Further, find a continuous seminorm $\|\cdot\|_{n+1}$ on B such that

$$\|b\|_{n+1} \geqslant \|b\|_n \qquad (b \in B),$$
$$\|a\|_{n+1} \geqslant |a|_{f(n+1)} \qquad (a \in A),$$
$$\|b\|_{n+1} \geqslant \max\{1, C_n\} q_n(b) \qquad (b \in B),$$
$$\|b\|_{n+1} \geqslant p_{n+1}(b) \qquad (b \in B).$$

Put

$$\|b\|'_{n+1} = \inf\{|a|_{f(n+1)} + \|b-a\|_{n+1}:\ a \in A\}.$$

Clearly $\|b\|'_{n+1} \geqslant \|b\|'_n$ for all $b \in B$ and $n = 1, 2, \ldots$ Let $b_1, b_2 \in B$ and $a_1, a_2 \in A$. Then

$$\begin{aligned}\|b_1 b_2\|'_n &\leqslant |a_1 a_2|_{f(n)} + \|b_1 b_2 - a_1 a_2\|_n \\ &\leqslant |a_1|_{f(n)+1} |a_2|_{f(n)+1} + \|a_1(b_2-a_2)\|_n \\ &\quad + \|(b_1-a_1)a_2\|_n + \|(b_1-a_1)(b_2-a_2)\|_n \\ &\leqslant |a_1|_{f(n+1)} |a_2|_{f(n+1)} + q_n(a_1) q_n(b_2-a_2) \\ &\quad + q_n(b_1-a_1) q_n(a_2) + q_n(b_1-a_1) q_n(b_2-a_2) \\ &\leqslant |a_1|_{f(n+1)} |a_2|_{f(n+1)} + |a_1|_{f(n+1)} \|b_2-a_2\|_{n+1} \\ &\quad + \|b_1-a_1\|_{n+1} |a_2|_{f(n+1)} + \|b_1-a_1\|_{n+1} \|b_2-a_2\|_{n+1} \\ &= \big[|a_1|_{f(n+1)} + \|b_1-a_1\|_{n+1}\big]\big[|a_2|_{f(n+1)} + \|b_2-a_2\|_{n+1}\big].\end{aligned}$$

Hence $\|b_1 b_2\|'_n \leqslant \|b_1\|'_{n+1} \|b_2\|'_{n+1}$.

It is a matter of routine to prove that $\|a\|'_n = |a|_{f(n)}$ ($a \in A$, $n \in \mathbf{N}$) and that the seminorms $\|\cdot\|'_n$, $n = 1, 2, \ldots$, define the topology of B. ∎

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Some remarks on the uniform approximation property in Banach spaces

by
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Abstract. We prove that if a Banach space X has the uniform approximation property with uniformity function $k_X(n, K) = O(n)$ (for some constant K), then X^* has weak type 2. Further, as an application of our method, we also show that the uniformity function of L_p $(2 < p < \infty)$ cannot be $O(n^{q/2})$ for any $q < p$.

1. Introduction. Given a Banach space X, a finite-dimensional subspace E of X and a constant $K \geqslant 1$, let

$$k_X(E, K) = \inf\{\operatorname{rank} T:\ T: X \to X,\ \|T\| \leqslant K,\ Te = e \text{ for all } e \in E\},$$
$$k_X(n, K) = \sup\{k_X(E, K):\ E \subseteq X,\ \dim E = n\}, \qquad n \in \mathbf{N}.$$

X has the *bounded approximation property* (B.A.P.) if there is a K such that $k_X(E, K) < \infty$ for every finite-dimensional subspace E of X. X has the *uniform approximation property* (U.A.P.) if there is a K such that $k_X(n, K) < \infty$ for each $n \in \mathbf{N}$. $k_X(n, K)$ is called the *uniformity function* of X.

The U.A.P. was introduced by Pełczyński and Rosenthal in the paper [17], where they proved that all L_p $(1 \leqslant p \leqslant \infty)$ have it. More precisely, in [17] we can find the estimate $k_{L_p}(n, 1+\varepsilon) = O((n/\varepsilon)^{cn})$ for some constant c (this was proved using an argument due to Kwapień).

Recently, Figiel, Johnson and Schechtman [4] proved that for $p \in \{1, \infty\}$ an upper exponential estimate is optimal in the sense that, in this case, $k_{L_p}(n, K) \geqslant \exp[\delta(K)n]$, where $\delta(K)$ is a constant depending only on K. On the other hand, trivially, we always have $k_{L_2}(n, K) = n$, and so it is conjectured in [4, 8] that, for $1 < p \neq 2 < \infty$, there exist constants $K = K(p)$ and $\alpha = \alpha(p, K)$ such that $k_{L_p}(n, K) = O(n^\alpha)$. Lower bounds for $k_{L_p}(n, K)$ $(1 \leqslant p < \infty)$ are not known (see [4] for the case $p = \infty$), but in Section 3 we will see that $k_{L_p}(n, K) \neq O(n^{q/2})$ for all K and all $2 \leqslant q < p < \infty$.

