STUDIA MATHEMATICA, T. XCVI (1990)

A method of approximation of Besov spaces

by

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Abstract. The paper deals with the approximation of Besov spaces $B^s_{pq}(\mathbf{R}^n)(s>0)$. For every $h\in[0,1]^n$ the space $B^s_{pq}(\mathbf{R}^n_h)$ of functions acting on a rectangular mesh is defined. The linear operators $r_h\colon B^s_{pq}(\mathbf{R}^n)\to B^s_{pq}(\mathbf{R}^n_h)$ and $p_h\colon B^s_{pq}(\mathbf{R}^n_h)\to B^s_{pq}(\mathbf{R}^n)$ are constructed with the use of multivariate box splines. It is proved that these operators are uniformly bounded. For every function $f\in B^s_{pq}(\mathbf{R}^n)$ the norm $\|f-p_hr_hf\|$ is estimated by an appropriate modulus of continuity of f.

1. Introduction and notation. The aim of this paper is the presentation of a method of approximation of Besov spaces $B^s_{pq}(\mathbf{R}^n)$, where s is a positive real number. This is an extension of the results obtained in [5] for Sobolev spaces $W^m_p(\mathbf{R}^n)$, m an integer.

As in [5], the following definition (introduced in [1]) of an approximation of a Banach space X is used. Assume that H' is a set possessing an accumulation point denoted by 0, and $H=H'\setminus\{0\}$. The system $\mathscr{A}(X,p,r)=\{(X_h,p_h,r_h)\}_{h\in H}$ is called a *convergent approximation* of X if the X_h are Banach spaces normed by $\|\cdot\|_h$, $p_h\colon X_h\to X$ (*prolongation*), $r_h\colon X\to X_h$ (*restriction*) are linear operators and

$$(1.1)\qquad \begin{aligned}&\exists M>0\ \forall h\in H\ \forall f\in X\quad \|r_hf\|_h\leqslant M\|f\|_X,\\ &\exists M>0\ \forall h\in H\ \forall u\in X_h\quad \|p_hu\|_X\leqslant M\|u\|_h,\end{aligned}$$

$$(1.2)\qquad \forall f\in X\quad \lim_{h\to 0}\|f-p_hr_hf\|_X=0.$$

Similarly to [5], our approximation is constructed with the use of multivariate box splines.

In this section some notation is introduced, the second one contains the definitions of the spaces $B^s_{pq}(\mathbf{R}^n)$ and of the moduli of continuity. In Section 3 the spaces of discrete functions are defined and the prolongations and restrictions are constructed.

Section 4 presents the main results; it follows from the theorems that conditions (1.1)–(1.2) are satisfied. The remaining part of the paper contains the proofs of all the results.

Now, let us introduce some notation which will be used in the paper.

1980 *Mathematics Subject Classification* (1985 *Revision*): Primary 41A15, 41A63; Secondary 41A25, 65D07.

The set of all positive real numbers is denoted by $\mathbf{R}_+$, the set of natur: numbers by $\mathbf{N}$, and $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

A vector $x \in \mathbf{R}^n$ is written as the column $[x_1, \ldots, x_n]^T$. If $x, y \in \mathbf{R}^n$, $z \in \mathbf{R}^n$ then

$$|x| = \sum_{i=1}^{n} |x_i|, \qquad z^x = \prod_{i=1}^{n} (z_i)^{x_i},$$

$$x \circ y = [x_1 y_1, \ldots, x_n y_n]^T, \qquad x/z = [x_1/z_1, \ldots, x_n/z_n]^T.$$

The symbol e_i denotes the unit vector of the ith axis, $e = [1, \ldots, 1]^T$.

2. Besov spaces. The spaces $L_p(\mathbf{R}^n)$, $L_p(\mathbf{R}^n)^{\mathrm{loc}}$, $W_p^m(\mathbf{R}^n)$ $(m \in \mathbf{Z}_+,\ 1 \leqslant p \leqslant$ o are defined as usual. The norm in $L_p(\mathbf{R}^n)$ is denoted by $\|\cdot\|_p$, the norm ar seminorms in $W_p^m(\mathbf{R}^n)$ are given by

$$\|f\|_p^{(m)} = \sum_{i=0}^{m} |f|_p^{(i)}, \qquad |f|_p^{(i)} = \sum_{|k|=i} \|D^k f\|_p,$$

where $D^k f$ $(k \in \mathbf{Z}_+^n)$ is the generalized derivative $\partial^{|k|} f / \partial^{k_1} x_1 \ldots \partial^{k_n} x_n$. The spa $C(\mathbf{R}^n)$ is the set of all bounded uniformly continuous functions on $\mathbf{R}^n$, C^m(F consists of all functions f such that $D^k f \in C(\mathbf{R}^n)$ for $|k| = m$; $\|\cdot\|_\infty^{(m)}$ is the norm $C^m(\mathbf{R}^n)$.

