Renormalizations of Banach and locally convex algebras

by

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Abstract. Let \((B, \|\cdot\|)\) be a normed algebra and \(A\) its subalgebra. If \(\|\cdot\|\) is an algebra norm on \(A\) equivalent to the restriction of \(\|\cdot\|\) to \(A\), then \(\|\cdot\|\) can be extended to an algebra norm on \(B\) equivalent to \(\|\cdot\|\). This generalizes the result of Lindberg [3] for commutative algebras. An analogous statement is also proved for locally convex algebras. As a corollary this gives an affirmative answer to a problem of Zelazko [4]: The topology of a locally convex algebra \(B\) with unit \(e\) can always be given by a submultiplicative system of seminorms \(\|\cdot\|_a\) satisfying \(\|e\|_a = 1\) for all \(a\).

1. Normed algebras. Let \(A, B\) be normed algebras and \(f: A \rightarrow B\) an isomorphic embedding. We prove that \(B\) can be renormed in such a way that \(f\) will become isometric. This means that the concepts of isomorphic and isometric extensions coincide. This result was known for commutative normed algebras [3].

THEOREM 1. Let \((B, \|\cdot\|)\) be a normed algebra and \(A\) its subalgebra. Suppose that \(\|\cdot\|\) is an algebra norm on \(A\) satisfying

\[\alpha\|a\| \leq \|a\| \leq \beta\|a\| \quad \text{for all } a \in A,\]

where \(0 < \alpha \leq 1 \leq \beta\). Then there exists an algebra norm \(\|\cdot\|'\) on \(B\) such that \(\|a\|' = \|a\|\) for all \(a \in A\), and

\[(\alpha\beta^2)\|b\|' \leq \|b\| \leq \beta\|b\|' \quad \text{for all } b \in B.\]

Proof. We may suppose that \(B\) has a unit \(e\) which belongs to \(A\) and \(\|e\| = \|e\| = 1\). If either of these conditions is not satisfied we consider the unitizations \(B_1 = \{b + \lambda: \lambda \in C\}\) and \(A_1 = \{a + \lambda: \lambda \in A\}\) and define operations and norms naturally on \(B_1\) with \(\|e\| = \|e\| = 1\) and \(\|a + \lambda\|_A = \|a + \lambda\|_A\).

For \(b \in B\) define \(q(b) = \sup\{\|ab\|: a \in A, \|a\| \leq 1\}\). Clearly \(q(b) \geq \|b\|\) and \(q(b) \leq \|b\|\sup\{\|a\|: a \in A, \|a\| \leq 1\} \leq \beta\|b\|\).
For \( b_1, b_2 \in B \) we have
\[
q(b_1, b_2) = \sup \{ \|ab\| : a \in A, \|a\| \leq 1 \} \\
\leq \|b_2\| \sup \{ \|a\| : a \in A, \|a\| \leq 1 \} = \|b_2\| q(b_1) \leq q(b_1) q(b_2).
\]
Hence \( q \) is a submultiplicative norm on \( B \) equivalent to \( \| \cdot \| \).

Let \( a_0 \in A, |a_0| = 1, b \in B \). Then
\[
q(a_0 b) = \sup \{ \|a_0 a\| : a \in A, \|a\| \leq 1 \} \\
\leq \sup \{ \|a\| : a \in A, \|a\| \leq 1 \} = q(b)
\]
so \( q(ab) \leq |a| q(b) \) for every \( a \in A \) and \( b \in B \).

Define further \( p(b) = \sup \{q(bc) : c \in B, q(c) \leq 1 \} \) for \( b \in B \). This means \( q(bc) \leq p(b) q(c) \) for \( b, c \in B \), and for \( b_1, b_2 \in B \)
\[
p(b_1 b_2) = \sup \{q(b_1 b_2 c) : c \in B, q(c) \leq 1 \} \\
\leq p(b_1) \sup \{q(b_2 c) : c \in B, q(c) \leq 1 \} = p(b_1) p(b_2).
\]

Let \( a \in A \). Then
\[
p(a) = \sup \{q(ac) : c \in B, \|c\| \leq 1 \} \\
\leq \sup \{|a| q(c) : c \in B, \|c\| \leq 1 \} \leq |a|.
\]

For \( b \in B \) we have
\[
p(b) = \sup \{q(bc) : c \in B, \|c\| \leq 1 \} \\
\leq \sup \{\|b\| \|c\| : c \in B, \|c\| \leq 1 \} \\
\leq \beta \|b\| \sup \{|c| : c \in B, \|c\| \leq 1 \} \leq \beta \|b\| \quad \text{and}
\]
\[
p(b) \geq \sup \{q(b) q(e) : \|b\| \}
\]

Finally, put \( \|b\|'' = \inf \{|a| + (\beta/a) p(b-a) : a \in A\} \).

