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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

On the essential spectrum and eigenvalue asymptotics of certain Schrödinger operators

by

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Abstract. The Fefferman estimate is used to examine the position of the essential spectrum and to obtain integral estimates for the number of eigenvalues for a wide range of Schrödinger operators.

Introduction. The Bohr–Sommerfeld quantization principle, according to which volume $\sim h^d$ in phase space corresponds to one bound state of the quantum system, has been fully mathematically expressed in the form of the Cwikel–Lieb–Rosenblum inequality.

For the Schrödinger operator $-\Delta+V$, denote the dimension of the image of the spectral projector $P(-\infty, \lambda)$ by $N(\lambda, V)$ and let $\mathrm{Vol}(\lambda, V) = |\{(x, \xi)\colon \xi^2+V(x) < \lambda\}|$.

THE CWIKEL–LIEB–ROSENBLUM INEQUALITY [7]. *For $d \geqslant 3$, there exists a constant $C = C(d)$ such that for every potential V on $\mathbf{R}^d$ and for every λ, $N(\lambda, V) \leqslant C\ \mathrm{Vol}(\lambda, V)$.*

The right side of this inequality is a "good" estimate only if $\mathrm{Vol}(\lambda, V) < \infty$. There exists a wide range of potentials (e.g. $V(x) = x_1^2 x_2^2 \ldots x_d^2$) for which $\mathrm{Vol}(\lambda, V) = \infty$ and $N(\lambda, V) < \infty$.

A more flexible version of the Heisenberg uncertainty principle, the Fefferman SAK-principle (which says that each box $B = \{|x-x_0| < \delta, |\xi-\xi_0| < \delta^{-1}\}$ which fits inside $\{(x, \xi)\colon \xi^2+V(x) < \lambda\}$ should count for one eigenvalue of the Schrödinger operator) is an idea which, together with some subtle techniques of harmonic analysis, makes it possible to obtain sharp estimates for the number of eigenvalues for nonpositive potentials.

THE FEFFERMAN INEQUALITY [1]. *For $d \geqslant 3$, $1 < p \leqslant d/2$, there exist constants $S = S(p, d)$, $A = A(p, d)$, $K = K(p, d)$ such that for every potential*

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$V \leqslant 0$ on $\mathbf{R}^d$ and every $E > 0$, if we denote by $N_p(E, V)$ the maximal number of disjoint balls B with radii r not larger than $E^{-1/2}$ for which

$$\Big(|B|^{-1} \int_B |V|^p\Big)^{1/p} \geqslant Sr^{-2},$$

then $N(-E, V) \leqslant AN_p(KE, V)$.

The aim of this paper is to use the Fefferman inequality to determine the position of the essential spectrum and to obtain integral estimates for the number of eigenvalues for Schrödinger operators. All the theorems of this paper concern the dimension $d \geqslant 3$ and the potential V on $\mathbf{R}^d$ for which:

(1) There exists a common core (an essential domain) for the operators $-\Delta$, $-\Delta+V$.

(2) There exists $E \in \boldsymbol{R}$ such that $-\Delta+V \geqslant E$.

The first condition has been thoroughly explored for $V \geqslant 0$ in [4]. Details concerning other potentials are found in [5]. The second condition will be discussed in a forthcoming paper.

Throughout the paper, the letters S, A, K are reserved for constants which appear in the Fefferman inequality. B is always used to denote a ball with radius r.

The position of the essential spectrum of Schrödinger operators. If H is a selfadjoint operator, $H \geqslant a > -\infty$ and

$$\mu_n(H) = \sup_{\dim K = n-1} \inf_{\substack{f \in D(H), f \perp K \\ \|f\| = 1}} (Hf, f)$$

then the mini-max principle says that either

(1) there exist n eigenvalues of H (counted in increasing order according to their multiplicities) and μ_n is the nth eigenvalue, or

(2) μ_n is the infimum of the essential spectrum of H, and then $\mu_n = \mu_{n+1} = \ldots$, and there exist at most $n-1$ eigenvalues of H (counted according to their multiplicities) less than μ_n.