Now, let $s = m + a$, $m \in \mathbf{Z}_+$, $0 < a \leqslant 1$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and let define the following seminorms:

$$|f|_{p,q}^{(a)} = \Big\{ \int_{\mathbf{R}^n} |z|^{-n-aq} \|\Delta_z^2 f\|_p^q \, dz \Big\}^{1/q} \quad \text{if } q < \infty,$$

$$|f|_{p,\infty}^{(a)} = \sup \{ |z|^{-a} \|\Delta_z^2 f\|_p \colon z \in \mathbf{R}^n \setminus \{0\} \},$$

$$|f|_{p,q}^{(s)} = \sum_{|k|=m} |D^k f|_{p,q}^{(a)},$$

where

$$\Delta_z^r f = \Delta^r(z) f = \sum_{i=0}^{r} (-1)^{r-i} \tbinom{r}{i} f(\cdot + iz), \qquad z \in \mathbf{R}^n. \tag{2.1}$$

The *Besov space* $B_{pq}^s(\mathbf{R}^n)$ is defined as the subspace of $W_p^m(\mathbf{R}^n)$ consisting of the functions f for which the seminorm $|f|_{p,q}^{(s)}$ is finite; the norm is given l

$$\|f\|_{p,q}^{(s)} = \|f\|_p^{(m)} + |f|_{p,q}^{(s)}.$$

Some other definitions of the norms in Besov spaces can be found, e.g., in [(Sect. 2.3) and [2] (Sect. 6.2). For instance, if $r \in \mathbf{N}$, $0 < s < r$, ω_p^r is the modul of continuity in $L_p(\mathbf{R}^n)$ defined by

$$\omega_p^r(t, f) = \sup \{ \|\Delta_z^r f\|_p \colon |z| < t \}, \qquad f \in L_p(\mathbf{R}^n),\ t > 0, \tag{2.2}$$

and

$$\Omega_{p,q}^{r,s}(f) = \Big\{ \int_0^\infty t^{-1-sq} \omega_p^r(t, f)^q \, dt \Big\}^{1/q} \quad (\text{modification for } q = \infty),$$

then $\|\cdot\|_p + \Omega_{p,q}^{r,s}$ is an equivalent norm in $B_{pq}^s(\mathbf{R}^n)$ (see [2], Theorem 6.2.5), and it follows from the results of Sections 5.5 and 5.6 in [6] that there are positive numbers M_1, M_2 such that

$$\forall f \in B_{pq}^s(\mathbf{R}^n) \qquad M_1 \Omega_{p,q}^{r,s}(f) \leqslant |f|_{p,q}^{(s)} \leqslant M_2 \Omega_{p,q}^{r,s}(f). \tag{2.3}$$

Let us now introduce the following moduli of continuity in $B_{pq}^s(\mathbf{R}^n)$:

$$\omega_{p,q}^{r,s}(t, f) = \sup \{ \|\Delta_z^r f\|_{p,q}^{(s)} \colon |z| < t \}, \qquad f \in B_{pq}^s(\mathbf{R}^n),\ t > 0,$$

$$\lambda_{p,q}^{r,s}(t, f) = \omega_{p,q}^{r,s}(t, f) + t^{-s} \omega_p^r(t, f), \qquad f \in B_{pq}^s(\mathbf{R}^n),\ t > 0.$$

We will also use the K-functional of Peetre (see [7], Sect. 2.4) in the special case: if A, B are normed spaces and $B \subset A$ (continuous embedding) then

$$K(t, f; A, B) = \inf \{ \|f - g\|_A + t \|g\|_B \colon g \in B \}, \qquad f \in A,\ t > 0. \tag{2.4}$$

Moreover, let us define a modification of the K-functional for $A = B_{pq}^s(\mathbf{R}^n)$, $B = W_p^r(\mathbf{R}^n)$: if $r \in \mathbf{N}$, $0 < s < r$, then

$$K_{p,q}^{r,s}(t, f) = \inf \{ \|f - g\|_{p,q}^{(s)} + t |g|_p^{(r)} \colon g \in W_p^r(\mathbf{R}^n) \}, \qquad f \in B_{pq}^s(\mathbf{R}^n),\ t > 0$$

(an analogous functional with $A = L_p(\mathbf{R}^n)$ was considered by Johnen and Scherer in [4]).