Let \( b_1, b_2 \in B \) and \( a_1, a_2 \in A \). Then
\[
\|b_1 b_2\|'' = |a_1 a_2 + a_1 (b_2 - a_2) + (b_1 - a_1) a_2 + (b_1 - a_1) (b_2 - a_2)|' \\
\leq |a_1 a_2| + (\beta/a) p(a_1 (b_2 - a_2) + (b_1 - a_1) a_2 + (b_1 - a_1) (b_2 - a_2)) \\
\leq |a_1 a_2| + (\beta/a) p(a_1 b_2 - a_2) + (\beta/a) p(b_1 - a_1) p(a_2) \\
+ (\beta/a) p(b_1 - a_1) p(b_2 - a_2) \\
\leq |a_1| + (\beta/a) p(b_1 - a_1)] + |a_2| + (\beta/a) p(b_2 - a_2)
\]
and, by passing to the infimum, \( \|b_1 b_2\|'' \leq \|b_1\|'' \|b_2\|'' \).

Also \( |a|'' \leq |a| \) and, for all \( a_0 \in A \),
\[
|a_0| + (\beta/a) p(a_0 - a_0) \geq |a_0| + (1/a) |a - a_0| \geq |a_0| + |a - a_0| \geq |a|
\]
Hence \( |a|'' = |a| \) for every \( a \in A \).

Further \( \|b\|' \leq (\beta/a) p(b) \leq (\beta^2/a) \|b\| \) and
\[
|a| + (\beta/a) p(b-a) \geq (1/\beta) |a| + (1/a) \|b-a\| \geq (1/\beta) \|b\|,
\]
i.e. \( \|b\|' \geq (1/\beta) \|b\| \) for all \( b \in B \).

Remark. The first part of the proof (the existence of an equivalent norm \( p \) on \( B \) satisfying \( p(a) \leq |a| \) for all \( a \in A \)) is a consequence of a general result in [2], p. 18. Here we repeat the argument for the sake of convenience.

The problem of extending norms in the case of matrix algebras was studied also in [1], Theorems 3.1.3 and 3.1.4. The proof given there contains some similar ideas.

2. Locally convex algebras. Let \( A \) be a locally convex algebra. Then the topology of \( A \) can be given by means of a system \( \{\| \cdot \|_x : x \in X\} \) of seminorms. Without loss of generality we can assume that this system satisfies the following properties:

(1) For every \( x \in X \) there exists \( \beta \in B \) such that \( x y \leq \|x\| \|y\| \) for all \( x, y \in A \).

(2) For each finite set \( \{x_1, \ldots, x_n\} \subseteq X \) there exists \( \beta \in B \) such that \( x_{i_1} \cdots x_{i_k} \leq \|x_{i_1}\| \cdots \|x_{i_k}\| \) for all \( i_1 = \cdots = i_k \leq n \) and \( x \in A \).

If the system of seminorms defining the topology of \( A \) satisfies (1) and (2) then a seminorm \( \| \cdot \| \) on \( A \) is continuous if and only if there exists an index \( x \in X \) and a constant \( C \) such that \( \|x\| \leq C \|x\|_x \) for all \( x \in A \). For further details in the theory of locally convex algebras we refer to [5].

Theorem 2. Let \( B \) be a locally convex algebra and \( A \) its subalgebra. Let \( \{\| \cdot \|_x : x \in X\} \) be a system of seminorms on \( A \) satisfying (1) and (2) which defines the topology of \( A \) inherited from \( B \). Then there exists a system of seminorms \( \{\| \cdot \|_x : x \in X\} \) in \( B \) satisfying (1) and (2) which defines the topology of \( B \) and such that for every \( x \in X \) the restriction of \( \| \cdot \|_x \) to \( A \) belongs to the system \( \{\| \cdot \|_x : x \in X\} \) (i.e. there exists \( x \in X \) such that \( \|a\|_x = \|a\|_x \) for all \( a \in A \)).