The mini-max principle and the Weyl theorem "about the essential spectrum" [6] lead to the well-known estimate

$$\lim_R \inf_{|x| \geqslant R} V(x) \leqslant \inf \sigma_{\mathrm{ess}}(-\Delta+V),$$

which in the case

$$\lim_R \inf_{|x| \geqslant R} V(x) = \lim_R \sup_{|x| \geqslant R} V(x) = a$$

gives the equality $\sigma_{\mathrm{ess}}(-\Delta+V) = [a, \infty)$.

The Fefferman estimate makes it possible to extend this mini-max method of determining the position of the essential spectrum to a considerably wider range of potentials. The assumptions of the Fefferman inequality suggest the following definition:

DEFINITION 1. Let Ω be a domain in $\mathbf{R}^d$, $r, \varepsilon > 0$. Set

$$R_\Omega(r, \varepsilon) = \inf\{R: \sup_{|x| \geqslant R} |\{y: |y| < r, \ x-y \in \Omega\}| < \varepsilon\}.$$

Call Ω a *tight domain* (written $\Omega \in \mathrm{Cs}$) if there exists $r_0 > 0$ such that $R_\Omega(r_0, \varepsilon) < \infty$ for every $\varepsilon > 0$.

It is easy to notice that if Ω is tight then $R_\Omega(r, \varepsilon) < \infty$ for any $r, \varepsilon > 0$. Let δ_Ω denote the indicator of Ω, i.e.

$$\delta_\Omega(x) = \begin{cases} 1 & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega. \end{cases}$$

LEMMA 1. *If Ω is a tight domain, then for all $t > 0$ and $E > 0$*

$$N(-E, -t\delta_\Omega) < \infty.$$

Proof. Let us fix $1 < p \leqslant d/2$ and examine the number of disjoint balls B with radii not larger than h, satisfying for an $\varepsilon > 0$ the inequality

$$|B \cap \Omega|/|B| \geqslant \varepsilon r^{-2p}.$$

Then obviously $r \geqslant \varepsilon^{1/2p}$ and therefore for any such ball

$$|B \cap \Omega| \geqslant C\varepsilon^{d/(2p)+1} h^{-2p}.$$

Since Ω is a tight domain, the number of disjoint balls satisfying the last condition is finite; putting $\varepsilon = S^p/t^p$, $h = K^{-1/2}E^{-1/2}$ and making use of the Fefferman theorem, we finish the proof of the lemma.

Define

$$\liminf_{\mathrm{Cs}} V = \sup_{\Omega \in \mathrm{Cs}} \inf_{x \in \Omega^c} V(x).$$

PROPOSITION 1. *For every potential V*

$$\liminf_{\mathrm{Cs}} V \leqslant \inf \sigma_{\mathrm{ess}}(-\Delta+V).$$

Proof. For $t < \liminf_{\mathrm{Cs}} V$, there exists $\Omega_t \in \mathrm{Cs}$ such that $V(x) \geqslant t$ for $x \notin \Omega_t$. If f is in the common core of the operators $-\Delta$, $-\Delta+V$ and $\|f\| = 1$ then

$$((-\Delta+V)f, f) \geqslant t + ((-\Delta - t\delta_{\Omega_t})f, f).$$

So $\mu_n(-\Delta+V) \geqslant t + \mu_n(-\Delta - t\delta_{\Omega_t})$.

For any $\varepsilon > 0$, by Lemma 1, $N(-\varepsilon, -t\delta_{\Omega_t}) < \infty$, and so $\mu_n(-\Delta+V) \geqslant t-\varepsilon$ for sufficiently large n. By the free choice of $t < \liminf_{\mathrm{Cs}} V$

and $\varepsilon > 0$, we see that there exists n_0 such that $\mu_n(-\Delta+V) \geqslant t$ for $n \geqslant n_0$. Therefore, according to the mini-max principle, $\liminf_{\mathrm{Cs}} V \leqslant \inf \sigma_{\mathrm{ess}}(-\Delta+V)$.

Let Cs_0 denote the class of those tight domains Ω for which the complement Ω^c contains balls with radii of arbitrary length, and let

$$\limsup_{\mathrm{Cs}_0} V = \inf_{\Omega \in \mathrm{Cs}_0} \sup_{x \in \Omega^c} V(x).$$

PROPOSITION 2. $\inf \sigma_{\mathrm{ess}}(-\Delta+V) \leqslant \limsup_{\mathrm{Cs}_0} V$.

