then the centre of any ball satisfying the Fefferman condition is in the ball of radius R and centre at zero. The bounds for the number of eigenvalues less than $-\varepsilon$ for the potential V and the Fefferman estimates for $N(-\varepsilon, \delta_{R+k\varepsilon^{-1/2}} V_\lambda)$ are, therefore, equal. When we repeat the argument fixing the radius R and substituting the potential $\delta_{R+k\varepsilon^{-1/2}} V_\lambda$ for V_λ, we see that the radius of any ball which we must count does not exceed $b^{1/2} E(\lambda|R+k\varepsilon^{-1/2})^{-1/2}$. Therefore, if $B_1, \ldots, B_n$ is a maximal family of disjoint balls satisfying the Fefferman condition then for $1 \leqslant j \leqslant n$

$$\int_{B_j} |V_\lambda|^p \geqslant c_2 E(\lambda|R+k\varepsilon^{-1/2})^{p-d/2}, \quad \text{i.e.}$$

$$N(-\varepsilon, V) \leqslant c_3 E(\lambda|R+k\varepsilon^{-1/2})^{d/2-p} \int_{|y|<R+k\varepsilon^{-1/2}} |V_\lambda|^p,$$

which completes the proof.

Remark. Choosing zero as the centre of a "big" ball in spite of its arbitrariness does not influence much the value of the bound for large λ.

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On weak $(r, 2)$-summing operators and weak Hilbert spaces

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Abstract. We study Pisier's concept of weak operator ideals and give applications, in particular to $(r, 2)$-summing operators. This leads to new characterizations of weak Hilbert spaces in terms of integral operators and Hilbert numbers. Moreover, a weak version of Grothendieck's inequality and the normability of certain Weyl number ideals are proved.

Introduction. We extend Pisier's [PS1] concept of weak properties of Banach spaces in the following way. If $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are quasi-Banach ideals and $0 < q < \infty$, an operator $T\colon X \to Y$ belongs to the ideal $\mathfrak{W}_q(\alpha, \beta)$ if there is a constant $c \geqslant 0$ such that for all $u \in \mathfrak{A}(l_2, X)$ and $v \in \mathfrak{B}^*(Y, l_2)$ (the conjugate ideal of $\mathfrak{B}$)

$$\sup_{k\in\mathbf{N}} k^{1/q} a_k(vTu) \leqslant c\alpha(u)\beta^*(v),$$

where a_k is the kth approximation number.

In this notation, by [PS2], X is a weak Hilbert space if and only if $\mathrm{id}_X \in \mathfrak{W}_1(\pi_2^{d}, \pi_2)$, where Π_2 is the ideal of 2-summing operators.

Given an ideal $(\mathfrak{B}, \beta)$, we call an operator $T\colon X \to Y$ a *weak$_q$ β-operator* if it belongs to $\mathfrak{W}_q\mathfrak{B} := \mathfrak{W}_q(\|\cdot\|, \beta)$ ($\mathfrak{W}\mathfrak{B} := \mathfrak{W}_1\mathfrak{B}$ are the *weak β-operators*).

Denote by $\Pi_{r,2}$ the $(r, 2)$-summing operators and by $\mathfrak{L}^x_{p,\infty}$ all operators with Weyl numbers in the Lorentz sequence space $l_{p,\infty}$. We prove that

$$\mathfrak{W}_q \Pi_{r,2} = \mathfrak{L}^x_{p,\infty}, \quad 1 + 1/p = 1/q + 1/r > 1.$$

For $q = 1$ this was shown in [PS1] and [MAS]. Even in this case, however, our proof is different and uses Grothendieck numbers.

As an application we prove that $\mathfrak{L}^x_{p,\infty}$ admits an equivalent Banach ideal norm if and only if $p > 2$.

Using equalities of the form $\mathfrak{W}_q \mathfrak{L} = \mathfrak{W}_p \Pi_2$ $(1/2 + 1/p = 1/q > 1)$, $\mathfrak{W}_q \Pi_{r,2} = \mathfrak{W}_p \Pi_2$ $(1/2 + 1/p = 1/q + 1/r)$ and an extension of a result of Geiss [GEI], we prove that the following are equivalent:

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1) X is a weak Hilbert space.
2) Every operator $T\colon l_1 \to X$ is weak 1-summing.
3) There exists a constant $c \geq 0$ such that for all integral operators $u\colon l_2 \to X$

$$\sup_{k\in\mathbf{N}} k a_k(u) \leq c i_1(u).$$

4) For every $1 \leq r \leq 2$ there is a constant $c \geq 0$ such that for all operators $u \in \mathfrak{L}^h_{r,\infty}(l_2, X)$

$$\sup_{k\in\mathbf{N}} k^{1/r} a_k(u) \leq c \sup_{k\in\mathbf{N}} k^{1/r} h_k(u),$$

where h_k is the kth Hilbert number.

Clearly the equivalence of 1) and 2) is a weak version of Grothendieck's inequality.

For the background material we refer in general to the monographs of König [KÖN], Pietsch [PI1], [PI2] and Pisier [PS2].

1. Preliminaries. We use standard Banach space notations. In particular, we have for all Banach spaces X and subspaces $E \subset X$

$$i_E\colon E \to X, \quad x \mapsto x.$$

The Lorentz spaces $l_{p,q}$ and $l^n_{p,q}$, $0 < p, q \leq \infty$, $n \in \mathbf{N}$, are defined in the usual way. Denote by $i^n_{p,q}\colon l^n_p \to l^n_q$ the formal identity map.

Standard references on s-numbers and operator ideals are the monographs of Pietsch [PI1] and [PI2].

Note that for two quasi-Banach ideals $(\mathfrak{A}, \alpha)$, $(\mathfrak{B}, \beta)$ the inclusion $\mathfrak{A} \subset \mathfrak{B}$ implies $\beta \leq c\alpha$ for some $c \geq 0$.

The ideals of bounded, finite-dimensional, absolutely $(r, 2)$-summing and integral operators are denoted by $\mathfrak{L}$, $\mathfrak{F}$, $\Pi_{r,2}$ and $\mathfrak{I}_1$, respectively ($\Pi_2 := \Pi_{2,2}$).

Let $(\mathfrak{A}, \alpha)$ be a quasi-Banach ideal. The component $\mathfrak{A}^*(X, Y)$ of the conjugate ideal $(\mathfrak{A}^*, \alpha^*)$ is the class of all operators $T \in \mathfrak{L}(X, Y)$ such that

$$\alpha^*(T) := \sup\{|\mathrm{tr}(TS)| \mid S \in \mathfrak{F}(X, Y),\ \alpha(S) \leq 1\} < \infty.$$

The component $\mathfrak{A}^{\mathrm{d}}(X, Y)$ of the dual ideal $(\mathfrak{A}^{\mathrm{d}}, \alpha^{\mathrm{d}})$ is the class of all operators $T \in \mathfrak{L}(X, Y)$ such that $T^* \in \mathfrak{L}(Y^*, X^*)$, and $\alpha^{\mathrm{d}}(T) := \alpha(T^*)$.

Next we recall the usual notation of some s-numbers of an operator $T \in \mathfrak{L}(X, Y)$:

$a_n(T)$: approximation numbers,
$c_n(T)$: Gelfand numbers,
$x_n(T)$: Weyl numbers,
$y_n(T)$: Chang numbers,
$h_n(T)$: Hilbert numbers.