In Section 2 we will give some characterizations of U.A.P. and, in Section 3, we will prove that, if $k_X(n, K) = O(n^\alpha)$, X has weak cotype 2α and X^* has finite cotype. A stronger result holds if $\alpha = 1$: in this case X is even K-convex and thus, since X has weak cotype 2, X^* has weak type 2. This fact may be

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considered as a step toward the solution of a problem posed by Pisier in [21]. As an application of our results, in Section 4 we will disprove a conjecture of Pietsch.

Let us now fix some notation. Given a subset S of X (resp. of X^*), let

$$S^{\perp} = \{y \in X^*\colon \langle y, x\rangle = 0 \text{ for all } x \in S\}$$
$$(\text{resp. } {}^{\perp}S = \{x \in X\colon \langle y, x\rangle = 0 \text{ for all } y \in S\}).$$

Given an ultrafilter $\mathscr{U}$, the corresponding ultrapower of X will be denoted by $(X)_{\mathscr{U}}$. For the (elementary) concepts and definitions from the ultraproduct theory we are going to use, we refer to the paper [6] by Heinrich.

If $u\colon X \to Y$ is an operator (= continuous linear map) and $n \in \mathbf{N}$, the nth approximation (resp. Gelfand, Kolmogorov) number of u is defined by

$$a_n(u) = \inf\{\|u-v\|\colon v\colon X \to Y,\ \operatorname{rank} v < n\},$$
$$c_n(u) = \inf\{\|uJ_Z^X\|\colon Z \subseteq X,\ \operatorname{codim} Z < n\},$$
$$d_n(u) = \inf\{\|Q_E^Y u\|\colon E \subseteq Y,\ \dim E < n\},$$

J_Z^X denoting the natural embedding $Z \to X$ and Q_E^Y the quotient map $Y \to Y/E$. We refer to the books [18, 19] of Pietsch for the main properties of these numbers.

If $0 < p < \infty$, $0 < q \leqslant \infty$ and $s \in \{a, c, d\}$, let $\mathscr{L}_{p,q}^s(X, Y)$ be the quasi-Banach ideal of all operators $u\colon X \to Y$ such that

$$l_{p,q}^s(u) = \Big(\sum_{k=1}^{\infty} (k^{1/p-1/q} s_k(u))^q\Big)^{1/q} < \infty$$

if $q < \infty$, resp.

$$l_{p,q}^s(u) = \sup_{k \in \mathbf{N}} k^{1/p} s_k(u) < \infty$$

if $q = \infty$. Once again, we refer to [18, 19] for the main properties of these ideals.

If $u\colon l_2^n \to X$ is an operator, let

$$l(u) = \Big(\int_{\mathbf{R}^n} \Big\|\sum_{k=1}^{n} \alpha_i u e_i\Big\|^2 d\gamma_n(\alpha)\Big)^{1/2},$$

where γ_n is the standard gaussian measure on $\mathbf{R}^n$ and $e_1, \ldots, e_n$ is an orthonormal basis of l_2^n. If $v\colon X \to l_2^n$, put

$$l^*(v) = \sup\{|\operatorname{tr}(vu)|\colon u\colon l_2^n \to X,\ l(u) \leqslant 1\}.$$

According to [15, 21], X has *weak cotype* q $(2 \leqslant q < \infty)$ if there is a constant c such that

$$l_{q,\infty}^a(u) \leqslant c\,l(u) \quad \forall u\colon l_2^n \to X,\ \forall n \in \mathbf{N}.$$

X is K-convex if and only if X does not contain l_1^n uniformly (see [22]). X has *weak type* p $(1 < p \leqslant 2)$ if X is K-convex and X^* has weak cotype $p^*(=p/(p-1))$. X is a *weak Hilbert space* if it has weak cotype 2 and weak type 2. Concerning these "weak" properties, we refer to [14, 15, 21, 22].

Part of this paper was written while I was visiting the University of Paris VII. I am grateful to Prof. G. Pisier for his encouragement and hints.