The relations between the functionals introduced above and the seminorms are described in the following lemmas, which are proved in Section 6.

Lemma 1. *Let $r \in \mathbf{N}$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $0 < s < d < r$. There exist constants M_3 to M_6, such that for every $t \in (0, 1)$*

$$\forall f \in W_p^r(\mathbf{R}^n) \qquad \omega_p^r(t, f) \leqslant M_3 t^r |f|_p^{(r)}, \qquad \omega_{p,q}^{r,s}(t, f) \leqslant M_4 t^{r-s} |f|_p^{(r)}, \tag{2.5}$$

$$\forall f \in B_{pq}^d(\mathbf{R}^n) \qquad \omega_p^r(t, f) \leqslant M_5 t^d |f|_{p,q}^{(d)}, \qquad \omega_{p,q}^{r,s}(t, f) \leqslant M_6 t^{d-s} |f|_{p,q}^{(d)}. \tag{2.6}$$

Lemma 2. *Let $r \in \mathbf{N}$, $0 < s < r$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. There exist constants M_7, M_8 such that for every function $f \in B_{pq}^s(\mathbf{R}^n)$ and each $t \in (0, 1)$*

$$M_7 \lambda_{p,q}^{r,s}(t, f) \leqslant K_{p,q}^{r,s}(t^{r-s}, f) \leqslant M_8 \lambda_{p,q}^{r,s}(t, f). \tag{2.7}$$

Moreover, if $q < \infty$ then

$$\lambda_{p,q}^{r,s}(t, f) \to 0 \quad \text{as } t \to 0. \tag{2.8}$$

3. Spaces of discrete functions; restrictions and prolongations. Similarly to [5], the operators of restriction and prolongation are built with the use of the integral and discrete partitions of unity, i.e., the functions belonging to the sets

$$\mathscr{P}_i = \{ A \in L_\infty(\mathbf{R}^n) \colon \operatorname{supp} A \text{ is bounded}, \ \int A(x)\, dx = 1 \},$$

$$\mathscr{P}_d = \{ B \in L_\infty(\mathbf{R}^n) \colon \operatorname{supp} B \text{ is bounded}, \ \sum_{k \in \mathbf{Z}^n} B(x + k) = 1 \text{ almost everywhere} \},$$

respectively. Examples of discrete partitions of unity are the box splines defined below (cf., e.g., [3]).

Let $x^1, \ldots, x^r \in \mathbf{R}^n$ and consider the matrix $X = [x^1, \ldots, x^r]$. Let $\langle X \rangle = \operatorname{span}\{x^1, \ldots, x^r\} = \mathbf{R}^n$. The multivariate box spline B_X is the function satisfying the identity

$$\int_{\mathbf{R}^n} B_X(x) f(x)\,dx = \int_{[0,1]^r} f(Xz)\,dz \quad \text{for every } f \in C(\mathbf{R}^n).$$

In the forthcoming considerations we restrict ourselves to the integer-valued matrices. Let $\mathscr{S} = \{X\colon X = [x^1, \ldots, x^r],\ x^i \in \mathbf{Z}^n\ (r \text{ arbitrary}, 1 \leqslant i \leqslant r)\}$. If $X, Y \in \mathscr{S}$ and X is a submatrix of Y, then we write $X \subset Y$ and we denote by $Y \backslash X$ the complementary submatrix of Y. It follows from the definition that if X is obtained from Y by a permutation of columns, then $B_X = B_Y$.

Finally, following [3] we define the number

$$d(X) = \max\{m\colon \text{for all } Y \subset X,\ |Y| = m \text{ implies } \langle X \backslash Y \rangle = \mathbf{R}^n\}$$

(where $|Y|$ is the number of columns of Y).

