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STUDIA MATHEMATICA, T. XCVI (1990)

Renormalizations of Banach and locally convex algebras

by

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Abstract. Let $(B, \|\cdot\|)$ be a normed algebra and A its subalgebra. If $|\cdot|$ is an algebra norm on A equivalent to the restriction of $\|\cdot\|$ to A, then $|\cdot|$ can be extended to an algebra norm on B equivalent to $\|\cdot\|$. This generalizes the result of Lindberg [3] for commutative algebras. An analogous statement is also proved for locally convex algebras. As a corollary this gives an affirmative answer to a problem of Żelazko [4]: The topology of a locally convex algebra B with unit e can always be given by a submultiplicative system of seminorms $|\cdot|_\alpha$ satisfying $|e|_\alpha = 1$ for all α.

1. Normed algebras. Let A, B be normed algebras and $f\colon A \to B$ an isomorphic embedding. We prove that B can be renormed in such a way that f will become isometric. This means that the concepts of isomorphic and isometric extensions coincide. This result was known for commutative normed algebras [3].

THEOREM 1. *Let $(B, \|\cdot\|)$ be a normed algebra and A its subalgera. Suppose that $|\cdot|$ is an algebra norm on A satisfying*

$$\alpha|a| \leqslant \|a\| \leqslant \beta|a| \quad \text{for all } a\in A,$$

where $0<\alpha\leqslant 1\leqslant\beta$. Then there exists an algebra norm $\|\cdot\|'$ on B such that $\|a\|' = |a|$ for all $a\in A$, and

$$(\alpha/\beta^2)\|b\|' \leqslant \|b\| \leqslant \beta\|b\|' \quad \text{for all } b\in B.$$

Proof. We may suppose that B has a unit e which belongs to A and $|e| = \|e\| = 1$. If either of these conditions is not satisfied we consider the unitizations $B_1 = \{b+\lambda\colon b\in B,\ \lambda\in\mathbf{C}\}$ and $A_1 = \{a+\lambda\colon a\in A,\ \lambda\in\mathbf{C}\} \subseteq B_1$ with naturally defined algebraic operations and the norms $\|b+\lambda\|_{B_1} = \|b\|_B + |\lambda|$ and $|a+\lambda|_{A_1} = |a|_A + |\lambda|$.

For $b\in B$ define $q(b) = \sup\{\|ab\|\colon a\in A,\ |a|\leqslant 1\}$. Clearly $q(b)\geqslant\|b\|$ and $q(b)\leqslant\|b\|\sup\{\|a\|\colon a\in A,\ |a|\leqslant 1\}\leqslant\beta\|b\|$.

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For $b_1, b_2 \in B$ we have

$$q(b_1 b_2) = \sup\{\|ab_1 b_2\|:\ a\in A,\ |a|\leqslant 1\}$$
$$\leqslant \|b_2\| \sup\{\|ab_1\|:\ a\in A,\ |a|\leqslant 1\} = \|b_2\|\, q(b_1) \leqslant q(b_1)\, q(b_2).$$

Hence q is a submultiplicative norm on B equivalent to $\|\cdot\|$.

Let $a_0\in A$, $|a_0| = 1$, $b\in B$. Then

$$q(a_0 b) = \sup\{\|aa_0 b\|:\ a\in A,\ |a|\leqslant 1\}$$
$$\leqslant \sup\{\|ab\|:\ a\in A,\ |a|\leqslant 1\} \leqslant q(b)$$

so $q(ab)\leqslant |a|\, q(b)$ for every $a\in A$ and $b\in B$.