2. Some characterizations of U.A.P. To prove the main result of this section, we need a lemma. Given a function $g\colon \mathbf{N} \to \mathbf{N}$ and $E \subseteq F \subseteq X$, $\dim E < \infty$, define

$$\tilde{\lambda}(E, F, g) = \inf\{\|T\|\colon T\colon F \to X,\ Te = e \text{ for all } e \in E,\ \operatorname{rank} T \leqslant g(\dim E)\}.$$

LEMMA 2.1. *For any finite-dimensional subspace E of X, and any function $g\colon \mathbf{N} \to \mathbf{N}$, we have*

$$\tilde{\lambda}(E, X, g) \leqslant \sup\{\tilde{\lambda}(E, F, g)\colon E \subseteq F \subseteq X,\ \dim F < \infty\}.$$

Proof. We follow an idea of Kürsten [9, Lemma 2.2]. Fix $E \subseteq X$, and let λ be such that

$$\sup\{\tilde{\lambda}(E, F, g)\colon E \subseteq F \subseteq X,\ \dim F < \infty\} < \lambda < \infty.$$

We will show that $\tilde{\lambda}(E, X, g) \leqslant \lambda$.

Consider the index set

$$I = \{F\colon E \subseteq F \subseteq X,\ \dim F < \infty\}$$

and let the ultrafilter $\mathscr{U}$ on I be such that $\{G \in I\colon F \subseteq G\} \in \mathscr{U}$ for any $F \in I$. Given $F \in I$, let $T_F\colon F \to X$ be such that $\|T_F\| < \lambda$, $\operatorname{rank} T_F \leqslant g(\dim E)$ and $T_F e = e$ for all $e \in E$. Define $T_0 = (T_F)_{\mathscr{U}}\colon (F)_{\mathscr{U}} \to (X)_{\mathscr{U}}$. We have then $\|T_0\| \leqslant \lambda$ and $\operatorname{rank} T_0 \leqslant g(\dim E)$ (use the same argument of the proof of Proposition 2.2(i)$\Rightarrow$(iii) below). Further, $T_0 e = e$ for all $e \in E$ (we write again e for the elements of $(F)_{\mathscr{U}}$ and $(X)_{\mathscr{U}}$ corresponding to the constant family $(e)_{F \in I}$). Let now, for $x \in X$ and for every $F \in I$,

$$x_F = \begin{cases} x, & x \in F, \\ 0, & x \notin F. \end{cases}$$

Then, by $x \mapsto (x_F)_{\mathscr{U}}$ we define a linear isometric embedding $J_1\colon X \to (F)_{\mathscr{U}}$. On the other hand, if $G = T_0 J_1(X)$ and $\varepsilon > 0$ is given, reasoning as in [6, Prop. 6.1] we can find an embedding $J_2\colon G \to X$ with $\|J_2\| \leqslant 1+\varepsilon$ and so that E is mapped pointwise onto itself by J_2.

Finally, letting $T = J_2 T_0 J_1$, we get $\|T\| \leqslant (1+\varepsilon)\lambda$, $\operatorname{rank} T \leqslant g(\dim E)$, and $Te = e$ for all $e \in E$. Since ε was arbitrary, we have $\tilde{\lambda}(E, X, g) \leqslant \lambda$, which proves the lemma. ∎

PROPOSITON 2.2. *Let X be a Banach space, $g\colon \mathbf{N}\to\mathbf{N}$ a function such that $g(n)\geqslant n$. Then the following properties are equivalent:*

(i) *$\exists K$ such that $k_X(n, K) = O(g(n))$.*

(ii) *$\exists K$, c such that, for all $n\geqslant 2$ and all operators u taking values in X, we have*

$$a_{[cg(n-1)]+1}(u)\leqslant Kd_n(u).$$

(iii) *$\exists K$ such that $k_{(X)_{\mathcal{U}}}(n, K) = O(g(n))$ for every ultrafilter $\mathcal{U}$.*

(iv) *$\exists K$ such that $k_{X^{**}}(n, K) = O(g(n))$.*

(v) *$\exists K$, c such that, given finite-dimensional subspaces E and F of X with $E\subseteq F$, we can find an operator $T\colon F\to X$ with*

a) $\|T\|\leqslant C$.

b) $\operatorname{rank} T\leqslant cg(\dim E)$.

c) *$Te = e$ for all $e\in E$.*

(vi) *$\exists K$, c such that, for any $n\in\mathbf{N}$ and any n-codimensional subspace Z of X^*, we can find an operator $T\colon X^*\to X^*$ such that*

a) $\|T\|\leqslant K$.

b) $T(X^*)\subseteq Z$.

c) $\operatorname{rank}(\mathrm{id}_{X^*}-T)\leqslant cg(n)$.