Proof. If $t > \limsup_{\mathrm{Cs}_0} V$, then there exists $\Omega_t \in \mathrm{Cs}_0$ such that $\sup_{x \in \Omega_t^c} V(x) < t$. Fix $\varepsilon > 0$ and a natural number n. Let $B_1, \ldots, B_n$ be a family of disjoint balls with radius $\varepsilon^{-1/2}$ included in Ω_t^c. Translating and dilating a fixed smooth function φ_0 with support in the unit ball, we obtain a sequence of smooth functions $\varphi_1, \ldots, \varphi_n$ such that $\operatorname{supp} \varphi_j \subset B_j$ for $j = 1, \ldots, n$. If $f = \sum_1^n a_j \varphi_j$ then

$$\begin{aligned}((-\Delta+V)f, f) &\leqslant \sum (|a_j|^2 \|\nabla \varphi_j\|^2 + |a_j|^2 (V\varphi_j, \varphi_j)) \\ &\leqslant \sum C\varepsilon |a_j|^2 \|\varphi_j\|^2 + t\|f\|^2 = (C\varepsilon + t)\|f\|^2.\end{aligned}$$

By the free choice of $\varepsilon > 0$ and $t > \limsup_{\mathrm{Cs}_0} V$ we come to the conclusion that for any natural n there exists an n-dimensional subspace $\mathscr{H} \subset L^2$ such that $((-\Delta+V)f, f) \leqslant t\|f\|^2$ for $f \in \mathscr{H}$. Therefore the image of the spectral projector $P(-\infty, t]$ has infinite dimension, i.e. $t \geqslant \inf \sigma_{\mathrm{ess}}(-\Delta+V)$, which finishes the proof.

PROPOSITION 3. *If* $\liminf_{\mathrm{Cs}_0} V = \limsup_{\mathrm{Cs}_0} V = a$ *then* $\sigma_{\mathrm{ess}}(-\Delta+V) = [a, \infty)$.

Proof. We may assume that $a = 0$. For any natural n choose domains $\Omega_-(n), \Omega_+(n) \in \mathrm{Cs}_0$ such that

$$\inf_{\Omega_-(n)^c} V \geqslant -1/n, \qquad \sup_{\Omega_+(n)^c} V \leqslant 1/n.$$

Let $x_n \in \mathbf{R}^d$ be such that $\{x\colon |x-x_n| < n^{1/2}\} \subset \Omega_-(n)^c \cap \Omega_+(n)^c$ (x_n exists by the Cs_0-condition). Take a function $\eta \geqslant 0$ such that $\eta(x) = 1$ for $|x| \leqslant 1/2$, $\eta(x) = 0$ for $|x| > 1$. Let $\eta_n(x) = \eta(n^{-1/2}(x-x_n))$. Then

$$|\nabla \eta_n| \leqslant Cn^{-1/2}, \qquad |\Delta \eta_n| \leqslant Cn^{-1}. \tag{1}$$

For fixed $\lambda \geqslant 0$ choose $k \in \mathbf{R}^d$ such that $|k| = \lambda^{1/2}$ and define $\varphi_n(x) = e^{ikx}\eta_n(x)$. We have

$$\limsup_n \{|V(x)|\colon x \in \operatorname{supp} \varphi_n\} = 0, \qquad \|\varphi_n\| = Cn^{-1/2}. \tag{2}$$

A straightforward calculation gives

$$(-\Delta+V-\lambda)\varphi_n = e^{ikx}((-\Delta+V)\eta_n - ik\nabla\eta_n).$$

So by (1) and (2)

$$\limsup_n |(-\Delta+V-\lambda)\varphi_n| = 0,$$

which implies that

$$\lim_n \|\varphi_n\|^{-1} \|(-\Delta+V-\lambda)\varphi_n\| = 0$$

and so $[0, \infty) \subset \sigma(-\Delta+V)$, which together with Proposition 1 proves the assertion.