Define further $p(b) = \sup\{q(bc):\ c\in B,\ q(c)\leqslant 1\}$ for $b\in B$. This means $q(bc)\leqslant p(b)\, q(c)$ for $b, c\in B$, and for $b_1, b_2\in B$

$$p(b_1 b_2) = \sup\{q(b_1 b_2 c):\ c\in B,\ q(c)\leqslant 1\}$$
$$\leqslant p(b_1)\sup\{q(b_2 c):\ c\in B,\ q(c)\leqslant 1\} = p(b_1)\, p(b_2).$$

Let $a\in A$. Then

$$p(a) = \sup\{q(ac):\ c\in B,\ q(c)\leqslant 1\}$$
$$\leqslant \sup\{|a|\, q(c):\ c\in B,\ q(c)\leqslant 1\} \leqslant |a|.$$

For $b\in B$ we have

$$p(b) = \sup\{q(bc):\ c\in B,\ q(c)\leqslant 1\}$$
$$\leqslant \sup\{\beta\|bc\|:\ c\in B,\ q(c)\leqslant 1\}$$
$$\leqslant \beta\|b\|\sup\{\|c\|:\ c\in B,\ q(c)\leqslant 1\} \leqslant \beta\|b\| \quad \text{and}$$
$$p(b)\geqslant q(b)/q(e)\geqslant \|b\|/\beta.$$

Finally, put $\|b\|' = \inf\{|a| + (\beta/\alpha)\, p(b-a):\ a\in A\}$.

Let $b_1, b_2\in B$ and $a_1, a_2\in A$. Then

$$\|b_1 b_2\|' = \|a_1 a_2 + a_1(b_2-a_2) + (b_1-a_1)\, a_2 + (b_1-a_1)(b_2-a_2)\|'$$
$$\leqslant |a_1 a_2| + (\beta/\alpha)\, p[a_1(b_2-a_2) + (b_1-a_1)\, a_2 + (b_1-a_1)(b_2-a_2)]$$
$$\leqslant |a_1|\,|a_2| + (\beta/\alpha)\, p(a_1)\, p(b_2-a_2) + (\beta/\alpha)\, p(b_1-a_1)\, p(a_2)$$
$$+ (\beta/\alpha)\, p(b_1-a_1)\, p(b_2-a_2)$$
$$\leqslant [|a_1| + (\beta/\alpha)\, p(b_1-a_1)]\, [|a_2| + (\beta/\alpha)\, p(b_2-a_2)]$$

and, by passing to the infimum, $\|b_1 b_2\|'\leqslant \|b_1\|'\,\|b_2\|'$.

Also $\|a\|'\leqslant |a|$ and, for all $a_0\in A$,

$$|a_0| + (\beta/\alpha)\, p(a-a_0)\geqslant |a_0| + (1/\alpha)\,\|a-a_0\|\geqslant |a_0| + |a-a_0|\geqslant |a|.$$

Hence $\|a\|' = |a|$ for every $a\in A$.

Further $\|b\|'\leqslant (\beta/\alpha)\, p(b)\leqslant (\beta^2/\alpha)\,\|b\|$ and

$$|a| + (\beta/\alpha)\, p(b-a)\geqslant (1/\beta)\,\|a\| + (1/\alpha)\,\|b-a\|\geqslant (1/\beta)\,\|b\|,$$

i.e. $\|b\|'\geqslant (1/\beta)\,\|b\|$ for all $b\in B$. ∎

Remark. The first part of the proof (the existence of an equivalent norm p on B satisfying $p(a)\leqslant |a|$ for all $a\in A$) is a consequence of a general result in [2], p. 18. Here we repeat the argument for the sake of convenience.

The problem of extending norms in the case of matrix algebras was studied also in [1], Theorems 3.1.3 and 3.1.4. The proof given there contains some similar ideas.

2. Locally convex algebras. Let A be a locally convex algebra. Then the topology of A can be given by means of a system $\{|\cdot|_\alpha:\ \alpha\in I\}$ of seminorms. Without loss of generality we can assume that this system satisfies the following properties:

(1) For every $\alpha\in I$ there exists $\beta\in I$ such that $|xy|_\alpha\leqslant |x|_\beta\, |y|_\beta$ for all $x, y\in A$.

(2) For each finite set $\{\alpha_1, \ldots, \alpha_n\}\subset I$ there exists $\beta\in I$ such that $|x|_{\alpha_i}\leqslant |x|_\beta$ for all $i = 1, \ldots, n$ and $x\in A$.

If the system of seminorms defining the topology of A satisfies (1) and (2) then a seminorm $\|\cdot\|$ on A is continuous if and only if there exists an index $\alpha\in I$ and a constant C such that $\|x\|\leqslant C|x|_\alpha$ for all $x\in A$. For further details in the theory of locally convex algebras we refer to [5].