The following matrices E_k are used in our considerations:

$$E_k = [\underbrace{e_1, \ldots, e_1}_{k_1 \text{ times}}, \underbrace{e_2, \ldots, e_2}_{k_2 \text{ times}}, \ldots, \underbrace{e_n, \ldots, e_n}_{k_n \text{ times}}] \quad (k \in \mathbf{Z}^n_+). \tag{3.1}$$

For the purpose of the construction of an approximation, we introduce the following classes of integer-valued matrices:

$$\begin{aligned} &\mathscr{S}_m = \{X \in \mathscr{S}\colon E_{me} \subset X,\ d(X) \geqslant m\} \quad (m \in \mathbf{Z}_+), \\ &\mathscr{S}'_m = \{X \in \mathscr{S}_m\colon |\det Y| \leqslant 1 \text{ for each } Y \subset X \text{ such that } |Y| = n\}. \end{aligned} \tag{3.2}$$

The spaces used for an approximation of Besov spaces consist of functions defined on a mesh. The mesh $\mathbf{R}^n_h$ is defined as in [5]. Let $H \subset \mathbf{R}^n_+$ be a bounded set of parameters with 0 as an accumulation point. For fixed $h \in H$, let

$$\mathbf{R}^n_h = \{x \in \mathbf{R}^n\colon x = l \circ h,\ l \in \mathbf{Z}^n\}.$$

The set of all functions $u\colon \mathbf{R}^n_h \to \mathbf{R}$ is denoted by $m(\mathbf{R}^n_h)$.

Now, let $A \in \mathscr{P}_i$, $X \in \mathscr{S}_m$, and define the operators

$$p^X_h\colon m(\mathbf{R}^n_h) \to L_\infty(\mathbf{R}^n)^{\text{loc}}, \qquad r^A_h\colon L_1(\mathbf{R}^n)^{\text{loc}} \to m(\mathbf{R}^n_h)$$

by the formulas

$$\forall u \in m(\mathbf{R}^n_h) \quad p^X_h u = \sum_{l \in \mathbf{Z}^n} B_X(\cdot/h - l)\, u(l \circ h), \tag{3.3}$$

$$\forall f \in L_1(\mathbf{R}^n)^{\text{loc}}\ \forall l \in \mathbf{Z}^n \quad (r^A_h f)(l \circ h) = h^{-e} \int_{\mathbf{R}^n} A(x/h - l) f(x)\,dx. \tag{3.4}$$

These operators can be used as prolongation and restriction in the construction of an approximation of $B^s_{pq}(\mathbf{R}^n)$.

Let us now define the spaces approximating $B^s_{pq}(\mathbf{R}^n)$. First, we recall some definitions from [5].

The space $L_p(\mathbf{R}^n_h)$ is defined as the set of all functions $u \in m(\mathbf{R}^n_h)$ for which the number

$$\|u\|_p = \{h^e \sum_{x \in \mathbf{R}^n_h} |u(x)|^p\}^{1/p}$$

(with the usual extension for the case $p = \infty$) is finite, normed by $\|\cdot\|_p$. The space $W^m_p(\mathbf{R}^n_h)$ $(m \in \mathbf{Z}_+)$ is the set $L_p(\mathbf{R}^n_h)$ normed by

$$\|u\|^{(m)}_p = \sum_{i=0}^{m} |u|^{(i)}_p, \qquad |u|^{(i)}_p = \sum_{|k|=i} \|\partial^k u\|_p,$$

where the operators of finite differences are defined in the following way: if $u \in m(\mathbf{R}^n_h)$, $k \in \mathbf{Z}^n_+$, then

$$\partial^k u = h^{-k} \sum_{0 \leqslant j \leqslant k} (-1)^{|k-j|} \tbinom{k}{j} u(\cdot + j \circ h).$$

Further, it is shown in [5] (Theorems 3 and 4) that if $Y \in \mathscr{S}'_m$ and p^Y_h is defined by (3.3) then

$$\exists C > 0\ \forall h \in H\ \forall u \in W^m_p(\mathbf{R}^n_h) \quad C\|u\|^{(m)}_p \leqslant \|p^Y_h u\|^{(m)}_p \leqslant \|u\|^{(m)}_p.$$

Hence, we introduce the following definition.

Definition 1. Let $s > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and let m be an integer satisfying $m > s$. Let Y be a fixed matrix from $\mathscr{S}'_m$. The space $B^s_{pq}(\mathbf{R}^n_h)$ is the set of all functions $u \in m(\mathbf{R}^n_h)$ such that $\|p^Y_h u\|^{(s)}_{p,q}$ is finite; the norm is defined by $\|u\|^{(s)}_{p,q} = \|p^Y_h u\|^{(s)}_{p,q}$.