The *Grothendieck numbers* were recently introduced by several authors. Let Det: $\mathbf{K}^n \to \mathbf{K}^n$ be the unique determinant. Then

$$\Gamma_n(T) := \sup\{|\mathrm{Det}(\langle Tx_i, y_j^*\rangle)|^{1/n} \mid (x_k)_{k=1}^n \subset B_X,\ (y_k^*)_{k=1}^n \subset B_{Y^*}\}.$$

Let s be any s-number. The component $\mathfrak{L}^s_{p,q}(X, Y)$ of the ideal $(\mathfrak{L}^s_{p,q}, l^s_{p,q})$ is the class of all operators $T \in \mathfrak{L}(X, Y)$ such that

$$l^s_{p,q}(T) := \|s(T)\|_{p,q} < \infty.$$

Since on Hilbert space components all s-numbers are equal, we put $\sigma_{p,q}(T) := l^s_{p,q}(T)$, provided that T acts between Hilbert spaces.

Let $a, b, c \in \mathbf{R}$. Then $a \sim_c b$ $(a \sim b)$ means that $(1/c)a \leq b \leq ca$.

We now make a list of those known results and facts which are essential for this paper.

(1.1) [CAR] *For all $n \in \mathbf{N}$ and $T \in \mathfrak{L}(X, Y)$ we have*

$$\Big(\prod_{k=1}^n c_k(T)\Big)^{1/n} \leq \Gamma_n(T).$$

(1.2) [GEI] *Let H, K be Hilbert spaces and $T \in \mathfrak{L}(H, K)$. Then for each $n \in \mathbf{N}$*

$$\Big(\prod_{k=1}^n a_k(T)\Big)^{1/n} = \Gamma_n(T).$$

In particular, when $T \in \mathfrak{L}(l_2^n, l_2^n)$ we have $|\mathrm{Det}(\langle T(e_i), e_j\rangle)|^{1/n} = \Gamma_n(T)$, where $(e_k)_{k=1}^n$ is the sequence of unit vectors in l_2^n.

(1.3) [GEI] *For all $n \in \mathbf{N}$, $T \in \mathfrak{L}(X, Y)$ and $S \in \Pi_2(Y, Z)$ we have $\Gamma_n(ST) \leq en^{-1/2}\pi_2(S)\Gamma_n(T)$.*

(1.4) [PI1] *Let $2 \leq r < \infty$. Then $\mathfrak{L}^x_{r,1} \subset \Pi_{r,2} \subset \mathfrak{L}^x_{r,\infty}$. Furthermore, there is a constant $c_r \geq 0$ such that $l^x_{r,\infty} \leq \pi_{r,2} \leq c_r l^x_{r,1}$.*

(1.5) [JOH] *Let $n \in \mathbf{N}$ and let E be a Banach space with $\dim E = n$. Then there exists an invertible operator $u \in \mathfrak{L}(l_2^n, E)$ such that $\|u\| = 1$ and $\pi_2(u) = \pi_2(u^{-1}) = \sqrt{n}$.*

For further information on (1.5) see [PS2], [TOJ] and [KÖN].

2. Weak operator ideals. In this chapter we introduce the concept of weak operator ideals. Since most of the proofs of this chapter are technical and not very interesting, we just state our results. Let $(\mathfrak{A}, \alpha)$, $(\mathfrak{B}, \beta)$ be quasi-Banach ideals, X, Y, Z Banach spaces and $0 < q < \infty$. We start with the following definitions:

(2.1) An operator $T \in \mathfrak{L}(X, Y)$ has the *property* $\mathfrak{P}(\alpha, \beta)$ ($T \in \mathfrak{P}(\alpha, \beta)(X, Y)$) if

$$p(\alpha, \beta)(T) := \sup\{\beta(TS) \mid Z \text{ Banach space},\ S \in \mathfrak{A}(Z, X),\ \alpha(S) \leq 1\} < \infty.$$

Observe that $(\mathfrak{P}(\alpha, \beta), p(\alpha, \beta)) = (\mathfrak{B} \circ \mathfrak{A}^{-1}, \beta \circ \alpha^{-1})$.

(2.2) An operator $T\in\mathfrak{L}(X,Y)$ has the *property* $\mathfrak{W}_q(\alpha,\beta)$ (written $T\in\mathfrak{W}_q(\alpha,\beta)(X,Y)$) if

$$\omega_q(\alpha,\beta)(T):=\sup\{\sigma_{q,\infty}(vTu)\mid u\in\mathfrak{A}(l_2,X),\ \alpha(u)\leqslant 1,\ v\in\mathfrak{B}^*(Y,l_2),\ \beta^*(v)\leqslant 1\}<\infty.$$

In the case $\alpha=\|\cdot\|$ we define

$$(\mathfrak{W}_q\mathfrak{B},\ \omega_q\beta):=(\mathfrak{W}_q(\|\cdot\|,\beta),\ \omega_q(\|\cdot\|,\beta)).$$

In the case $q=1$ we define

$$(\mathfrak{W}(\alpha,\beta),\ \omega(\alpha,\beta)):=(\mathfrak{W}_1(\alpha,\beta),\ \omega_1(\alpha,\beta)),\quad (\mathfrak{W}\mathfrak{B},\ \omega\beta):=(\mathfrak{W}_1\mathfrak{B},\ \omega_1\beta).$$

2.3. Proposition. (i) *$(\mathfrak{W}_q(\alpha,\beta),\ \omega_q(\alpha,\beta))$ is a quasi-Banach ideal and for all $1<q<\infty$ there exists an equivalent Banach ideal norm.*

(ii) $\mathfrak{W}_q(\alpha,\beta)=\mathfrak{W}_q(\alpha,\beta^{**})$.

(iii) *$T\in\mathfrak{W}_q(\alpha,\beta)(X,Y)$ if and only if*

$$M(T):=\sup\{n^{1/q}a_n(vTu)\mid n\in\mathbf{N},\ u\in\mathfrak{L}(l_2^n,X),\ \alpha(u)\leqslant 1,\ v\in\mathfrak{L}(Y,l_2^n),\ \beta^*(v)\leqslant 1\}<\infty.$$

In this case we have $\omega_q(\alpha,\beta)(T)=M(T)$.

(iv) $\mathfrak{W}_q(\alpha,\beta)=\mathfrak{P}(\alpha,\omega_q\beta)=(\mathfrak{B}^*)^{-1}\circ\mathfrak{W}_q(\alpha,i_1)$.

(v) *$\mathfrak{P}(\alpha,\beta)\subset\mathfrak{W}_q(\alpha,\beta)$ for all $1\leqslant q<\infty$.*

(vi) *Suppose that for all operators $R\in\mathfrak{L}(Z,Y)$ we have $R\in\mathfrak{B}$ if and only if*

$$\sup\{\beta^{**}(Rw)\mid w\in\mathfrak{L}(l_2,Z),\ \|w\|\leqslant 1\}<\infty.$$

Then $\mathfrak{W}_q(\alpha,\beta)\subset\mathfrak{P}(\alpha,\beta)$ for all $0<q<1$.

