\eta f) + (\partial_i \varrho)\eta f$ yields

$$A(\eta f) - \eta(Af) = \sum \partial^\beta (a_I \cdot \prod (\partial^{\gamma(j)} \varrho) \cdot \partial^\alpha(\eta f))$$

where the $a_I$ are functions which depend only on the coefficients of $A$, and the summation runs over $|\alpha| \leqslant m/2$, $|\beta| \leqslant m/2$, $|\alpha| + |\beta| + \sum|\gamma(j)| \leqslant m$, $|\gamma(j)| > 0$. By Lemma (8.4), we have

$$\sup\left|\prod (\partial^{\gamma(j)} \varrho)\right| \leqslant Cs^{|\gamma|},$$

where $\gamma = \sum \gamma(j)$. Consequently,

$$\begin{aligned}\left|\left(\partial^\beta(a_I \cdot \prod(\partial^{\gamma(j)} \varrho) \cdot \partial^\alpha(\eta f)), \eta f\right)\right| &\leqslant Cs^{|\gamma|} \cdot \sup|a_I| \cdot \|\eta f\|_{|\alpha|} \|\eta f\|_{|\beta|}\\ &\leqslant Cs^{|\gamma|} \|\eta f\|_{m/2}^{2(|\alpha|+|\beta|)/m} \|\eta f\|_0^{2a/m}\\ &\leqslant \varepsilon \|\eta f\|^2_{m/2} + C(\varepsilon) s^m \|\eta f\|_0^2,\end{aligned}$$

where $a = m - |\alpha| - |\beta| \geqslant |\gamma|$.

Proof of Theorem (8.1). Lemma (8.5) and Gårding's inequality yield

(8.6) $$\begin{aligned}\operatorname{Re}(\eta Af, \eta f) &\geqslant \operatorname{Re}(A(\eta f), \eta f) - |([\eta, A]f, \eta f)|\\ &\geqslant C_1 \|\eta f\|^2_{m/2} - C_2 s^m \|\eta f\|_0^2.\end{aligned}$$

Hence

$$\operatorname{Re}(\eta Af, \eta f) + Cs^m \|\eta f\|_0^2 \geqslant c\|\eta f\|^2_{m/2}$$

and since $\operatorname{Im}(\eta Af, \eta f) = \operatorname{Im}([\eta, A]f, \eta f)$, we have

$$\operatorname{Re}(\eta Af, \eta f) + Cs^m \|\eta f\|_0^2 \geqslant |\operatorname{Im}(\eta Af, \eta f)|.$$

This means that for $\operatorname{Re} z > |\operatorname{Im} z|$ the operator

$$-z(A+Cs^m I) \tag{8.7}$$

is dissipative on $L^2(E, \eta^2 \mu)$. Since $M$ is compact, $L^2(E, \eta^2 \mu)$ is isomorphic to $L^2(E, \mu)$, whence the semigroup generated by (8.7) on $L^2(E, \mu)$ defines a semigroup on $L^2(E, \eta^2 \mu)$ and thus, since (8.7) is dissipative on $L^2(E, \eta^2 \mu)$ it is a semigroup of contractions. Therefore $-(A+Cs^m I)$ is the infinitesimal generator of a holomorphic semigroup of contractions in $\{z\colon \operatorname{Re} z > |\operatorname{Im} z|\}$, and

$$\|e^{-zA}\|_{L^2(E,\eta^2\mu),L^2(E,\eta^2\mu)} \leqslant e^{Cs^m \operatorname{Re} z}. \tag{8.8}$$

Hence by the Cauchy integral formula, for all $t > 0$, we have

$$\|A^k e^{-tA}\|_{L^2(E,\eta^2\mu),L^2(E,\eta^2\mu)} \leqslant C_1 t^{-k} e^{C_2 ts^m}. \tag{8.9}$$

Now for a $k \geqslant 1$ such that $mk > q/2$, by (8.2) we have

$$|\eta(x_0)(e^{-tA}f)(x_0)| \leqslant C\lambda^{-q/2}(\lambda^{mk}\|\eta e^{-tA}f\|_{mk} + \|\eta e^{-tA}f\|_0).$$

Therefore by an application of (8.6) with $A^{2k}$ in place of $A$, we obtain

$$\begin{aligned}|\eta(x_0)(e^{-tA}f)(x_0)| &\leqslant C_1 \lambda^{q/2}\{\lambda^{mk}[\operatorname{Re}(\eta A^{2k} e^{-tA}f, \eta e^{-tA}f) \\ &\quad + C_2 s^m \|\eta e^{-tA}f\|_0^2]^{1/2} + \|\eta e^{-tA}f\|_0\} \\ &\leqslant C\lambda^{-q/2}\{\lambda^{mk}\|\eta A^{2k} e^{-tA}f\|_0^{1/2}\|\eta e^{-tA}f\|_0^{1/2} \\ &\quad + (\lambda^{km}s^m + 1)\|\eta e^{-tA}f\|_0^{1/2}\}.\end{aligned}$$

Thus with $\lambda^m = t \leqslant 1$, using (8.8) and (8.9) we obtain

$$\begin{aligned}|\eta(x_0)(e^{-tA}f)(x_0)| &\leqslant C_1 t^{-q/(2m)}(1+ts^m+1)e^{C_2 ts^m}\|\eta f\|_0 \\ &\leqslant C_3 t^{-q/(2m)} e^{cts^m}\|\eta f\|_0.\end{aligned}$$

This yields

$$\left(\int |e^{-tA}(x_0, y)|^2 \eta^{-2}(y)\,d\mu(y)\right)^{1/2} \leqslant C_3 t^{-q/(2m)} e^{cs^m t}. \tag{8.10}$$

Since, by (8.3), $\int \eta\,d\mu \leqslant Cs^{-q}$, for $s^m t \geqslant 1$ the Schwarz inequality implies

$$\int |e^{-tA}(x_0, y)|\eta^{-1/2}(y)\,d\mu(y) \leqslant Ct^{-q/(2m)} s^{-q/2} e^{cs^m t} \leqslant Ce^{cs^m t}.$$

To summarize,

$$\int |e^{-tA}(x_0, y)| e^{sd(x_0,y)}\,d\mu(y) \leqslant Ce^{cs^m t}. \tag{8.11}$$

Putting $s^m t = 1$ and dropping $\eta$ in (8.10) we obtain

$$\int |e^{-tA}(x, y)|^2\,d\mu(y) \leqslant Ct^{-q/m}, \quad \text{for } t \leqslant 1. \tag{8.12}$$



Now (8.12) and (8.9), in virtue of Gårding's inequality, yield

$$\|e^{-tA}(x, \cdot)\|_k \leqslant ct^{-q/(2m)} t^{-k/m}, \quad \text{for } t \leqslant 1.$$

Since for $0 < \alpha = k - q/2 < 1$ the space $H^k$ is contained in $C^\alpha$ (cf. [5], p. 123) we obtain for $t \leqslant 1$

$$|e^{-tA}(x, y) - e^{-tA}(x, z)| \leqslant ct^{-(q+\alpha)/m} d^\alpha(y, z). \tag{8.13}$$

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