Proof. (i)$\Rightarrow$(ii). Let E be a subspace of X with $\dim E<n$ and let $u\colon Y\to X$ be an operator defined on some Banach space Y. Since X satisfies (i), there are constants K, c, a function $g(n)$ and an operator $T\colon X\to X$ such that $\|T\|\leqslant K$, $Te=e$ for all $e\in E$ and $\operatorname{rank} T\leqslant cg(n-1)$. By definition of the approximation numbers, we have

$$a_{[cg(n-1)]+1}(u)\leqslant \|u-Tu\| = \|(\mathrm{id}_X-T)u\|.$$

Since $E\subseteq\ker(\mathrm{id}_X-T)$, there exists an operator $R\colon X/E\to X$ such that $\mathrm{id}_X-T=RQ_E^X$, $\|R\|\leqslant\|\mathrm{id}_X-T\|\leqslant 1+K$. Then

$$a_{[cg(n-1)+1]}(u)\leqslant (1+K)\,\|Q_E^X u\|,$$

and thus (ii) follows after taking the infimum on the right-hand side over all E as above.

(ii)$\Rightarrow$(i). Let E be an n-dimensional subspace of X and let $P\colon X\to X$ be a projection onto E. Since $E=\ker(\mathrm{id}_X-P)$, there is an operator $S\colon X/E\to X$ such that $\mathrm{id}_X-P=SQ_E^X$. By (ii) and the definition of approximation numbers, there is an operator $L\colon X/E\to X$ with $\operatorname{rank} L\leqslant cg(n)$ such that, for some constant K,

$$\|S-L\|\leqslant Kd_{n+1}(S)\leqslant K\|Q_E^X S\| = K,$$

since $Q_E^X S=\mathrm{id}_{X/E}$. Define $T=P+LQ_E^X$. We have

$$\|T\| = \|\mathrm{id}_X-(S-L)\,Q_E^X\|\leqslant 1+\|S-L\|\leqslant K+1,$$

$Te=e$ for all $e\in E$, and $\operatorname{rank} T\leqslant \operatorname{rank} P+\operatorname{rank} L\leqslant cg(n)+n$, therefore $k_X(n, K+1) = O(cg(n)+n) = O(g(n))$.

(i)$\Rightarrow$(iii). We use the argument of [11, Prop. 1]. Let X satisfy $k_X(n, K)\leqslant cg(n)$ for some constants K, c, and let $\mathcal{U}$ be a free ultrafilter on an index set I. Let E be an n-dimensional subspace of $(X)_{\mathcal{U}}$, and let $\{x^k=(x_\iota^k)_{\iota\in I}\colon 1\leqslant k\leqslant n\}$ be a basis of E. Let $E_\iota=\operatorname{span}\{x_\iota^1,\ldots,x_\iota^n\}$ and, for each $\iota\in I$, let $T_\iota\colon X\to X$ be such that $\|T_\iota\|\leqslant K$, $T_\iota x_\iota^k=x_\iota^k$ for all $1\leqslant k\leqslant n$, and $\operatorname{rank} T_\iota\leqslant cg(n)$.

Define the operator $T\colon (X)_{\mathcal{U}}\to(X)_{\mathcal{U}}$ by

$$T(y_\iota)_{\iota\in I} = (T_\iota y_\iota)_{\iota\in I}.$$

From the definition of an ultraproduct it follows first that $\|T\|\leqslant K$ and $Tx^k=x^k$ for all $1\leqslant k\leqslant n$, where we put $x^k=(x_\iota^k)_{\iota\in I}$. It remains to prove that $\operatorname{rank} T\leqslant cg(n)$. For every $\iota\in I$, let $\{b_\iota^1, b_\iota^2,\ldots\}$ be an Auerbach basis of $T_\iota(X)$, that is, a basis satisfying

$$\max_k|\alpha^k|\leqslant\Big\|\sum_k\alpha^k b_\iota^k\Big\|$$

for all scalars α^k.

Now, if $y=(y_\iota)_{\iota\in I}\in(X)_{\mathcal{U}}$ we have $T_\iota y_\iota=\sum_k\alpha_\iota^k b_\iota^k$ where

$$|\alpha_\iota^k|\leqslant\|T_\iota y_\iota\|\leqslant K\|y_\iota\|,$$

so that, for all k, $|\alpha_\iota^k|\leqslant K\|y\|$. This shows that $\alpha^k=\lim_{\mathcal{U}}\alpha_\iota^k$ exists for all k, hence

$$Ty = \Big(\sum_k\alpha_\iota^k b_\iota^k\Big)_{\iota\in I} = \sum_k\alpha^k(b_\iota^k)_{\iota\in I},$$

which shows that

$$T\big((X)_{\mathcal{U}}\big)\subseteq\operatorname{span}\{(b_\iota^k)_{\iota\in I}\colon k\leqslant cg(n)\},$$

i.e., $\operatorname{rank} T\leqslant cg(n)$.

(iii)$\Rightarrow$(iv)$\Rightarrow$(i). By [6, Prop. 6.7] there is an ultrapower $(X)_{\mathcal{U}}$ of X such that X^{**} is 1-complemented in $(X)_{\mathcal{U}}$. Clearly, since $k_{(X)_{\mathcal{U}}}(n, K)=O(g(n))$, we must have $k_{X^{**}}(n, K)=O(g(n))$, too, and so (iv) holds. (i) follows now easily from local reflexivity.