2.4. Proposition. (i) *Let $\mathfrak{B}$ be an injective Banach ideal. Then $\mathfrak{W}_q(\alpha,\beta)$ is injective.*

(ii) *Let $\mathfrak{A}$ be an injective Banach ideal. Then $\mathfrak{W}_q(\alpha^{*d},\beta)$ is surjective.*

(iii) *Let $\mathfrak{A}$ be an injective quasi-Banach ideal and $\mathfrak{B}$ an injective or $\mathfrak{B}^*$ a surjective Banach ideal. Then $\mathfrak{W}_q(\alpha,\beta)$ is maximal.*

(iv) *Let $\mathfrak{A}$ be an injective Banach ideal and $\mathfrak{B}$ an injective or $\mathfrak{B}^*$ a surjective Banach ideal. Then $\mathfrak{W}_q(\alpha^{*d},\beta)$ is maximal.*

(v) *Let $\mathfrak{A}$ be an injective Banach ideal and $\mathfrak{B}$ an injective quasi-Banach ideal. Then*

$$\mathfrak{W}_q(\alpha^{*d},\beta)=\mathfrak{W}_q(\beta^{*d},\alpha)^{d}.$$

If additionally $\mathfrak{B}$ is an injective Banach ideal, then

$$\mathfrak{W}_q(\alpha,\beta)^{d}=\mathfrak{W}_q(\beta^{*d},\alpha^{*d}).$$

(vi) *Let $\mathfrak{A}$ be an injective Banach ideal. Then*

$$\mathfrak{W}_q(\alpha^{*d},i_1)=(\mathfrak{W}_q\mathfrak{A})^{d}.$$

2.5. Proposition. *Every maximal quasi-Banach ideal $(\mathfrak{A},\alpha)$ with $\mathfrak{W}\mathfrak{A}=\mathfrak{A}$ admits an equivalent Banach ideal norm.*

Proof. By Proposition 2.3 (v) and (ii) we have $\mathfrak{A}\subset\mathfrak{A}^{**}\subset\mathfrak{W}\mathfrak{A}^{**}=\mathfrak{W}\mathfrak{A}=\mathfrak{A}$. Since $(\mathfrak{A}^{**},\alpha^{**})$ is a Banach ideal, the assertion is proved. ■

3. Some new aspects of Grothendieck numbers. We generalize Geiss' result (1.3) as follows:

3.1. Proposition. *For all $0<r<\infty$, $n\in\mathbf{N}$, $T\in\mathfrak{L}(X,Y)$ and $S\in\mathfrak{L}_{r,\infty}^{x}(Y,Z)$ we have*

$$\Gamma_n(ST)\leqslant(2e)^{1/2+1/r}\,en^{-1/r}\,l_{r,\infty}^{x}(S)\,\Gamma_n(T).$$

Proof. W.l.o.g. we may assume that $\dim Y\geqslant n$. Fix $(x_k)_{k=1}^n\subset B_X$, $(z_k^*)_{k=1}^n\subset B_{Z^*}$ and define

$$u:=\sum_{k=1}^n e_k\otimes x_k\in\mathfrak{L}(l_2^n,X),\qquad v:=\sum_{k=1}^n z_k^*\otimes f_k\in\mathfrak{L}(Z,l_\infty^n),$$

where $(e_k)_{k=1}^n\subset l_2^n$, $(f_k)_{k=1}^n\subset l_\infty^n$ are the unit vectors. Then we have $\|v\|=\sup_{k=1,\ldots,n}\|z_k^*\|\leqslant 1$.

Choose a subspace $Tu(l_2^n)\subset E\subset Y$ with $\dim E=n$. By (1.5) there exists an invertible operator $w\in\mathfrak{L}(E,l_2^n)$ such that $\pi_2(w^{-1})=\pi_2(w)=\sqrt{n}$ and $\|w^{-1}\|=1$. Hence there exists an extension $W\in\mathfrak{L}(Y,l_2^n)$ of w (i.e. $W|_E=w$) with $\pi_2(W)=\sqrt{n}$ (e.g. [PI1]).

The multiplicativity of the Weyl numbers and (1.4) imply

$$\begin{aligned}
a_k(i_{\infty,2}^n vSi_E w^{-1}) &\leqslant x_{2[(k+1)/2]-1}(i_{\infty,2}^n vSi_E w^{-1})\\
&\leqslant x_{[(k+1)/2]}(i_{\infty,2}^n)\,x_{[(k+1)/2]}(S)\,\|v\|\,\|i_E\|\,\|w^{-1}\|\\
&\leqslant\left[\frac{k+1}{2}\right]^{-1/2}\pi_2(i_{\infty,2}^n)\left[\frac{k+1}{2}\right]^{-1/r} l_{r,\infty}^{x}(S)\\
&\leqslant 2^{1/2+1/r}k^{-1/2-1/r}n^{1/2}\,l_{r,\infty}^{x}(S).
\end{aligned}$$

Hence by (1.2)–(1.4) we obtain

$$\begin{aligned}
|\mathrm{Det}(\langle STx_i,z_j^*\rangle)|^{1/n} &=|\mathrm{Det}(\langle i_{\infty,2}^n vSTu(e_i),e_j\rangle)|^{1/n}\\
&=|\mathrm{Det}(\langle i_{\infty,2}^n vSi_E w^{-1}WTu(e_i),e_j\rangle)|^{1/n}\\
&=|\mathrm{Det}(\langle i_{\infty,2}^n vSi_E w^{-1}(e_i),e_j\rangle)|^{1/n}\,|\mathrm{Det}(\langle WTu(e_i),e_j\rangle)|^{1/n}\\
&=\Big(\prod_{k=1}^n a_k(i_{\infty,2}^n vSi_E w^{-1})\Big)^{1/n}|\mathrm{Det}(\langle WTx_i,e_j\rangle)|^{1/n}\\
&\leqslant 2^{1/2+1/r}n^{1/2}\Big(\prod_{k=1}^n k^{-1/2-1/r}\Big)^{1/n} l_{r,\infty}^{x}(S)\,\Gamma_n(WT)\\
&\leqslant 2^{1/2+1/r}n^{1/2}(e/n)^{1/2+1/r}\,l_{r,\infty}^{x}(S)\,en^{-1/2}\pi_2(W)\Gamma_n(T)\\
&=(2e)^{1/2+1/r}\,en^{-1/r}\,l_{r,\infty}^{x}(S)\,\Gamma_n(T).\ ■
\end{aligned}$$

The next proposition is a kind of converse of Proposition 3.1.

3.2. PROPOSITION. *Let* $(\mathfrak{C}, \gamma)$ *be a quasi-Banach ideal and* $T \in \mathfrak{L}(X, Y)$. *Then we have for all* $n \in \mathbf{N}$

(i) $\Gamma_n(T) \leqslant \sup\{\Gamma_n(vT) \mid v \in \mathfrak{C}(Y, l_2),\ \gamma(v) \leqslant \gamma(i^n_{\infty,2})\}$.

(ii) $\Gamma_n(T) \leqslant \sup\{\Gamma_n(Ti_E u) \mid E \subset X,\ \dim E = n,\ u \in \mathfrak{L}(l^n_2, E),\ \gamma^d(u) \leqslant \gamma(i^n_{\infty,2})\}$.

Proof. (ii) Let $(x_k)^n_{k=1} \subset B_X$ and $(y^*_k)^n_{k=1} \subset B_{Y^*}$. Choose a subspace $\{x_k\}^n_{k=1} \subset E \subset X$ with $\dim E = n$ (w.l.o.g. $\dim X \geqslant n$). We define $S := \sum^n_{k=1} f_k \otimes x_k \in \mathfrak{L}(l^n_1, E)$, where $(f_k)^n_{k=1}$ are unit vectors in l^n_∞. Furthermore, let $u := Si^n_{2,1} \in \mathfrak{L}(l^n_2, E)$ and hence

$$\gamma^d(u) \leqslant \gamma^d(i^n_{2,1})\, \|S\| = \gamma(i^n_{\infty,2}) \sup_{k=1,\ldots,n} \|x_k\| \leqslant \gamma(i^n_{\infty,2}).$$

Let $(e_k)^n_{k=1}$ be the sequence of unit vectors in l^n_2. Then

$$|\mathrm{Det}(\langle Tx_i, y^*_j\rangle)|^{1/n} = |\mathrm{Det}(\langle Ti_E u(e_i), y^*_j\rangle)|^{1/n} \leqslant \Gamma_n(Ti_E u).$$

(i) can be proved similarly. ∎

4. A criterion to compute weak norms. The purpose of this chapter is to give a necessary and sufficient criterion for an operator to be in a weak operator ideal.