THEOREM 2. *Let B be a locally convex algebra and A its subalgebra. Let $\{|\cdot|_\alpha:\ \alpha\in I\}$ be a system of seminorms on A satisfying* (1) *and* (2) *which defines the topology of A inherited from B. Then there exists a system of seminorms $\{\|\cdot\|'_\beta:\ \beta\in K\}$ in B satisfying* (1) *and* (2) *which defines the topology of B and such that for every $\beta\in K$ the restriction of $\|\cdot\|'_\beta$ to A belongs to the system $\{|\cdot|_\alpha:\ \alpha\in I\}$ (i.e. there exists $\alpha\in I$ such that $|a|_\alpha = \|a\|'_\beta$ for all $a\in A$).*

Proof. Let $\|\cdot\|_\beta$, $\beta\in J$, be the system of all continuous seminorms in B. The restrictions of the seminorms $\|\cdot\|_\beta$, $\beta\in J$, to A define the same topology as the seminorms $|\cdot|_\alpha$, $\alpha\in I$, therefore for every $\beta\in J$ there exist $\alpha\in I$ and a constant $k(\beta, \alpha)$ such that $\|a\|_\beta\leqslant k(\beta, \alpha)|a|_\alpha$ for all $a\in A$, and for every $\alpha\in I$ there exists $\gamma\in J$ such that $|a|_\alpha\leqslant \|a\|_\gamma$ for all $a\in A$.

Consider the set

$$K = \{(\alpha, \beta)\in I\times J:\ |a|_\alpha\leqslant \|a\|_\beta \text{ for all } a\in A\}.$$

For $(\alpha, \beta)\in K$ define the seminorm $\|\cdot\|'_{\alpha\beta}$ on B by

$$\|b\|'_{\alpha\beta} = \inf\{|a|_\alpha + \|b-a\|_\beta:\ a\in A\}.$$

Let $a_0\in A$. Clearly $\|a_0\|'_{\alpha\beta}\leqslant |a_0|_\alpha$ and, for $a\in A$,

$$|a|_\alpha + \|a_0-a\|_\beta\geqslant |a|_\alpha + |a_0-a|_\alpha\geqslant |a_0|_\alpha.$$

Therefore $\|a\|'_{\alpha\beta} = |a|_\alpha$ for all $a \in A$. Further, $\|b\|'_{\alpha\beta} \leqslant \|b\|_\beta$ for all $b \in B$, i.e. $\|\cdot\|'_{\alpha\beta}$ is a continuous seminorm on B.

Let $\beta \in J$. Then there exist $\lambda \in I$ and a constant $k(\beta, \lambda) \geqslant 1$ such that $\|a\|_\beta \leqslant k(\beta, \lambda)|a|_\lambda$ for all $a \in A$. Also, there exists $\gamma \in J$ such that $\|a\|_\gamma \geqslant |a|_\lambda$ for all $a \in A$ and $\|b\|_\gamma \geqslant \|b\|_\beta$ for all $b \in B$. Let $a \in A$, $b \in B$. Then

$$|a|_\lambda + \|b-a\|_\gamma \geqslant (1/k(\beta, \lambda))\|a\|_\beta + \|b-a\|_\beta \geqslant (1/k(\beta, \lambda))\|b\|_\beta,$$

therefore $\|b\|'_{\lambda\gamma} \geqslant (1/k(\beta, \lambda))\|b\|_\beta$ and the system K defines the topology of B.

It remains to prove that K satisfies conditions (1) and (2). Let $b_1, b_2 \in B$, $a_1, a_2 \in A$ and $(\alpha, \beta) \in K$. Then

$$b_1 b_2 = a_1 a_2 + a_1(b_2 - a_2) + (b_1 - a_1)a_2 + (b_1 - a_1)(b_2 - a_2) \quad \text{and}$$