Nontrivial examples of tight domains are given by

LEMMA 2. *If p is a polynomial on $\mathbf{R}^d$ such that for every $x \neq 0$ there exists y with $p(x+y) \neq p(y)$ then for every $\lambda > 0$ the domain $\Omega_\lambda = \{x\colon |p(x)| < \lambda\}$ is tight.*

Proof. We first show that the operator $-\Delta + p^2$ has a compact resolvent. Set $\sigma_x p(y) = p(x+y)$, $W = \operatorname{lin}\{\sigma_x p\colon x \in \mathbf{R}^d\}$. We define a multiplication on the product $\mathbf{R}^d \times W$ by

$$(x_1, w_1)(x_2, w_2) = (x_1+x_2, w_1+\sigma_{x_1} w_2).$$

It is easy to see that $\mathbf{R}^d \times W$ with this multiplication is a finite-dimensional nilpotent Lie group whose Lie algebra is isomorphic to the Lie algebra of unbounded operators on $L^2(\mathbf{R}^d)$ of the form

$$\partial_x + iw, \qquad x \in \mathbf{R}^d, \ w \in W.$$

Let χ be the unitary character of W given by

$$\chi\colon w \mapsto \exp(iw(0)).$$

χ induces a representation π of $\mathbf{R}^d \times W$ in $L^2(\mathbf{R}^d)$. One can see that for any $f \in L^2$, $x, y \in \mathbf{R}^d$, $w \in W$

$$\pi_{(x,w)} f(y) = e^{iw(y)} f(y+x).$$

By the Kirillov theory [3], the representation π is irreducible if and only if for every x there exists $w \in W$ such that $\partial_x w \neq 0$.

Suppose that π is not irreducible. Let x be such that $\partial_x w = 0$ for every w. Then, in particular, $\partial_x^n w = 0$, i.e. for every $n \in \mathbf{N}$ and $y \in \mathbf{R}^d$, $\partial_x^n \sigma_y p(0) = 0$. Using the Taylor formula, we see that

$$p(x+y) = \sum k!^{-1} \partial_x^k \sigma_y p(0) = p(y),$$

contrary to the assumption of the lemma. Therefore π is an irreducible representation.

Let $\mathfrak{g}$ be the free nilpotent Lie algebra of the same nilpotence class as $\mathbf{R}^d \times W$, and let $X_1, \ldots, X_{d+1}$ be a system of free generators of $\mathfrak{g}$. The correspondence

$$X_j \mapsto \partial_j, \quad j = 1, \ldots, d, \qquad X_{d+1} \mapsto ip$$

defines a homomorphism h: $\exp(\mathfrak{g}) \to \mathbf{R}^d \times W$ which is onto.

Set $\tilde{\pi} = \pi \circ h$, $L = -\sum X_j^2$. Then $d\tilde{\pi}(L) = -\Delta + p^2$. The operator $(L-z)^{-1}$ is the convolution with an integrable function for any $z \notin [0, \infty)$ [2]. According to the Kirillov theory, the image of an operator of convolution with an integrable function by an irreducible representation of a nilpotent Lie group is a compact operator. Hence the resolvent of $-\Delta + p^2$ is compact.

Note that this implies that for all ε, $a > 0$ the number of disjoint balls with radius ε included in $\{x: p(x)^2 < a\}$ is finite. Indeed, if there are n such disjoint balls, then dilating and translating a fixed smoth function with support in the unit ball we conclude, in the same way as in the proof of Proposition 2, that there exists an n-dimensional subspace $\mathscr{H} \subset L^2$ such that for every $f \in \mathscr{H}$

$$((-\Delta + p^2) f, f) \leqslant (C\varepsilon^{-1/2} + a) \|f\|^2.$$

If there existed an infinite number of such balls, then the dimension of the image of the spectral projector $P(0, C\varepsilon^{-1/2} + a]$ would be infinite, which contradicts the compactness of the resolvent of $-\Delta + p^2$.

Let $U = \operatorname{int} \bar{U}$ be a connected subset of a ball B such that each connected component of ∂U is smooth. If, for a $\delta > 0$, no ball with radius δ is included in U, then comparing the curvature of the components of ∂U to the curvature of the sphere, we notice that at least one of the following properties is true:

(1) there exists a ball $b \subset B$ for which $b \cap U$ is disconnected,

(2) $|U| \leqslant \tau_d (r^d - (r-\delta)^d)$.

Since p is a polynomial, the set of those points of the hypersurface $\{x: p(x)^2 = \lambda\}$ at which the tangent component of the gradient $\nabla(p^2)$ vanishes does not contain an infinite set of isolated points. We may therefore find pairwise disjoint domains $U_1, \ldots, U_n$ such that for every $1 \leqslant j \leqslant n$ and every ball B, $B \cap U_j$ is connected and $\overline{\bigcup U_j} = \{x: p(x)^2 \leqslant \lambda\}$. Thus the Dirichlet box principle finishes the proof.