4.1. PROPOSITION. *Let* $(\mathfrak{A}, \alpha)$, $(\mathfrak{C}, \gamma)$ *be quasi-Banach ideals and let* $(f(n))_{n \in \mathbf{N}}$ *be a positive sequence such that*

$$c := \sup_{n \in \mathbf{N}} f(n) \Big(\prod_{k=1}^n 1/f(k)\Big)^{1/n} < \infty.$$

Then we have for all $T \in \mathfrak{L}(X, Y)$ *the implications* (i)$\Rightarrow$(ii)$\Rightarrow$(iii). *Furthermore, we have the estimates* $M_3(T) \leqslant M_2(T) \leqslant cM_1(T)$.

(i) *There exists a constant* $M_1(T) \geqslant 0$ *such that for all* $u \in \mathfrak{A}(l_2, X)$ *and* $v \in \mathfrak{C}(Y, l_2)$

$$\sup_{k \in \mathbf{N}} f(k)\, a_k(vTu) \leqslant M_1(T)\, \alpha(u)\, \gamma(v).$$

(ii) *There exists a constant* $M_2(T) \geqslant 0$ *such that for all* $u \in \mathfrak{A}(l_2, X)$

$$\sup_{k \in \mathbf{N}} \frac{1}{\gamma(i^k_{\infty,2})} f(k)\, \Gamma_k(Tu) \leqslant M_2(T)\, \alpha(u).$$

(iii) *There exists a constant* $M_3(T) \geqslant 0$ *such that for all* $u \in \mathfrak{A}(l_2, X)$

$$\sup_{k \in \mathbf{N}} \frac{1}{\gamma(i^k_{\infty,2})} f(k)\, a_k(Tu) \leqslant M_3(T)\, \alpha(u).$$

If additionally

$$b := \sup_{k \in \mathbf{N}} \{f(2k)/f(k),\ f(2k-1)/f(k)\} < \infty$$

and $\sup_{k \in \mathbf{N}} \gamma(i^k_{\infty,2})\, h_k(v) \leqslant \gamma(v)$ *for all* $v \in \mathfrak{C}(Y, l_2)$, *then* (i), (ii) *and* (iii) *are equivalent and we have* $(1/b)\, M_1(T) \leqslant M_3(T) \leqslant M_2(T) \leqslant cM_1(T)$.

Proof. (i)$\Rightarrow$(ii). Let $u \in \mathfrak{A}(l_2, X)$. We obtain from Proposition 3.2 and (1.2) for all $n \in \mathbf{N}$

$$\frac{1}{\gamma(i^n_{\infty,2})} f(n)\, \Gamma_n(Tu)$$

$$\leqslant \frac{1}{\gamma(i^n_{\infty,2})} f(n) \sup\Big\{\Big(\prod_{k=1}^n a_k(wTu)\Big)^{1/n} \,\Big|\, w \in \mathfrak{C}(Y, l_2),\ \gamma(w) \leqslant \gamma(i^n_{\infty,2})\Big\}$$

$$\leqslant \frac{1}{\gamma(i^n_{\infty,2})} f(n) \Big(\prod_{k=1}^n 1/f(k)\Big)^{1/n}$$

$$\times \sup\Big\{\sup_{k=1,\ldots,n} f(k)\, a_k(wTu) \,\Big|\, w \in \mathfrak{C}(Y, l_2),\ \gamma(w) \leqslant \gamma(i^n_{\infty,2})\Big\}$$

$$\leqslant cM_1(T)\, \alpha(u).$$

(ii)$\Rightarrow$(iii) follows from (1.1) and we have $M_3(T) \leqslant M_2(T)$.

(iii)$\Rightarrow$(i). Let the additional assumptions hold. Then we obtain for all $u \in \mathfrak{A}(l_2, X)$ and $v \in \mathfrak{C}(Y, l_2)$

$$\sup_{k \in \mathbf{N}} f(k)\, a_k(vTu) \leqslant b \sup_{k \in \mathbf{N}} f\big([(k+1)/2]\big)\, x_{[(k+1)/2]}(v)\, x_{[(k+1)/2]}(Tu)$$

$$\leqslant b\gamma(v) \sup_{k \in \mathbf{N}} \frac{1}{\gamma(i^k_{\infty,2})} f(k)\, x_k(Tu) \leqslant bM_3(T)\, \alpha(u)\, \gamma(v). \ \blacksquare$$

Remark. Using the second part of Proposition 3.2 it is possible to prove a "left-hand" version of the preceding result.

For applications in Chapter 5 the following corollary is useful.

4.2. COROLLARY. *Let* $(\mathfrak{A}, \alpha)$ *be a quasi-Banach ideal. Then for all* $0 < q < \infty$ *and* $T \in \mathfrak{W}_q \mathfrak{A}$

$$\sup_{k \in \mathbf{N}} k^{1/q} \frac{1}{\alpha^*(i^k_{\infty,2})} x_k(T) \leqslant e^{1/q}\, \omega_q \alpha(T).$$

Proof. Choose $(\mathfrak{C}, \gamma) := (\mathfrak{A}^*, \alpha^*)$ and $f(n) := n^{1/q}$, $n \in \mathbf{N}$. Then

$$c := \sup_{n \in \mathbf{N}} f(n) \Big(\prod_{k=1}^n 1/f(k)\Big)^{1/n} = \sup_{n \in \mathbf{N}} n^{1/q} (1/n!)^{1/nq} \leqslant e^{1/q} \quad \text{and}$$

$$\sup_{k \in \mathbf{N}} k^{1/q}\, a_k(vTu) \leqslant \omega_q \alpha(T)\, \|u\|\, \alpha^*(v) \quad \text{for all } u \in \mathfrak{L}(l_2, X) \text{ and } v \in \mathfrak{A}^*(Y, l_2).$$

Proposition 4.1 (i)⇒(iii) yields for all $u \in \mathfrak{L}(l_2, X)$

$$\sup_{k\in\mathbb{N}} k^{1/q} \frac{1}{\alpha^*(i_{\infty,2}^k)} a_k(Tu) \leqslant e^{1/q} \omega_q \alpha(T) \|u\|. \ \blacksquare$$

5. Weak $(r, 2)$**-summing operators.** The following proposition follows immediately from Proposition 2.4.

5.1. PROPOSITION. *Let* $2 \leqslant r < \infty$ *and* $0 < q < \infty$. *Then*

(i) $\mathfrak{W}_q \Pi_{r,2}$ *is an injective and maximal quasi-Banach ideal.*
(ii) $(\mathfrak{W}_q \Pi_{r,2})^{\mathrm{d}} = \mathfrak{W}_q(\pi_{r,2}^{*\mathrm{d}}, i_1)$.