(i)$\Rightarrow$(v) is trivial.

(v)$\Rightarrow$(i) follows immediately from Lemma 2.1. In fact, if X satisfies (v) then there is a constant c such that $\sup\{\tilde\lambda(E, F, cg)\colon E\subseteq F\subseteq X, \dim F<\infty\}<\infty$, so that, by Lemma 2.1, $\tilde\lambda(E, X, cg)$ is uniformly bounded for all finite-dimensional subspaces E of X. This means that X satisfies (i).

(i)$\Rightarrow$(vi). By (i)$\Rightarrow$(iv), there are constants K, c such that $k_{X^{**}}(n, K)\leqslant cg(n)$. Given an n-codimensional subspace Z of X^*, let $T_0\colon X^{**}\to X^{**}$ be such that $\|T_0\|\leqslant K$, $T_0w=w$ for all $w\in Z^\perp$, $\operatorname{rank} T_0\leqslant cg(n)$. Let $S\colon X^*\to X^*$ be such that $S^*=T_0$ and define $T=\mathrm{id}_{X^*}-S$. Then it is easy to see that $\|T\|\leqslant K+1$, $T(X^*)\subseteq Z$, and $\operatorname{rank}(\mathrm{id}_{X^*}-T)=\operatorname{rank} S=\operatorname{rank} T_0\leqslant cg(n)$.

(vi)$\Rightarrow$(i). Let $E \subseteq X$ be n-dimensional. Then $E^\perp$ is n-codimensional in X^*. By (vi), there are constants K, c and an operator T_0: $X^* \to X^*$ with $\|T_0\| \leqslant K$, $T_0(X^*) \subseteq E^\perp$ and $\operatorname{rank}(\mathrm{id}_{X^*} - T_0) \leqslant cg(n)$. Let $T_1 = \mathrm{id}_{X^{**}} - T_0^*$ so that, trivially, $\|T_1\| \leqslant 1+K$. Then, since $T_0(X^*) \subseteq E^\perp$, for any $e \in E$ and any $y \in X^*$ we have

$$\langle T_0^* e, y\rangle = \langle e, T_0 y\rangle = 0,$$

so that $E \subseteq \ker T_0^*$ and thus $T_1 e = e$ for all $e \in E$. Further,

$$\operatorname{rank} T_1 = \operatorname{rank}(\mathrm{id}_{X^*} - T_0) \leqslant cg(n).$$

By local reflexivity, let S: $T_1(X^{**}) \to X$ be an embedding with $\|S\| \leqslant 2$ (say). Then it is easy to see that $T = ST_1 J_X^{X^{**}}$ has the properties which show that X satisfies (i). ∎

The next proposition characterizes the property which is dual to U.A.P. (see Proposition 2.2 (i)$\Leftrightarrow$(vii)):

Proposition 2.3. *Let X be a Banach space and let g: $\mathbf{N} \to \mathbf{N}$ be such that $g(n) \geqslant n$. Then the following are equivalent:*

(i) *There exists a constant K such that, for any $n \in \mathbf{N}$ and any n-codimensional subspace $Z \subseteq X$, we can find an operator T: $X \to X$ with*

a) $\|T\| \leqslant K$.
b) $T(X) \subseteq Z$.
c) $\operatorname{rank}(\mathrm{id}_X - T) = O(g(n))$.

(ii) *There exist constants K, c such that, for all $n \in \mathbf{N}$ and for every operator u defined on X,*

$$a_{[cg(n-1)]+1}(u) \leqslant Kc_n(u).$$

Proof. (i)$\Rightarrow$(ii). Let $Z \subseteq X$ have codimension $< n$, and let u: $X \to Y$ be an operator to an arbitrary Banach space Y. Since X satisfies (i), there are constants K, c, a function g: $\mathbf{N} \to \mathbf{N}$ and an operator T: $X \to X$ such that $\|T\| \leqslant K$, $T(X) \subseteq Z$ and $\operatorname{rank}(\mathrm{id}_X - T) \leqslant cg(n-1)$. So we have

$$a_{[cg(n-1)]+1}(u) \leqslant \|u - u(\mathrm{id}_X - T)\| = \|uT\| = \|uJ_Z^X T\| \leqslant K\|uJ_Z^X\|.$$

We get (ii) after taking the infimum on the right-hand side over all n-codimensional $Z \subseteq X$.