$$\begin{aligned}
\|b_1 b_2\|'_{\alpha\beta} &\leqslant |a_1 a_2|_\alpha + \|a_1(b_2 - a_2) + (b_1 - a_1)a_2 + (b_1 - a_1)(b_2 - a_2)\|_\beta \\
&\leqslant |a_1|_{\alpha_1}|a_2|_{\alpha_1} + \|a_1\|_{\beta_1}\|b_2 - a_2\|_{\beta_1} + \|b_1 - a_1\|_{\beta_1}\|a_2\|_{\beta_1} \\
&\quad + \|b_1 - a_1\|_{\beta_1}\|b_2 - a_2\|_{\beta_1} \\
&\leqslant |a_1|_{\alpha_2}|a_2|_{\alpha_2} + k(\beta_1, \alpha_2)|a_1|_{\alpha_2}\|b_2 - a_2\|_{\beta_1} \\
&\quad + k(\beta_1, \alpha_2)\|b_1 - a_1\|_{\beta_1}|a_2|_{\alpha_2} + \|b_1 - a_1\|_{\beta_1}\|b_2 - a_2\|_{\beta_1} \\
&\leqslant |a_1|_{\alpha_2}|a_2|_{\alpha_2} + |a_1|_{\alpha_2}\|b_2 - a_2\|_{\beta_2} + \|b_1 - a_1\|_{\beta_2}|a_2|_{\alpha_2} \\
&\quad + \|b_1 - a_1\|_{\beta_2}\|b_2 - a_2\|_{\beta_2} \\
&= [|a_1|_{\alpha_2} + \|b_1 - a_1\|_{\beta_2}][|a_2|_{\alpha_2} + \|b_2 - a_2\|_{\beta_2}]
\end{aligned}$$

where $\alpha_1 \in I$ is a seminorm satisfying

$$|a_1 a_2|_\alpha \leqslant |a_1|_{\alpha_1}|a_2|_{\alpha_1} \quad \text{for all } a_1, a_2 \in A,$$

$\beta_1 \in J$ satisfies

$$\|b_1 b_2\|_\beta \leqslant \|b_1\|_{\beta_1}\|b_2\|_{\beta_1} \quad \text{for all } b_1, b_2 \in B,$$

$\alpha_2 \in I$ and $k(\beta_1, \alpha_2) > 0$ satisfy

$$|a_1|_{\alpha_1} \leqslant |a_1|_{\alpha_2}, \quad \|a_1\|_{\beta_1} \leqslant k(\beta_1, \alpha_2)|a_1|_{\alpha_2} \quad \text{for all } a_1 \in A,$$

and $\beta_2 \in J$ satisfies

$$\max\{1, k(\beta_1, \alpha_2)\}\|b_1\|_{\beta_1} \leqslant \|b_1\|_{\beta_2} \quad \text{for all } b_1 \in B \text{ and}$$

$$|a_1|_{\alpha_2} \leqslant \|a_1\|_{\beta_2} \quad \text{for all } a_1 \in A.$$

Taking the infimum over all $a_1, a_2 \in A$ we obtain

$$\|b_1 b_2\|'_{\alpha\beta} \leqslant \|b_1\|'_{\alpha_2\beta_2}\|b_2\|'_{\alpha_2\beta_2},$$

hence (1) is satisfied.

Let $\alpha_1, \ldots, \alpha_n \in I$, $\beta_1, \ldots, \beta_n \in J$ satisfy

$$|a|_{\alpha_i} \leqslant \|a\|_{\beta_i} \quad (a \in A,\ i = 1, \ldots, n).$$

Then there exists $\alpha \in I$ such that $|a|_{\alpha_i} \leqslant |a|_\alpha$ ($a \in A$, $i = 1, \ldots, n$) and $\beta \in J$ such that $\|b\|_{\beta_i} \leqslant \|b\|_\beta$ ($b \in B$, $i = 1, \ldots, n$) and $|a|_\alpha \leqslant \|a\|_\beta$ for all $a \in A$. Then clearly $(\alpha, \beta) \in K$ satisfies condition (2) for the n-tuple $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in K$. ∎

COROLLARY. *Let B be a locally convex algebra with unit e. Then there exists a system K of seminorms on B defining the topology of B such that $\|e\|'_\alpha = 1$ for all $\alpha \in K$.*