Remark. For all $2 \leqslant r < \infty$ and $v \in \mathfrak{L}(X, l_2)$, $v \in \mathfrak{W}_q(\pi_{r,2}^{*\mathrm{d}}, i_1)(X, l_2)$ if and only if

$$\omega_q(\pi_{r,2}^{*\mathrm{d}}, i_1)(v) = \sup\{\sigma_{q,\infty}(vu) \mid u \in \Pi_{r,2}^{*\mathrm{d}}(l_2, X),\ \pi_{r,2}^{*\mathrm{d}}(u) \leqslant 1\} < \infty.$$

Mascioni [MAS] denotes the above supremum by "$\omega\pi_{r,2}^{\mathrm{d}}(v)$". Hence $(\omega\pi_{r,2})^{\mathrm{d}} =$ "$\omega\pi_{r,2}^{\mathrm{d}}$".

The following lemma was essentially proved by Mascioni [MAS].

5.2. LEMMA. *Let* $v \in \mathfrak{L}(Y, l_2)$ *and* $1 < q, q' < \infty$ *with* $1/q + 1/q' = 1$. *Then*

(i) $l_{q',\infty}^x(v) \leqslant (l_{q,1}^x)^*(v) \leqslant 2 l_{q',\infty}^x(v)$.
(ii) $l_{1,\infty}^x(v) \leqslant i_1(v)$.

Proof. (ii) Let $u \in \mathfrak{F}(l_2, Y)$ with $\|u\| \leqslant 1$. Then $\sup_{k\in\mathbb{N}} k a_k(vu) \leqslant \sigma_1(vu) \leqslant i_1(v)\|u\| \leqslant i_1(v)$. ■

5.3. LEMMA. *There exists a constant* $c > 0$ *such that for all* $n \in \mathbb{N}$

(i) $(l_{2,\infty}^x)^*(i_{1,2}^n) \sim_c (1 + \ln n)$.
(ii) $\omega l_{2,\infty}^x(i_{2,1}^n) \sim_c n/(1 + \ln n)$.

Proof. (i) Let $u \in \mathfrak{L}(l_2^n, l_1^n)$. Hence by (1.4)

$$\begin{aligned}|\mathrm{tr}\, i_{1,2}^n u| &\leqslant \sigma_1(i_{1,2}^n u) \leqslant (1+\ln n)\sigma_{1,\infty}(i_{1,2}^n u)\\ &\leqslant 2(1+\ln n) l_{2,\infty}^x(i_{1,2}^n) l_{2,\infty}^x(u)\\ &\leqslant 2(1+\ln n)\pi_2(i_{1,2}^n) l_{2,\infty}^x(u) \leqslant 2(1+\ln n) l_{2,\infty}^x(u).\end{aligned}$$

To prove the reverse estimate consider the diagonal operator $D_\sigma\colon l_2 \to l_1$, $(x_k)_k \mapsto (k^{-1} x_k)_k$. Since $D_\sigma \in \mathfrak{L}_{2,\infty}^x(l_2, l_1)$ (e.g. [PI1]), we obtain

$$(l_{2,\infty}^x)^*(i_{1,2}^n) \geqslant c^{-1} |\mathrm{tr}\, i_{1,2}^n D_\sigma| \geqslant (3c)^{-1}(1+\ln n).$$

(ii) The lower estimate follows from (i):

$$\frac{1}{c}\frac{n}{1+\ln n} \leqslant \frac{\sigma_{1,\infty}(i_{1,2}^n i_{2,1}^n i_{2,2}^n)}{\|i_{2,2}^n\|\,(l_{2,\infty}^x)^*(i_{1,2}^n)} \leqslant \omega l_{2,\infty}^x(i_{2,1}^n).$$

Since $(\mathfrak{L}_{2,\infty}^x)^*$ is a Banach ideal, $(l_{2,\infty}^x)^*(i_{1,2}^n)(l_{2,\infty}^x)^{**}(i_{2,1}^n) = n$ (e.g. [PI2]). Hence from Proposition 2.3 and (i) we obtain the upper estimate:

$$\omega l_{2,\infty}^x(i_{2,1}^n) = \omega(l_{2,\infty}^x)^{**}(i_{2,1}^n) \leqslant (l_{2,\infty}^x)^{**}(i_{2,1}^n) = \frac{n}{(l_{2,\infty}^x)^*(i_{1,2}^n)} \leqslant c\frac{n}{1+\ln n}. \ \blacksquare$$

5.4. LEMMA. *For all* $0 < r < \infty$ *there is a constant* $c_r > 0$ *such that for all* $n \in \mathbb{N}$

(i)
$$(l_{r,\infty}^x)^*(i_{\infty,2}^n) \sim_{c_r} \begin{cases} \sqrt{n} & \text{if } r < 2,\\ \sqrt{n}(1+\ln n) & \text{if } r = 2,\\ n^{1-1/r} & \text{if } r > 2,\end{cases}$$

(ii)
$$\omega l_{r,\infty}^x(i_{2,\infty}^n) \sim_{c_r} \begin{cases} \sqrt{n} & \text{if } r < 2,\\ \sqrt{n}/(1+\ln n) & \text{if } r = 2,\\ n^{1/r} & \text{if } r > 2.\end{cases}$$

Proof. (i) For all $k \in \{1, \ldots, n\}$ we have by (1.4)

$$x_k(i_{\infty,2}^n) \leqslant \pi_2(i_{\infty,2}^n) k^{-1/2} = (n/k)^{1/2}.$$

This implies for all $u \in \mathfrak{L}_{r,\infty}^x(l_2^n, l_\infty^n)$

$$|\mathrm{tr}\, i_{\infty,2}^n u| \leqslant \sigma_1(i_{\infty,2}^n u) \leqslant 2\sum_{k=1}^n k^{-1/r} x_k(i_{\infty,2}^n) l_{r,\infty}^x(u) \leqslant 2\sqrt{n}\sum_{k=1}^n k^{-1/2-1/r} l_{r,\infty}^x(u).$$

This proves the upper estimate.

If $r < 2$, then $(l_{r,\infty}^x)^*(i_{\infty,2}^n) \geqslant \|i_{\infty,2}^n\| = \sqrt{n}$. If $r > 2$, then

$$(l_{r,\infty}^x)^*(i_{\infty,2}^n) \geqslant \frac{n}{l_{r,\infty}^x(i_{2,\infty}^n)} \geqslant \frac{n}{n^{1/r}\|i_{2,\infty}^n\|} = n^{1-1/r}.$$

Now let $r = 2$. By [PI1], we have $a_n(i_{2,1}^{2n}) = (n+1)^{1/2}$. By Corollary 4.2 and Lemma 5.3

$$\begin{aligned}\frac{n}{(l_{2,\infty}^x)^*(i_{\infty,2}^n)}(n+1)^{1/2} &= \frac{n}{(l_{2,\infty}^x)^*(i_{\infty,2}^n)} x_n(i_{2,1}^{2n})\\ &\leqslant e\omega l_{2,\infty}^x(i_{2,1}^{2n}) \leqslant ec\,2n/(1+\ln 2n).\end{aligned}$$

Therefore $(l_{2,\infty}^x)^*(i_{\infty,2}^n) \geqslant (2ec)^{-1}\sqrt{n}(1+\ln n)$.

(ii) follows from (i) as the corresponding implication in the proof of Lemma 5.3. ■

5.5. THEOREM. *Let* $0 < q, p < \infty$, $1 \leqslant t \leqslant \infty$ *and* $2 \leqslant r < \infty$. *Then*

(i) $\mathfrak{W}_q \mathfrak{L}_{r,t}^x = \mathfrak{W}_q \Pi_{r,2} = \mathfrak{L}$ *if* $1/q + 1/r \leqslant 1$.
(ii) $\mathfrak{W}_q \mathfrak{L}_{r,1}^x = \mathfrak{W}_q \Pi_{r,2} = \mathfrak{L}_{p,\infty}^x$ *if* $1/q + 1/r > 1$ *and* $1 + 1/p = 1/q + 1/r$.