Proof. Let \( \| \cdot \|_x, \beta \in X \) be the system of all continua as seminorms in \( B \). The restrictions of the seminorms \( \| \cdot \|_x, \beta \in X \), to \( A \) define the same topology as the seminorms \( \| \cdot \|_x, x \in X \), therefore for every \( \beta \in X \) there exists \( x \in X \) and a constant \( k(\beta, x) \) such that \( \|a\|_x \leq k(\beta, x) \|a\|_x \) for all \( a \in A \), and for every \( x \in X \) there exists \( y \in X \) such that \( \|a\|_y \leq \|a\|_y \) for all \( a \in A \).

Consider the set
\[
K = \{x, \beta \in X \times X : \|a\|_x \leq \|a\|_y \text{ for all } a \in A\}.
\]

For \( x, \beta \in K \) define the seminorm \( \| \cdot \|_x \) on \( B \) by
\[
\|b\|_x = \inf \{|a| + (\beta/a) p(b-a) : a \in A\}.
\]

Let \( a_0 \in A \). Clearly \( \|a_0\|_x \leq \|a_0\|_x \) and, for \( a \in A \),
\[
\|a\|_x \leq \|a\|_x \leq \|a\|_x \leq \|a\|_x
\]

Hence \( \|a\|'_x = \|a\| \) for every \( a \in A \).
Therefore \( \|a\|_p = \|a\|_p \) for all \( a \in A \). Further, \( \|b\|_q = \|b\|_q \) for all \( b \in B \), i.e. \( \|\cdot\|_q \) is a continuous seminorm on \( B \).

Let \( \beta \in J \). Then there exist \( \lambda \in I \) and a constant \( \kappa(\beta, \lambda) \geq 1 \) such that \( \|a\|_p \leq \kappa(\beta, \lambda) \|a\|_p \) for all \( a \in A \). Also, there exists \( \gamma \in J \) such that \( \|a\|_p \geq \|a\|_p \) for all \( a \in A \) and \( \|a\|_p \geq \|a\|_p \) for all \( b \in B \). Let \( a, b \in A \). Then
\[
\|a\|_p + \|b - a\|_p \geq (1/k(\beta, \lambda)) \|a\|_p + \|b - a\|_p \geq (1/k(\beta, \lambda)) \|b\|_p,
\]
therefore \( \|b\|_p \geq (1/k(\beta, \lambda)) \|b\|_p \) and the system \( K \) defines the topology of \( B \).

It remains to prove that \( K \) satisfies conditions (1) and (2). Let \( b_1, b_2 \in B \), \( a_1, a_2 \in A \) and \( (\alpha_1, \beta_1) \in K \). Then
\[
\|b_1 b_2\|_p \leq \|a_1 a_2\|_p + \|a_1 (b_2 - a_2) + (b_1 - a_2) a_2 + (b_1 - a_1) b_1 - a_2\|_p
\]
\[
\leq \|a_1\|_p \|a_2\|_p + \|a_1\|_p \|b_2 - a_2\|_p + \|b_1 - a_1\|_p \|a_2\|_p + \|b_1 - a_1\|_p \|b_2 - a_2\|_p + \|b_1 - a_1\|_p \|b_2 - a_2\|_p
\]
\[
= \|a_1\|_p \|a_2\|_p + \|a_1\|_p \|b_2 - a_2\|_p + \|b_1 - a_1\|_p \|a_2\|_p + \|b_1 - a_1\|_p \|b_2 - a_2\|_p
\]
where \( \alpha_1, \beta_1 \) is a seminorm satisfying
\[
\|a_1 a_2\|_p \leq \|a_1\|_p \|a_2\|_p \quad \text{for all} \quad a_1, a_2 \in A,
\]
\( \beta_1 \) satisfies
\[
\|b_1 b_2\|_p \leq \|b_1\|_p \|b_2\|_p \quad \text{for all} \quad b_1, b_2 \in B,
\]
\( \alpha_2 \in I \) and \( \kappa(\beta_1, \alpha_2) > 0 \) satisfy
\[
\|a_1\|_p \leq \|a_1\|_p \|b_2 - a_2\|_p \quad \text{for all} \quad a_1 \in A,
\]
and \( \beta_2 \) satisfies
\[
\max \{1, \kappa(\beta_1, \alpha_2)\} \|b_1\|_p \leq \|b_1\|_p \quad \text{for all} \quad b_1 \in B \text{ and}
\]
\[
\|a_1\|_p \leq \|a_1\|_p \|b_2 - a_2\|_p \quad \text{for all} \quad a_1 \in A.
\]

Taking the infimum over all \( a_1, a_2 \in A \) we obtain
\[
\|b_1 b_2\|_p \leq \|b_1\|_p \|b_2\|_p \quad \text{for all} \quad b_1, b_2 \in B,
\]
hence (1) is satisfied.