Proof. Consider the subalgebra $A = \{\lambda e\colon \lambda \in \mathbf{C}\}$. The topology of A is determined by one seminorm $|\lambda e| = |\lambda|$. By the preceding theorem there exists a system of seminorms $\{\|\cdot\|'_\alpha\colon \alpha \in K\}$ on B satisfying (1) and (2) which defines the topology of B and satisfies $\|e\|'_\alpha = 1$ for all $\alpha \in K$. ∎

If A is a metrizable locally convex algebra, then its topology can be given by a countable system of seminorms $|\cdot|_n$, $n = 1, 2, \ldots$, such that

$$(3) \qquad \begin{cases} |a|_n \leqslant |a|_{n+1} \\ |a_1 a_2|_n \leqslant |a_1|_{n+1}|a_2|_{n+1} \end{cases} \quad \text{for all } a, a_1, a_2 \in A,\ n = 1, 2, \ldots$$

The following result is the analogue of Theorem 2 for metrizable locally convex algebras.

THEOREM 3. *Let B a metrizable locally convex algebra, A its subalgebra, and let $|\cdot|_n$, $n = 1, 2, \ldots$, be a system of seminorms on A which defines the topology of A inherited from B and satisfies (3). Then there exist seminorms $\|\cdot\|'_n$, $n = 1, 2, \ldots$, on B defining the topology of B which satisfy (3) and $\|a\|'_n = |a|_{f(n)}$ ($a \in A$, $n = 1, 2, \ldots$) where $f\colon \mathbf{N} \to \mathbf{N}$ is an increasing function.*

Proof. Let $\{p_n\colon n = 1, 2, \ldots\}$ be a countable system of continuous seminorms on B defining the topology of B. We define seminorms $\|\cdot\|_n$, $\|\cdot\|'_n$, $n = 1, 2, \ldots$, on B and the value of $f(n)$ by induction on n.

Put $f(1) = 1$, and choose a continuous seminorm $\|\cdot\|_1$ on B such that $\|a\|_1 \geqslant |a|_1$ for all $a \in A$. Define $\|\cdot\|'_1$ by $\|b\|'_1 = \inf\{|a|_1 + \|b - a\|_1\colon a \in A\}$.

Suppose that we have already defined continuous seminorms $\|\cdot\|_k$, $k = 1, \ldots n$, on B, the values $1 = f(1) < \ldots < f(n)$, and let

$$\|b\|'_k = \inf\{|a|_{f(k)} + \|b - a\|_k\colon a \in A\}, \quad b \in B,\ k = 1, \ldots, n.$$

Choose a continuous seminorm q_n on B satisfying

$$\|b_1 b_2\|_n \leqslant q_n(b_1)q_n(b_2) \quad (b_1, b_2 \in B).$$

Find $f(n+1) > f(n)$ and a constant C_n such that

$$q_n(a) \leqslant C_n|a|_{f(n+1)} \quad \text{for all } a \in A.$$

Further, find a continuous seminorm $\|\cdot\|_{n+1}$ on B such that

$$\|b\|_{n+1} \geqslant \|b\|_n \qquad (b\in B),$$
$$\|a\|_{n+1} \geqslant |a|_{f(n+1)} \qquad (a\in A),$$
$$\|b\|_{n+1} \geqslant \max\{1, C_n\}\, q_n(b) \qquad (b\in B),$$
$$\|b\|_{n+1} \geqslant p_{n+1}(b) \qquad (b\in B).$$

Put

$$\|b\|'_{n+1} = \inf\{|a|_{f(n+1)} + \|b-a\|_{n+1}\colon\ a\in A\}.$$

Clearly $\|b\|'_{n+1} \geqslant \|b\|'_n$ for all $b\in B$ and $n = 1, 2, \ldots$ Let $b_1, b_2\in B$ and $a_1, a_2\in A$. Then