If additionally $r > 2$, *then* $\mathfrak{W}_q \mathfrak{L}_{r,t}^x = \mathfrak{L}_{p,\infty}^x$.

(iii) $\mathfrak{W}_q\,\mathfrak{L} = \mathfrak{L}^x_{p,\infty}$ *if* $0 < q < 1$ *and* $1+1/p = 1/q$.
(iv) $\mathfrak{W}_q\,\mathfrak{L}^s_1 = \mathfrak{L}^h_{q,\infty}$ *for each s-number s.*
(v) $\mathfrak{W}\,\mathfrak{L}^x_{p,\infty} \supsetneq \mathfrak{L}^x_{p,\infty}$ *if* $0 < p \leqslant 2$.

Proof. (i) Let $1/q+1/r \leqslant 1$. Hence $r' \leqslant q$. By Lemma 5.2 we have for all $T \in \mathfrak{L}(X, Y)$, $u \in \mathfrak{L}(l_2, X)$ and $v \in (\mathfrak{L}^x_{r,1})^*(Y, l_2)$

$$\sigma_{q,\infty}(vTu) \leqslant \sigma_{r',\infty}(vTu) \leqslant \|T\|\,\|u\|\, l^x_{r',\infty}(v) \leqslant \|T\|\,\|u\|\,(l^x_{r,1})^*(v).$$

This proves $\mathfrak{L} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,1} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,t} \subset \mathfrak{L}$ and $\mathfrak{L} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,1} \subset \mathfrak{W}_q\,\Pi_{r,2} \subset \mathfrak{L}$.

(ii) Let $1/q+1/r > 1$. By Lemma 5.2 we have for all $T \in \mathfrak{L}^x_{p,\infty}(X, Y)$, $u \in \mathfrak{L}(l_2, X)$ and $v \in (\mathfrak{L}^x_{r,1})^*(Y, l_2)$

$$\sigma_{q,\infty}(vTu) \leqslant 2^{1/q}\, l^x_{p,\infty}(Tu)\, l^x_{r',\infty}(v) \leqslant 2^{1/q}\, l^x_{p,\infty}(T)\,\|u\|\,(l^x_{r,1})^*(v).$$

This shows $\mathfrak{L}^x_{p,\infty} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,1} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,t} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,\infty}$ and, by (1.4), $\mathfrak{L}^x_{p,\infty} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,1} \subset \mathfrak{W}_q\,\Pi_{r,2} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,\infty}$.

Note that $\pi^*_2(i^k_{\infty,2}) = \sqrt{k}$ for all $k \in \mathbf{N}$. By (1.4) and Lemma 5.4 for all $r > 2$ there exists $c_r \geqslant 0$ such that for all $k \in \mathbf{N}$

$$\pi^*_{r,2}(i^k_{\infty,2}) \leqslant (l^x_{r,\infty})^*(i^k_{\infty,2}) \leqslant c_r\, k^{1-1/r}.$$

Hence $\mathfrak{W}_q\,\Pi_2 \subset \mathfrak{L}^x_{p,\infty}$ by Corollary 4.2 and $\mathfrak{W}_q\,\Pi_{r,2} \subset \mathfrak{W}_q\,\mathfrak{L}^x_{r,\infty} \subset \mathfrak{L}^x_{p,\infty}$ for all $r > 2$. This completes the proof of (ii).

(iii) By Lemma 5.2 we have for all $T \in \mathfrak{L}^x_{p,\infty}(X, Y)$, $u \in \mathfrak{L}(l_2, X)$ and $v \in \mathfrak{L}^*(Y, l_2) = \mathfrak{I}_1(Y, l_2)$

$$\sigma_{q,\infty}(vTu) \leqslant 2^{1/q}\, l^x_{p,\infty}(Tu)\, l^x_{1,\infty}(v) \leqslant 2^{1/q}\, l^x_{p,\infty}(T)\,\|u\|\, i_1(v).$$

Hence $\mathfrak{L}^x_{p,\infty} \subset \mathfrak{W}_q\,\mathfrak{L}$. The reverse inclusion follows again from Corollary 4.2, since $i_1(i^k_{\infty,2}) = k$ for all $k \in \mathbf{N}$.

(iv) Observe that $(\mathfrak{L}^s_1)^*(Y, l_2) = \mathfrak{L}(Y, l_2)$. Hence (iv) is obvious.

(v) Since $l^x_{p,\infty}(i^n_{2,\infty}) \sim n^{1/p}$ (e.g. [PI1]), (v) follows from Lemma 5.4. ∎

5.6. Corollary. *The following operator ideals possess an equivalent Banach ideal norm:*

(i) $\mathfrak{L}^h_{p,\infty}$ *if* $1 < p \leqslant \infty$,
(ii) $\mathfrak{L}^x_{p,\infty}$ *and* $\mathfrak{L}^y_{p,\infty}$ *if* $2 < p \leqslant \infty$.

In all other cases of p, there exists no equivalent Banach ideal norm.

Proof. Since by Proposition 2.3 for all $1 < q < \infty$ $\mathfrak{W}_q(\alpha, \beta)$ has an equivalent Banach ideal norm the assertion follows from Theorem 5.5.

The Lorentz spaces $l_{p,\infty}$, $0 < p \leqslant 1$, possess no equivalent norm. Hence the operator ideal $\mathfrak{L}^s_{p,\infty}$ has no equivalent Banach ideal norm for any s-number s.

Lemma 5.4 yields for $\alpha = l^x_{r,\infty}$ that

$$\alpha^*(i^n_{\infty,2})\,\alpha(i^n_{2,\infty}) \sim \begin{cases} n(1+\ln n) & \text{if } r = 2, \\ n^{-1/2+1/r} & \text{if } 1 < r < 2. \end{cases}$$

For Banach ideals $(\mathfrak{W}, \beta)$, by [PI2], $\beta^*(i^n_{\infty,2})\,\beta(i^n_{2,\infty}) = n$. Thus $l^x_{r,\infty}$ cannot be equivalent to any Banach ideal norm for $1 \leqslant r \leqslant 2$. ∎

The following corollary contains a quotient formula for Weyl numbers.

5.7. Corollary. *Let* $0 < p, q, s, t < \infty$, $0 < z \leqslant \infty$ *and* $2 \leqslant r < \infty$ *such that* $1+1/s+1/p = 1/q+1/r = 1+1/t$. *Then*

$$\mathfrak{L}^x_{p,\infty} = \mathfrak{P}(l^x_{s,z}, l^x_{t,\infty}) = \mathfrak{W}_q(l^x_{s,z}, \pi_{r,2}).$$

In particular, $T \in \mathfrak{L}^x_{p,\infty}(X, Y)$ *if and only if there exists a constant* $c \geqslant 0$ *such that* $l^x_{t,\infty}(Tu) \leqslant c\, l^x_{s,z}(u)$ *for all* $u \in \mathfrak{L}^x_{s,z}(l_2, X)$.