(ii)$\Rightarrow$(i). Let Z be an n-codimensional subspace of X, and let P: $X \to Z$ be a projection onto Z. By definition of the Gelfand numbers, we have $c_{n+1}(P) \leqslant \|PJ_Z^X\| = 1$ and so, by (ii), there are constants K, c and a function $g(n)$ such that $a_{[cg(n)]+1}(P) \leqslant K$. This means that there is an operator L: $X \to Z$ with rank $\leqslant [cg(n)]$ such that $\|P - L\| \leqslant K+1$. Let $T = J_Z^X(P-L)$. Then $\|T\| \leqslant K+1$, $T(X) \subseteq Z$, and

$$\operatorname{rank}(\mathrm{id}_X - T) \leqslant \operatorname{rank}(\mathrm{id}_X - J_Z^X P) + \operatorname{rank} L = O([cg(n)] + n) = O(g(n)),$$

so X satisfies (i). ∎

3. Spaces with $k_X(n, K) = O(n^\alpha)$. As was mentioned in the introduction, the property $k_X(n, K) = n$ is equivalent to X being isomorphic to a Hilbert space [10]. On the other hand, if we take g: $\mathbf{N} \to \mathbf{N}$ with $\sup_{n\in\mathbf{N}}(g(n) - n) = \infty$, Johnson [7, Example 2.2] has constructed nonhilbertian weak Hilbert spaces X satisfying $k_X(n, K) = O(g(n))$. In the next theorem we will see that a polynomial estimate of $k_X(n, K)$ has consequences on the cotype of X and X^*:

Theorem 3.1. *If $k_X(n, K) = O(n^\alpha)$ for some $1 \leqslant \alpha < \infty$, then X has weak cotype 2α and X^* has finite cotype.*

Proof. Let u: $Y \to X$ be an operator, Y being an arbitrary Banach space. By Proposition 2.2(i)$\Rightarrow$(ii) we have then

$$a_{k_X(n-1,K)+1}(u) \leqslant Kd_n(u) \tag{1}$$

for all $n \geqslant 2$ and for some constant K. Putting $k_X(0, K) = 0$ and using (1), we have

$$\begin{aligned} l_{2\alpha,\infty}^a(u) &= \sup_{n\geqslant 1} n^{1/2\alpha} a_n(u) = \sup_{n\geqslant 1} \sup_{k_X(n-1,K) < i \leqslant k_X(n,K)} i^{1/2\alpha} a_i(u) \\ &\leqslant \sup_{n\geqslant 1} (k_X(n, K))^{1/2\alpha} a_{k_X(n-1,K)+1}(u) \\ &\leqslant K\left(\sup_{n\geqslant 1} \frac{(k_X(n, K))^{1/2\alpha}}{n^{1/2}}\right) \sup_{n\geqslant 1} n^{1/2} d_n(u) \leqslant Kc^{1/2\alpha} l_{2,\infty}^d(u), \end{aligned} \tag{2}$$

where c is a constant such that $k_X(n, K) \leqslant cn^\alpha$. Now, if we take $Y = l_2^n$ we can apply an inequality of Pajor and Tomczak-Jaegermann [16] which states that

$$l_{2,\infty}^d(u) \leqslant \varkappa l(u) \tag{3}$$

for some universal constant $\varkappa$. (2) and (3) together give finally

$$l_{2\alpha,\infty}^a(u) \leqslant Kc^{1/2\alpha} \varkappa l(u), \quad \forall u: l_2^n \to X,$$

which says that X has weak cotype 2α.

Let us now prove that X does not contain the l_1^n uniformly complemented. Suppose the contrary. Since $k_X(n, K) = O(n^\alpha)$, it is easy to see that there must be constants K, c (not depending on n) such that, for any subspace E_n of $l_1^{2^n}$, there is an operator T_n: $l_1^{2^n} \to l_1^{2^n}$ such that $\|T_n\| \leqslant K$, $T_n e = e$ for all $e \in E_n$, and $\operatorname{rank} T_n \leqslant c(\dim E)^\alpha$. Taking E_n to be the space spanned by the first n Rademacher functions in $l_1^{2^n}$ we find, by a special case of [4, Cor. 1.5], that $\operatorname{rank} T_n \geqslant \exp[c(K)n]$, where $c(K) > 0$ is a constant depending only on K. Since $\exp[c(K)n] \leqslant cn^\alpha$ cannot hold for all n, we have a contradiction.