Then there exists \( \alpha \in I \) such that \( \|a\|_p \leq \|a\|_p \) for all \( a \in A \), and \( \beta \in J \) such that \( \|b\|_p \leq \|b\|_p \) for all \( b \in B \). Then clearly \( (\alpha, \beta) \in K \) satisfies condition (2) for the \( n \)-tuple \( (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \in K \).

**Corollary.** Let \( B \) be a locally convex algebra with unit \( e \). Then there exists a system \( K \) of seminorms on \( B \) defining the topology of \( B \) such that \( \|e\|_p = 1 \) for all \( e \in K \).

**Proof.** Consider the subalgebra \( A = \{ \lambda e : \lambda \in \mathbb{C} \} \). The topology of \( A \) is determined by one seminorm \( \|\cdot\|_p = \|\cdot\|_p \). By the preceding theorem there exists a system of seminorms \( \{ \|\cdot\|_p : \lambda \in \mathbb{C} \} \) on \( B \) satisfying (1) and (2) which defines the topology of \( B \) and satisfies \( \|e\|_p = 1 \) for all \( e \in K \).

If \( A \) is a metrizable locally convex algebra, then its topology can be given by a countable system of seminorms \( \|\cdot\|_n, n = 1, 2, \ldots \), such that
\[
\begin{align*}
\|a\|_n &\leq \|a\|_{n+1} \\
\|a\|_{n+1} &\leq \|a\|_n \|a\|_{n+1}
\end{align*}
\]
for all \( a, a_1, a_2 \in A, n = 1, 2, \ldots \).

The following result is the analogue of Theorem 2 for metrizable locally convex algebras.

**Theorem 3.** Let \( B \) be a metrizable locally convex algebra, \( A \) its subalgebra, and \( \|\cdot\|_p = \|\cdot\|_p \) be a system of seminorms on \( A \) which defines the topology of \( A \) inherited from \( B \) and satisfies (3). Then there exist seminorms \( \|\cdot\|_k, k = 1, 2, \ldots, n = 1, 2, \ldots, \) on \( B \) defining the topology of \( B \) which satisfy (3) and \( \|a\|_k = \|a\|_{f(n)} \) (\( a \in A, n = 1, 2, \ldots \)) where \( f : \mathbb{N} \to \mathbb{N} \) is an increasing function.

**Proof.** Let \( \{p_n : n = 1, 2, \ldots \} \) be a countable system of continuous seminorms on \( B \) defining the topology of \( B \). We define seminorms \( \|\cdot\|_k, k = 1, 2, \ldots, \) on \( B \) and the value of \( f(n) \) by induction on \( n \).

Put \( f(1) = 1 \), and choose a continuous seminorm \( \|\cdot\|_1 \), \( \|\cdot\|_2, n = 1, 2, \ldots, \) on \( B \) and the value of \( f(n) \) by induction on \( n \).

Put \( f(1) = 1 \), and choose a continuous seminorm \( \|\cdot\|_1 \), \( \|\cdot\|_2, n = 1, 2, \ldots, \) on \( B \) and the value of \( f(n) \) by induction on \( n \).

Suppose that we have already defined continuous seminorms \( \|\cdot\|_k, k = 1, \ldots, n, \) on \( B \), the values \( f = 1 < \ldots < f(n) \), and let
\[
\|b\|_k = \inf \{\|a\|_f + \|b - a\|_k : a \in A\}, \quad b \in B, \quad k = 1, \ldots, n.
\]

Choose a continuous seminorm \( q_n \) on \( B \) satisfying
\[
\|b_1 b_2\|_n \leq q_n(b_1) q_n(b_2) \quad (b_1, b_2 \in B).
\]

Find \( f(n+1) > f(n) \) and a constant \( C_n \) such that
\[
q_n(a) \leq C_n q_{f(n+1)}(a) \quad \text{for all} \quad a \in A.
\]
Further, find a continuous seminorm $\| \cdot \|_{a+1}$ on $B$ such that

$$
\begin{align*}
\| b \|_{a+1} & \geq \| b \|_a & (b \in B), \\
\| a \|_{a+1} & \geq |a|_{f(a+1)} & (a \in A), \\
\| b \|_{a+1} & \geq \max \{ 1, C_n \} q_n(b) & (b \in B), \\
\| b \|_{a+1} & \geq p_{a+1}(b) & (b \in B).
\end{align*}
$$