Proof. By the multiplicativity of the Weyl numbers, Proposition 2.3 (iv) and Theorem 5.5 we have obviously

$$\mathfrak{L}^x_{p,\infty} \subset \mathfrak{P}(l^x_{s,\infty}, l^x_{t,\infty}) \subset \mathfrak{P}(l^x_{s,z}, l^x_{t,\infty}) = \mathfrak{P}(l^x_{s,z}, \omega_q\,\pi_{r,2}) = \mathfrak{W}_q(l^x_{s,z}, \pi_{r,2}).$$

Let $0 < e, f < \infty$ such that $1/2+1/e = 1+1/t$ and $1/2+1/f = 1+1/p$. Then again by Proposition 2.3 (iv) and Theorem 5.5

$$\mathfrak{W}_q(l^x_{s,z}, \pi_{r,2}) = \mathfrak{P}(l^x_{s,z}, l^x_{t,\infty}) = \mathfrak{W}_e(l^x_{s,z}, \pi_2) \quad\text{and}\quad \mathfrak{W}_f(\|\cdot\|, \pi_2) = \mathfrak{L}^x_{p,\infty}.$$

So it remains to show that $\mathfrak{W}_e(l^x_{s,z}, \pi_2) \subset \mathfrak{W}_f(\|\cdot\|, \pi_2)$.

For this, let $T \in \mathfrak{W}_e(l^x_{s,z}, \pi_2)(X, Y)$, $n \in \mathbf{N}$, $u \in \mathfrak{L}(l^n_2, X)$ and $v \in \mathfrak{L}(Y, l^n_2)$. Then

$$\begin{aligned} n^{1/f}\, a_n(vTu) &\leqslant n^{1/f-1/e}\,\sigma_{e,\infty}(vTu) \leqslant n^{1/f-1/e}\,\omega_e(l^x_{s,z}, \pi_2)(T)\, l^x_{s,z}(u)\,\pi_2(v) \\ &\leqslant c n^{1/f-1/e+1/s}\,\omega_e(l^x_{s,z}, \pi_2)(T)\,\|u\|\,\pi_2(v) \\ &= c\omega_e(l^x_{s,z}, \pi_2)\,\|u\|\,\pi_2(v), \end{aligned}$$

where the constant $c \geqslant 0$ only depends on s and z. Hence Proposition 2.3 (iii) completes the proof. ∎

We finish this chapter with the following easy proposition:

5.8. Proposition. *For all* $0 < q < \infty$ *and* $1 \leqslant p \leqslant 2$ *we have* $\mathfrak{W}_q\,\Pi_p = \mathfrak{W}_q\,\Pi_2$.

Proof. We only have to observe that by the Grothendieck–Pietsch factorization theorem $\Pi_p(l_2, X) = \Pi_2(l_2, X)$ for all $1 \leqslant p \leqslant 2$. ∎

6. Some applications to weak Hilbert spaces. Weak Hilbert spaces were introduced and investigated by Pisier [PS1]. We use one of their various characterizations as a formal definition:

(6.1) We call $\mathfrak{W}(\pi^d_2, \pi_2)$ the ideal of *weak Hilbert operators.* A Banach space X is called a *weak Hilbert space* if id_X is a weak Hilbert operator.

Proposition 2.4 yields immediately the following

6.2. PROPOSITION. *For all $0 < q < \infty$ the quasi-Banach ideal $\mathfrak{W}_q(\pi_2^{\mathrm{d}}, \pi_2)$ is injective, surjective, maximal and completely symmetric.*

The next theorem is a weak version of Grothendieck's inequality.

6.3. PROPOSITION. *A Banach space X is a weak Hilbert space if and only if $\mathfrak{W}\Pi_1(l_1, X) = \mathfrak{L}(l_1, X)$.*

Proof. First of all note that by the "little Grothendieck theorem" $\Pi_2^{\mathrm{d}}(l_2, L) = \mathfrak{L}(l_2, L)$ for each $\mathfrak{L}_1^{\lambda}$-space L and therefore we have $\mathfrak{W}(\pi_2^{\mathrm{d}}, \pi_2)(L, X) = \mathfrak{W}\Pi_2(L, X)$ for each Banach space X.
Now let X be a weak Hilbert space. Then by the ideal properties and Proposition 5.8

$$\mathfrak{L}(l_1, X) = \mathfrak{W}(\pi_2^{\mathrm{d}}, \pi_2)(l_1, X) = \mathfrak{W}\Pi_2(l_1, X) = \mathfrak{W}\Pi_1(l_1, X).$$

For the reverse direction let X be a Banach space such that $\mathfrak{W}\Pi_1(l_1, X) = \mathfrak{L}(l_1, X)$. By the maximality of $\mathfrak{W}\Pi_2$ (Proposition 5.1) and by Proposition 5.8 we obtain therefore

$$\mathfrak{W}(\pi_2^{\mathrm{d}}, \pi_2)(l_1(B_X), X) = \mathfrak{W}\Pi_2(l_1(B_X), X) = \mathfrak{L}(l_1(B_X), X).$$

But this implies that the canonical surjection $Q: l_1(B_X) \to X$ is a weak Hilbert operator. Hence the surjectivity of $\mathfrak{W}(\pi_2^{\mathrm{d}}, \pi_2)$ (Proposition 6.2) completes the proof. ∎

Now we want to characterize weak Hilbert spaces only by Weyl numbers and integral operators.

6.4. THEOREM. *Let $0 < q, p < \infty$ such that $1/p = 1 + 1/q$. Then*

$$\mathfrak{W}_q(\pi_2^{\mathrm{d}}, \pi_2) = \mathfrak{W}_p(i_1, \|\cdot\|) = \mathfrak{P}(i_1, l_{q,\infty}^x).$$

In particular, for all Banach spaces X the following are equivalent:

(1) *X is a weak Hilbert space.*
(2) *There exists a constant $c \geq 0$ such that for all $u \in \mathfrak{I}_1(l_2, X)$ and $v \in \mathfrak{I}_1(Y, l_2)$*

$$\sup_{k \in \mathbb{N}} k^2 a_k(vu) \leq c i_1(u) i_1(v).$$

(3) *There exists a constant $c \geq 0$ such that for all $u \in \mathfrak{I}_1(l_2, X)$*

$$\sup_{k \in \mathbb{N}} k a_k(u) \leq c i_1(u).$$

Proof. Let $0 < r, s, t < \infty$ such that $1/r = 1 + 1/s = 1/2 + 1/q$. Then Proposition 2.4 (v) and Theorem 5.5 imply

$$\begin{aligned}\mathfrak{W}_q(\pi_2^{\mathrm{d}}, \pi_2) &= \mathfrak{P}(\pi_2^{\mathrm{d}}, \omega_q \pi_2) = \mathfrak{P}(\pi_2^{\mathrm{d}}, l_{s,\infty}^x) = \mathfrak{P}(\pi_2^{\mathrm{d}}, \omega_r \|\cdot\|) = \mathfrak{W}_r(\pi_2^{\mathrm{d}}, \|\cdot\|)\\ &= \mathfrak{W}_r(\pi_2^{*\mathrm{d}}, \|\cdot\|) = \mathfrak{W}_r(i_1, \pi_2)^{\mathrm{d}} = \mathfrak{P}(i_1, \omega_r \pi_2)^{\mathrm{d}} = \mathfrak{P}(i_1, l_{q,\infty}^x)^{\mathrm{d}}\\ &= \mathfrak{P}(i_1, \omega_p \|\cdot\|)^{\mathrm{d}} = \mathfrak{W}_p(i_1, \|\cdot\|)^{\mathrm{d}} = \mathfrak{W}_p(i_1, \|\cdot\|).\end{aligned}$$

Now let $T \in \mathfrak{W}_p(i_1, \|\cdot\|)(X, Y)$. Hence we have for all $u \in \mathfrak{I}_1(l_2, X)$ and $v \in \mathfrak{I}_1(Y, l_2)$

$$\sup_{k \in \mathbb{N}} k^{1/p} a_k(vTu) \leq \omega_p(i_1, \|\cdot\|) i_1(u) i_1(v).$$

Since $i_1(i_{\infty,2}^k) = k$ for all $k \in \mathbb{N}$ and $l_{1,\infty}^h(v) \leq i_1(v)$ for all $v \in \mathfrak{I}_1(Y, l_2)$ (Lemma 5.2), by Proposition 4.1 (i)⇔(iii) the inequality above is equivalent to: there exists a constant $c \geq 0$ such that for all $u \in \mathfrak{I}_1(l_2, X)$

$$\sup_{k \in \mathbb{N}} k^{1/p - 1} a_k(Tu) \leq c i_1(u).$$

But this means $T \in \mathfrak{P}(i_1, l_{q,\infty}^x)(X, Y)$. ∎

The following proposition is proved in the special case of weak Hilbert spaces by Pisier [PS1]. The general case was obtained by Pietsch [PI3].