So, X does not contain the l_1^n uniformly complemented and thus, by duality, X^* does not contain the l_∞^n uniformly. By the Maurey–Pisier Theorem [14], this means that X^* has cotype q for some $q < \infty$. ∎

Theorem 3.1 has a direct application to the local theory of L_p spaces. To illustrate the meaning of the next corollary, let us first recall that, for a subspace E of L_p and $K \geqslant 1$,

$$m_p(E, K) = \inf\{m\colon \exists F \subseteq L_p \text{ with } E \subseteq F \text{ and } d(l_p^m, F) \leqslant K\},$$

$$m_p(n, K) = \sup\{m_p(E, K)\colon \dim E = n\}.$$

As remarked in [4], by the Dor–Schechtman Theorem [3, 23], for each $1 \leqslant p \leqslant \infty$ there is a constant $K_p > 1$ such that every K-isomorph of an l_p^m space in L_p with $K < K_p$ is $f(p, K)$-complemented in L_p, and $f(p, K) \to 1$ as $K \to 1$. It follows that, if $K < K_p$, then

$$k_{L_p}(n, f(p, K)) \leqslant m_p(n, K).$$

As for $m_p(n, K)$, it follows from an euclidean section argument that

$$\delta(p, K) n^{\max\{1, p/2\}} \leqslant m_p(n, K).$$

The above inequalities suggest that the lower bound for $k_{L_p}(n, K)$ might also be of the form $\tilde{\delta}(p, K) n^{\max\{1, p/2\}}$, if K is big enough (in fact, for small values of K the situation might dramatically change, as is shown in [1]). If $p > 2$, this conjecture is supported by the next corollary:

Corollary 3.2. *If $2 < p < \infty$ we have $k_{L_p}(n, K) \neq O(n^{q/2})$ for all K and all $2 \leqslant q < p < \infty$. In other words,*

$$\limsup_{n\to\infty} k_{L_p}(n, K)/n^{q/2} = \infty$$

for all K, q as above.

Proof. If we had $k_{L_p}(n, K) = O(n^{q/2})$ for some $q < p$ and some K, from Theorem 3.1 we would deduce that L_p has weak cotype q, which could hold only if $q \geqslant p$. ∎

As might be expected, assuming $k_X(n, K) = O(n)$ has deep implications on the geometry of X (recall that $k_X(n, K) = n$ if and only if X is isomorphic to a Hilbert space [10]). In fact, Pisier [21] asks if the property $k_X(n, K) = O(n)$ has something to do with X being a weak Hilbert space (see also [2, Problem Af13]). The following theorem can be regarded as a first step toward an answer to this question.

Theorem 3.3. *If $k_X(n, K) = O(n)$, then X^* has weak type 2.*

Proof. Since X^* has weak type 2 if and only if X has weak cotype 2 and is K-convex, by Theorem 3.1 we only have to show that X is K-convex. Now, by Theorem 3.1, X^* has cotype q for some finite q. Consequently, X having weak cotype 2, it has cotype $2+\varepsilon$ for all $\varepsilon > 0$, and thus we can choose ε such that $(2+\varepsilon)^{-1} + q^{-1} > 2^{-1}$. This condition together with the B.A.P. of X finally imply that X is K-convex, by the main result of [20]. ∎

Remarks. (i) Notice that if there is K such that $k_X(n, K) = O(n)$ and $k_{X^*}(n, K) = O(n)$, then X must be a weak Hilbert space. So, if the property $k_X(n, K) = O(n)$ were self-dual, we would have solved part of Pisier's problem. Unfortunately, this self-duality is open.

(ii) It may well be that a polynomial estimate for $k_X(n, K)$ forces X to be K-convex. In this direction, because of the same result of Pisier quoted above [20], we have:

Corollary 3.4. *If $k_X(n, K) = O(n^\alpha)$ and $k_{X^*}(n, K) = O(n^\beta)$ for some $1 < \alpha$, $\beta < \infty$ such that $\alpha^{-1} + \beta^{-1} > 1$, then X is K-convex (and thus has weak cotype 2α and weak type $2\beta/(2\beta-1)$, by Theorem 3.1).*

(iii) By the definitions and the method we used to deduce inequality (2) in the proof of Theorem 3.1, it is not hard to see that the following holds:

Corollary 3.5. *If $k_X(n, K) = O(n)$, Y is any Banach space, and $0 < p < \infty$, $0 < q \leqslant \infty$, we have*

$$\mathscr{L}^a_{p,q}(Y, X) = \mathscr{L}^d_{p,q}(Y, X),$$

i.e., there is a constant $\varkappa$ depending only on X such that, for all operators u taking values in X, we have

$$l^a_{p,q}(u) \leqslant \varkappa l^d_{p,q}(u).$$

(iv) Of course, we have a similar statement for the property discussed in Proposition 2.3: *if a Banach space X satisfies* (i) *of Prop.* 2.3, *then*

$$\mathscr{L}^a_{p,q}(X, Y) = \mathscr{L}^c_{p,q}(X, Y)$$

for all $0 < p < \infty$, $0 < q \leqslant \infty$, and all Banach spaces Y.