Combining Lemma 2 and Propositions 1–3 we obtain

THEOREM 1. *If p is a polynomial satisfying the assumptions of Lemma 2 and V is a potential on $\mathbf{R}^d$, then*

$$\lim_R \inf_{|p(x)| \geqslant R} V(x) \leqslant \inf \sigma_{\text{ess}}(-\Delta + V) \leqslant \lim_R \sup_{|p(x)| \geqslant R} V(x).$$

Moreover, if $\liminf V = \limsup V = a$ *then* $\sigma_{\text{ess}}(-\Delta + V) = [a, \infty)$.

Integral estimates for the number of eigenvalues. Let $V \leqslant 0$ be a locally p-integrable function and let B be the ball with centre at zero and radius r. Fix $0 < b < S$ and put

$$E_p = E_p(V) = \sup_{x,r} \Big(\big(|B|^{-1} \int_B |V(x-y)|^p \, dy \big)^{1/p} - (S-b) r^{-2} \Big).$$

If $E_p < \infty$ then for every $r < b^{1/2} E_p^{-1/2}$ and every $x \in \mathbf{R}^d$

$$\big(|B|^{-1} \int_B |V(x-y)|^p \, dy \big)^{1/p} < S r^{-2}.$$

Therefore, for any $\varepsilon > 0$

$$N(K^{-1} b^{-1} (E_p + \varepsilon), V) \leqslant A N_p (b^{1/2} (E_p + \varepsilon)^{-1/2}, V) = 0,$$

i.e. $-\Delta + V \geqslant -C E_p(V)$ (see [1]).

Fix $1 < p \leqslant d/2$ and a potential V on $\mathbf{R}^d$. Using the Fefferman inequality, we may obtain integral estimates for $N(\lambda, V)$. To this end, let us introduce the following notation:

$$\delta_R(x) = 1 \quad \text{if } |x| < R, \qquad \delta_R(x) = 0 \quad \text{if } |x| \geqslant R,$$

$$V_\lambda = \min(V - \lambda, 0), \qquad E(\lambda | R) = E_p(\delta_R V_\lambda), \qquad E(\lambda) = E(\lambda | \infty),$$

$$\sigma_x V(y) = V(x+y), \qquad \operatorname{Vol}(\lambda, V | R) = \big| \{ (\xi, x): |x| < R, \ |\xi|^{d/p} + V(x) < \lambda \} \big|.$$

THEOREM 2. *There exist constants $s = s(p, d)$, $a = a(p, d)$, $k = k(p, d)$ such that, for every $\varepsilon > 0$, if for fixed $R > 0$*

$$\sup_{|x| \geqslant R} \operatorname{Vol}(\lambda, \sigma_x V | k\varepsilon^{-1/2}) < s E(\lambda)^{p-d/2}$$

then

$$N(\lambda - \varepsilon, V) \leqslant a E(\lambda | R + k\varepsilon^{-1/2})^{d/2-p} \operatorname{Vol}(\lambda, V | R + k\varepsilon^{-1/2}).$$

Proof. First notice that $((-\Delta + V) f, f) \geqslant \lambda + ((-\Delta + V_\lambda) f, f)$, so $\mu_n(-\Delta + V) \geqslant \lambda + \mu_n(-\Delta + V_\lambda)$, which means that

$$N(\lambda - \varepsilon, V) \leqslant N(-\varepsilon, V_\lambda).$$

Let us estimate the number of disjoint balls of radii not larger than $k^{-1/2} \varepsilon^{-1/2}$ satisfying the Fefferman condition

$$\big(|B|^{-1} \int_B |V_\lambda|^p \big)^{1/p} \geqslant S r^{-2}.$$

For such a ball $r \geqslant b^{1/2} E(\lambda)^{-1/2}$, so

$$\int_B |V_\lambda|^p \geqslant c_1 E(\lambda)^{p-d/2}.$$

If for every $|x| \geqslant R$ we have

$$\operatorname{Vol}(\lambda, \sigma_x V | k\varepsilon^{-1/2}) < c_1 E(\lambda)^{p-d/2}$$