Put

$$
\| b \|_{a+1} = \inf \{ |a|_{f(a+1)} + \| b-a \|_{a+1} : a \in A \}.
$$

Clearly $\| b \|_{a+1} \geq \| b \|_n$ for all $b \in B$ and $n = 1, 2, \ldots$. Let $b_1, b_2 \in B$ and $a_1, a_2 \in A$. Then

$$
\| b_1 b_2 \|_n \leq |a_1|_{f(a_1+1)} |a_2|_{f(a_2+1)} + \| b_1 b_2 - a_1 a_2 \|_n
$$

$$
+ \| (b_1 - a_1) b_2 \|_n + \| (b_2 - a_2) b_1 \|_n
$$

$$
\leq |a_1|_{f(a_1+1)} |a_2|_{f(a_2+1)} + |a_1 (b_2 - a_2)\|_n
$$

$$
+ q_n(b_1 - a_1) q_n(b_2 - a_2) + q_n(b_1 - a_1) q_n(b_2 - a_2)
$$

$$
\leq \max \{ 1, C_n \} q_n(b_1 - a_1) q_n(b_2 - a_2)
$$

$$
= \| b_1 - a_1 \|_{a+1} \| b_2 - a_2 \|_{a+1}.
$$

Hence $\| b_1 b_2 \|_n \leq \| b_1 \|_{a+1} \| b_2 \|_{a+1}$.

It is a matter of routine to prove that $\| a \|_n = |a|_{f(a)}$ (a.e $A$, $n \in \mathbb{N}$) and that the seminorms $\| \cdot \|_n$, $n = 1, 2, \ldots$, define the topology of $B$.

Some remarks on the uniform approximation property in Banach spaces

by

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Abstract. We prove that if a Banach space $X$ has the uniform approximation property with uniformity function $k_X(n, K) = O(n)$ for some constant $K$, then $X^*$ has weak type 2. Further, as an application of our method, we also show that the uniformity function of $l_p(2 < p < \infty)$ cannot be $O(n^{p/2})$ for any $q < p$.

1. Introduction. Given a Banach space $X$, a finite-dimensional subspace $E$ of $X$ and a constant $K \geq 1$, let

$$
k_X(E, K) = \inf \{ \text{rank } T : T : X \rightarrow X, \| T \| \leq K, \text{Te} = e \text{ for all } e \in E \},
$$

$$
k_X(n, K) = \sup \{ k_X(E, K) : E \subseteq X, \dim E = n \}, \quad n \in \mathbb{N}.
$$

$X$ has the bounded approximation property (B.A.P.) if there is a $K$ such that $k_X(E, K) < \infty$ for every finite-dimensional subspace $E$ of $X$. $X$ has the uniform approximation property (U.A.P.) if there is a $K$ such that $k_X(n, K) < \infty$ for each $n \in \mathbb{N}$. $k_X(n, K)$ is called the uniformity function of $X$.

The U.A.P. was introduced by Pełczyński and Rosenthal in the paper [17], where they proved that all $l_p(1 \leq p \leq \infty)$ have it. More precisely, in [17] we can find the estimate $k_X(n, 1 + \varepsilon) = O(n^{p-2})$ for some constant $c$ (this was proved using an argument due to Kwapień).

Recently, Figiel, Johnson and Schechtman [4] proved that for $p \in (1, \infty)$ an upper exponential estimate is optimal in the sense that, in this case, $k_X(n, K) \geq \exp \{ \delta(K, n) \}$, where $\delta(K)$ is a constant depending only on $K$. On the other hand, trivially, we always have $k_X(n, K) = n$, and so it is conjectured in [4, 8] that, for $1 < p \neq 2 < \infty$, there exist constants $K = K(p)$ and $\delta = \delta(p)$ (K) such that $k_X(n, K) = O(n^{p-2})$. Lower bounds for $k_X(n, K)(1 \leq p < \infty)$ are not known (see [4] for the case $p = \infty$), but in Section 3 we will see that $k_X(n, K) \neq O(n^{p-2})$ for all $K$ and all $2 < q < p < \infty$.

In Section 2 we will give some characterizations of U.A.P. and, in Section 3, we will prove that, if $k_X(n, K) = O(n^2)$, $X$ has weakly $2k$ and $X^*$ has finite cotype. A stronger result holds if $n = 1$: in this case $X$ is even K-convex and thus, since $X$ has weak cotype $2$, $X^*$ has weak type 2. This fact may be

References