4. Two examples and a conjecture of Pietsch. Concerning the study of the property $k_X(n, K) = O(n)$, it may be useful to keep a couple of examples in mind. Both of them were constructed by Johnson in [7]. Let us briefly recall the definitions:

(i) $T^{(2)}$ is the completion of the finitely nonzero sequences of scalars under the norm $\|\cdot\|$ satisfying the identity

$$\|x\| = \max\{\|x\|_{c_0}, 2^{-1}\sup(\sum_{i=1}^{k_n} \|A_i x\|^2)^{1/2}\},$$

where the sup is over all n and all pairwise disjoint sequences $(A_i)_{i=1}^{k_n}$ of subsets of N for which

$$\bigcup_{i=1}^{k_n} A_i \subseteq \{n+j\}_{j=1}^\infty,$$

and $(k_n)_{n\in\mathbb{N}}$ is a sequence which tends to ∞ sufficiently fast (see [7] for details).

(ii) $X_2 = (\sum_{n=1}^{\infty} l_{p_n}^{k_n})_2$, where $p_n \geq 2$, and $p_n \to 2$, $k_n \to \infty$ fast enough (this example is also described in [12, 1.g.7]).

Remarks. (i) The properties of $T^{(2)}$ show that it is a nonhilbertian weak Hilbert space satisfying $k_{X^{(2)}}(n, K) = O(n)$. By the way, $T^{(2)}$ is the so-called "2-convexified Tsirelson space" (see [2, 22]).

(ii) X_2 is not a weak Hilbert space (though having type 2 and cotype $2+\varepsilon$ for all positive ε), but has the remarkable property that every subspace of every quotient of X_2 has U.A.P. Now, by Theorem 3.1, $k_{X_2}(n, K) \neq O(n)$ since X_2 does not have weak cotype 2.

The results of Section 3 together with the space $T^{(2)}$ above allow us to disprove (and update) two conjectures made by A. Pietsch several years before the introduction of the "weak" properties [18, 28.3.7]: he conjectured that

(i) *X is isomorphic to a Hilbert space whenever* $\mathcal{L}_{2,2}^{a}(\cdot, X) = \mathcal{L}_{2,2}^{d}(\cdot, X)$.

(ii) *X is isomorphic to a Hilbert space whenever* $\mathcal{L}_{2,2}^{a}(X, \cdot) = \mathcal{L}_{2,2}^{c}(X, \cdot)$.

We have the following

PROPOSITION 4.1. *There is a nonhilbertian weak Hilbert space satisfying*

$$\mathcal{L}_{p,q}^{a}(\cdot, X) = \mathcal{L}_{p,q}^{d}(\cdot, X), \qquad \mathcal{L}_{p,q}^{a}(X^*, \cdot) = \mathcal{L}_{p,q}^{c}(X^*, \cdot),$$

for all $0 < p < \infty$, $0 < q \leq \infty$.

Proof. Consider the space $X = T^{(2)}$. Then (among several other properties), there is a constant K such that, for every finite-dimensional subspace E of X, we can find a projection P_1 from X onto a subspace F of E with $\dim F \geq (\dim E)/2$ and a projection P_2 from X onto a subspace G containing E with $\dim G \leq 3(\dim E)/2$, both projections having norm $\leq K$.

Now, the existence of P_1 for all E means that X is a weak Hilbert space [21, Th. 2.8], and the existence of P_2 for all E implies that X has the property $k_X(n, K) = O(n)$. By Corollary 3.5 we have, in particular, $\mathcal{L}_{2,2}^{a}(\cdot, X) = \mathcal{L}_{2,2}^{d}(\cdot, X)$. Further, since X^{**} satisfies $k_{X^{**}}(n, K) = O(n)$ (by Proposition 2.2) and by the duality between the ideals $\mathcal{L}_{2,2}^{d}$ and $\mathcal{L}_{2,2}^{c}$ (see [19]), we have $\mathcal{L}_{2,2}^{a}(X^*, \cdot) = \mathcal{L}_{2,2}^{c}(X^*, \cdot)$. Finally, since X does not contain isomorphic copies of l_2 [7], it certainly fails to be hilbertian, and thus the proposition holds. ■

We conclude with an "updated" version of Pietsch's conjectures (compare with Corollary 3.5):

CONJECTURE 4.2. (i) *X satisfies $k_X(n, K) = O(n)$ for some K whenever*

$$\mathcal{L}_{p,q}^{a}(\cdot, X) = \mathcal{L}_{p,q}^{d}(\cdot, X)$$

for all $0 < p < \infty$, $0 < q \leq \infty$.

(ii) *X satisfies $k_X(n, K) = O(n)$ for some K whenever*

$$\mathcal{L}_{p,q}^{a}(X^*, \cdot) = \mathcal{L}_{p,q}^{c}(X^*, \cdot)$$

for all $0 < p < \infty$, $0 < q \leq \infty$.

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