then the centre of any ball satisfying the Fefferman condition is in the ball of radius R and centre at zero. The bounds for the number of eigenvalues less than $-\varepsilon$ for the potential V and the Fefferman estimates for $N(-\varepsilon, \delta_{R+k\varepsilon^{-1/2}} V_\lambda)$ are, therefore, equal. When we repeat the argument fixing the radius R and substituting the potential $\delta_{R+k\varepsilon^{-1/2}} V_\lambda$ for V_λ, we see that the radius of any ball which we must count does not exceed $b^{1/2} E(\lambda|R+k\varepsilon^{-1/2})^{-1/2}$. Therefore, if $B_1, \ldots, B_n$ is a maximal family of disjoint balls satisfying the Fefferman condition then for $1 \leqslant j \leqslant n$

$$\int_{B_j} |V_\lambda|^p \geqslant c_2 E(\lambda|R+k\varepsilon^{-1/2})^{p-d/2}, \quad \text{i.e.}$$

$$N(-\varepsilon, V) \leqslant c_3 E(\lambda|R+k\varepsilon^{-1/2})^{d/2-p} \int_{|y|<R+k\varepsilon^{-1/2}} |V_\lambda|^p,$$

which completes the proof.

Remark. Choosing zero as the centre of a "big" ball in spite of its arbitrariness does not influence much the value of the bound for large λ.

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On weak $(r, 2)$-summing operators and weak Hilbert spaces

by

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Abstract. We study Pisier's concept of weak operator ideals and give applications, in particular to $(r, 2)$-summing operators. This leads to new characterizations of weak Hilbert spaces in terms of integral operators and Hilbert numbers. Moreover, a weak version of Grothendieck's inequality and the normability of certain Weyl number ideals are proved.

Introduction. We extend Pisier's [PS1] concept of weak properties of Banach spaces in the following way. If $(\mathfrak{A}, \alpha)$ and $(\mathfrak{B}, \beta)$ are quasi-Banach ideals and $0 < q < \infty$, an operator $T\colon X \to Y$ belongs to the ideal $\mathfrak{W}_q(\alpha, \beta)$ if there is a constant $c \geqslant 0$ such that for all $u \in \mathfrak{A}(l_2, X)$ and $v \in \mathfrak{B}^*(Y, l_2)$ (the conjugate ideal of $\mathfrak{B}$)

$$\sup_{k \in \mathbf{N}} k^{1/q} a_k(vTu) \leqslant c\alpha(u)\beta^*(v),$$

where a_k is the kth approximation number.

In this notation, by [PS2], X is a weak Hilbert space if and only if $\mathrm{id}_X \in \mathfrak{W}_1(\pi_2^d, \pi_2)$, where Π_2 is the ideal of 2-summing operators.

Given an ideal $(\mathfrak{B}, \beta)$, we call an operator $T\colon X \to Y$ a *weak*$_q$ *β-operator* if it belongs to $\mathfrak{W}_q\mathfrak{B} := \mathfrak{W}_q(\|\cdot\|, \beta)$ ($\mathfrak{W}\mathfrak{B} := \mathfrak{W}_1\mathfrak{B}$ are the *weak β-operators*).

Denote by $\Pi_{r,2}$ the $(r, 2)$-summing operators and by $\mathfrak{L}^x_{p,\infty}$ all operators with Weyl numbers in the Lorentz sequence space $l_{p,\infty}$. We prove that

$$\mathfrak{W}_q \Pi_{r,2} = \mathfrak{L}^x_{p,\infty}, \quad 1 + 1/p = 1/q + 1/r > 1.$$

For $q = 1$ this was shown in [PS1] and [MAS]. Even in this case, however, our proof is different and uses Grothendieck numbers.

As an application we prove that $\mathfrak{L}^x_{p,\infty}$ admits an equivalent Banach ideal norm if and only if $p > 2$.

Using equalities of the form $\mathfrak{W}_q \mathfrak{L} = \mathfrak{W}_p \Pi_2$ $(1/2 + 1/p = 1/q > 1)$, $\mathfrak{W}_q \Pi_{r,2} = \mathfrak{W}_p \Pi_2$ $(1/2 + 1/p = 1/q + 1/r)$ and an extension of a result of Geiss [GEI], we prove that the following are equivalent:

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