6.5. PROPOSITION. *Let $0 < q < \infty$ and $T \in \mathfrak{L}$. Then $T \in \mathfrak{W}_q(\pi_2^{\mathrm{d}}, \pi_2)$ if and only if*

$$G_q(T) := \sup_{k \in \mathbb{N}} k^{1/q - 1} \Gamma_k(T) < \infty.$$

Remark. Using Proposition 4.1, Proposition 3.2 and (1.3) we get a different proof of Proposition 6.5, yielding the following estimates:

$$G_q(T) \leq \omega_q(\pi_2^{\mathrm{d}}, \pi_2)(T) \leq 2^{1/q} e G_q(T).$$

In the last theorem we characterize weak Hilbert spaces by Weyl and Hilbert numbers.

6.6. THEOREM. *Let $0 < q, s < \infty$ and $1 \leq r \leq 2$ such that $1 + 1/s = 1/q + 1/r$. Then*

$$\mathfrak{W}_q(\pi_2^{\mathrm{d}}, \pi_2) = \mathfrak{P}(l_{r,\infty}^h, l_{s,\infty}^x).$$

In particular, for all Banach spaces X the following are equivalent:

(1) *X is a weak Hilbert space.*
(2) *For all $1 \leq r \leq 2$ there exists a constant $c \geq 0$ such that for all $u \in \mathfrak{L}_{r,\infty}^h(l_2, X)$*

$$l_{r,\infty}^x(u) \leq c l_{r,\infty}^h(u).$$

(3) *There exist $1 \leq r \leq 2$ and a constant $c \geq 0$ such that for all $u \in \mathfrak{L}_{r,\infty}^h(l_2, X)$*

$$l_{r,\infty}^x(u) \leq c l_{r,\infty}^h(u).$$

Proof. Let $T \in \mathfrak{P}(l_{r,\infty}^h, l_{s,\infty}^x)(X, Y)$. If $r > 1$, we have by Lemma 5.2 and (1.4) for all $u \in \mathfrak{L}_{r,\infty}^h(l_2, X)$

$$l_{s,\infty}^x(Tu) \leq c l_{r,\infty}^h(u) = c l_{r,\infty}^h(u^*) = c l_{r,\infty}^x(u^*) \leq c (l_{r',1}^x)^*(u^*) \leq c c_r \pi_{r',2}^{*\mathrm{d}}(u).$$

Let $0 < p < \infty$ such that $1/2+1/p = 1+1/s$. Hence by Propositions 2.3 and 2.4, Theorem 5.5 and Proposition 6.2

$$\mathfrak{P}(l^h_{r,\infty}, l^x_{s,\infty}) \subset \mathfrak{P}(\pi^{*\mathrm{d}}_{r',2}, l^x_{s,\infty}) = \mathfrak{P}(\pi^{*\mathrm{d}}_{r',2}, \omega_p \pi_2) = \mathfrak{W}_p(\pi^{*\mathrm{d}}_{r',2}, \pi_2)$$
$$= \mathfrak{W}_p(\pi^{\mathrm{d}}_2, \pi_{r',2})^{\mathrm{d}} = \mathfrak{P}(\pi^{\mathrm{d}}_2, \omega_p \pi_{r',2})^{\mathrm{d}} = \mathfrak{P}(\pi^{\mathrm{d}}_2, \omega_q \pi_2)^{\mathrm{d}}$$
$$= \mathfrak{W}_q(\pi^{\mathrm{d}}_2, \pi_2)^{\mathrm{d}} = \mathfrak{W}_q(\pi^{\mathrm{d}}_2, \pi_2).$$

In the case $r = 1$, by Lemma 5.2 we have $l^x_{s,\infty}(Tu) \leqslant c l^h_{1,\infty}(u) \leqslant c i_1(u)$ for all $u \in \mathfrak{L}^h_{1,\infty}(l_2, X)$. Hence $T \in \mathfrak{W}_s(\pi^{\mathrm{d}}_2, \pi_2) = \mathfrak{W}_q(\pi^{\mathrm{d}}_2, \pi_2)$ by Theorem 6.4.

Conversely, let $T \in \mathfrak{W}_q(\pi^{\mathrm{d}}_2, \pi_2)(X, Y)$. Hence by Proposition 6.5 there exists a constant $c \geqslant 0$ such that $\sup_{k\in\mathbf{N}} k^{1/q-1} \Gamma_k(T) \leqslant c$.

Now let $S \in \mathfrak{L}^h_{r,\infty}(Z, X)$ and $u \in \mathfrak{L}(l_2, X)$. Then we obtain by (1.1) and Proposition 3.1

$$n^{1/s} a_n(TSu) = n^{1/s} c_n(TSu) \leqslant n^{1/s} \Gamma_n(TSu) = n^{1/s} \Gamma_n((Su)^* T^*)$$
$$\leqslant n^{1/s} (2e)^{1/2+1/r} e n^{-1/r} l^x_{r,\infty}((Su)^*) \Gamma_n(T^*)$$
$$\leqslant (2e)^{1/2+1/r} e \|u\| l^h_{r,\infty}(S^*) n^{1/q-1} \Gamma_n(T)$$
$$\leqslant (2e)^{1/2+1/r} e c l^h_{r,\infty}(S) \|u\|. \ \blacksquare$$

Remark. 1) Theorem 6.6 yields that the eigenvalue type of $\mathfrak{W}_q(\pi^{\mathrm{d}}_2, \pi_2) \circ \mathfrak{L}^h_{r,\infty}$ is $l_{s,\infty}$. This implies the known eigenvalue estimate for weak Hilbert spaces ([PS1]).

2) Since $\mathfrak{L}^x_{2,\infty} = \mathfrak{W}\Pi_2$ by Theorem 5.5 and $\mathfrak{L}^h_{2,\infty}(l_2, X) = \mathfrak{L}^y_{2,\infty}(l_2, X) = (\mathfrak{W}\Pi_2)^{\mathrm{d}}(l_2, X)$ we have the following reformulation of Theorem 6.6:

X is a weak Hilbert space if and only if there is a constant $c \geqslant 0$ such that $\omega\pi_2(u) \leqslant c(\omega\pi_2)^{\mathrm{d}}(u)$ for all $u \in (\mathfrak{W}\Pi_2)^{\mathrm{d}}(l_2, X)$.

This is the analogue of the well-known Hilbert space characterization, due to Kwapień [KWA]: X is a Hilbert space if and only if there is a constant $c \geqslant 0$ such that $\pi_2(u) \leqslant c\pi^{\mathrm{d}}_2(u)$ for all $u \in \Pi^{\mathrm{d}}_2(l_2, X)$.

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