$$\begin{aligned}\|b_1 b_2\|'_n &\leqslant |a_1 a_2|_{f(n)} + \|b_1 b_2 - a_1 a_2\|_n\\
&\leqslant |a_1|_{f(n)+1}\, |a_2|_{f(n)+1} + \|a_1(b_2-a_2)\|_n\\
&\quad + \|(b_1-a_1)a_2\|_n + \|(b_1-a_1)(b_2-a_2)\|_n\\
&\leqslant |a_1|_{f(n+1)}\, |a_2|_{f(n+1)} + q_n(a_1)\, q_n(b_2-a_2)\\
&\quad + q_n(b_1-a_1)\, q_n(a_2) + q_n(b_1-a_1)\, q_n(b_2-a_2)\\
&\leqslant |a_1|_{f(n+1)}\, |a_2|_{f(n+1)} + |a_1|_{f(n+1)}\, \|b_2-a_2\|_{n+1}\\
&\quad + \|b_1-a_1\|_{n+1}\, |a_2|_{f(n+1)} + \|b_1-a_1\|_{n+1}\, \|b_2-a_2\|_{n+1}\\
&= \big[|a_1|_{f(n+1)} + \|b_1-a_1\|_{n+1}\big]\big[|a_2|_{f(n+1)} + \|b_2-a_2\|_{n+1}\big].\end{aligned}$$

Hence $\|b_1 b_2\|'_n \leqslant \|b_1\|'_{n+1}\, \|b_2\|'_{n+1}$.

It is a matter of routine to prove that $\|a\|'_n = |a|_{f(n)}$ $(a\in A,\ n\in \mathbf{N})$ and that the seminorms $\|\cdot\|'_n$, $n = 1, 2, \ldots$, define the topology of B. ∎

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Some remarks on the uniform approximation property in Banach spaces

by

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Abstract. We prove that if a Banach space X has the uniform approximation property with uniformity function $k_X(n, K) = O(n)$ (for some constant K), then X^* has weak type 2. Further, as an application of our method, we also show that the uniformity function of L_p $(2 < p < \infty)$ cannot be $O(n^{q/2})$ for any $q < p$.

1. Introduction. Given a Banach space X, a finite-dimensional subspace E of X and a constant $K \geqslant 1$, let

$$k_X(E, K) = \inf\{\operatorname{rank} T\colon\ T\colon X\to X,\ \|T\| \leqslant K,\ Te = e \text{ for all } e\in E\},$$
$$k_X(n, K) = \sup\{k_X(E, K)\colon\ E\subseteq X,\ \dim E = n\}, \qquad n\in\mathbf{N}.$$

X has the *bounded approximation property* (B.A.P.) if there is a K such that $k_X(E, K) < \infty$ for every finite-dimensional subspace E of X. X has the *uniform approximation property* (U.A.P.) if there is a K such that $k_X(n, K) < \infty$ for each $n\in\mathrm{N}$. $k_X(n, K)$ is called the *uniformity function* of X.

The U.A.P. was introduced by Pełczyński and Rosenthal in the paper [17], where they proved that all L_p $(1 \leqslant p \leqslant \infty)$ have it. More precisely, in [17] we can find the estimate $k_{L_p}(n, 1+\varepsilon) = O\big((n/\varepsilon)^{cn}\big)$ for some constant c (this was proved using an argument due to Kwapień).

Recently, Figiel, Johnson and Schechtman [4] proved that for $p\in\{1, \infty\}$ an upper exponential estimate is optimal in the sense that, in this case, $k_{L_p}(n, K) \geqslant \exp[\delta(K)n]$, where $\delta(K)$ is a constant depending only on K. On the other hand, trivially, we always have $k_{L_2}(n, K) = n$, and so it is conjectured in [4, 8] that, for $1 < p \neq 2 < \infty$, there exist constants $K = K(p)$ and $\alpha = \alpha(p, K)$ such that $k_{L_p}(n, K) = O(n^\alpha)$. Lower bounds for $k_{L_p}(n, K)$ $(1 \leqslant p < \infty)$ are not known (see [4] for the case $p = \infty$), but in Section 3 we will see that $k_{L_p}(n, K) \neq O(n^{q/2})$ for all K and all $2 \leqslant q < p < \infty$.

In Section 2 we will give some characterizations of U.A.P. and, in Section 3, we will prove that, if $k_X(n, K) = O(n^\alpha)$, X has weak cotype 2α and X^* has finite cotype. A stronger result holds if $\alpha = 1$: in this case X is even K-convex and thus, since X has weak cotype 2, X^* has weak type 2. This fact may be

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