The norm defined above depends on the choice of the matrix $Y \in \mathscr{S}'_m$, but it follows from Theorem 1 that the norms generated by different matrices are equivalent, that means, if $Y, Z \in \mathscr{S}'_m$ then

$$\exists C', C'' > 0\ \forall h \in H\ \forall u \in B^s_{pq}(\mathbf{R}^n_h) \quad C'\|p^Z_h u\|^{(s)}_{p,q} \leqslant \|p^Y_h u\|^{(s)}_{p,q} \leqslant C''\|p^Z_h u\|^{(s)}_{p,q}.$$

4. Main results. In this section, the properties of the operators defined by (3.3) and (3.4) are investigated. The first theorem shows that the operators r^A_h and p^X_h satisfy condition (1.1).

Theorem 1. *Let $A \in \mathscr{P}_i$, $X \in \mathscr{S}_m$, $0 < s < m$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$. If $f \in B^s_{pq}(\mathbf{R}^n)$ then $r^A_h f \in B^s_{pq}(\mathbf{R}^n_h)$; if $u \in B^s_{pq}(\mathbf{R}^n_h)$ then $p^X_h u \in B^s_{pq}(\mathbf{R}^n)$. Moreover, there exist constants C_1, C_2 independent of h and such that*

$$\begin{aligned} &\forall f \in B^s_{pq}(\mathbf{R}^n) \quad \|r^A_h f\|^{(s)}_{p,q} \leqslant C_1 \|f\|^{(s)}_{p,q}, \\ &\forall u \in B^s_{pq}(\mathbf{R}^n_h) \quad \|p^X_h u\|^{(s)}_{p,q} \leqslant C_2 \|u\|^{(s)}_{p,q}. \end{aligned} \tag{4.1}$$

If $X \in \mathscr{S}'_m$, then

$$\exists C_3 > 0\ \forall u \in B^s_{pq}(\mathbf{R}^n_h) \quad \|u\|^{(s)}_{p,q} \leqslant C_3 \|p^X_h u\|^{(s)}_{p,q}. \tag{4.2}$$

The rate of convergence of the approximation (condition (1.2)) can be if the function $A\in\mathscr{P}_i$ and the matrix $X\in\mathscr{S}_m$ satisfy the additional condition obtained

$$\forall g\in \Pi_r(\mathbf{R}^n)\qquad p_h^X r_h^A g = g \tag{4.3}$$

(where $\Pi_r(\mathbf{R}^n)$ is the set of all polynomials of degree not greater than r). Note that for every A and X condition (4.3) is satisfied with $r=0$; it is proved in [5] that for every $X\in\mathscr{S}_m$ there exists $A\in\mathscr{P}_i$ such that (4.3) is fulfilled with $r=m$.

THEOREM 2. *Let $A\in\mathscr{P}_i$, $X\in\mathscr{S}_m$, $0<s<m$, $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$, and let condition* (4.3) *be satisfied with $r\geqslant m-1$. Then*

$$\exists C_4>0\ \ \forall h\in H\ \ \forall f\in B^s_{pq}(\mathbf{R}^n)\qquad \|f-p_h^X r_h^A f\|^{(s)}_{p,q}\leqslant C_4\lambda^{m,s}_{p,q}(|h|,f). \tag{4.4}$$

Thus, if $q<\infty$ then it follows from Lemma 2 that for every $f\in B^s_{pq}(\mathbf{R}^n)$, $\|f-p_h^X r_h^A f\|^{(s)}_{p,q}\to 0$ as $h\to 0$. Hence, condition (1.2) is satisfied. Combining this with Theorem 1 we obtain the following corollary.

COROLLARY. *If $q<\infty$ and the assumptions of Theorem 2 are satisfied, then the system $\{(B^s_{pq}(\mathbf{R}^n_h),\ p_h^X,\ r_h^A)\}_{h\in H}$ is a convergent approximation of $B^s_{pq}(\mathbf{R}^n)$.*

5. Auxiliary definitions and formulas. The following difference operator will be used together with Δ_z^r: if $f\in L_1(\mathbf{R}^n)^{\mathrm{loc}}$, $z\in\mathbf{R}^n$, $k\in\mathbf{Z}^n_+$, then

$$S_z^k f=\sum_{0\leqslant j\leqslant k}(-1)^{|k-j|}\binom{k}{j}f(\cdot+j\circ z).$$

Let $r=|k|$, and denote the columns of the matrix E_k (defined by (3.1)) by $[\lambda_1,\ldots,\lambda_r]$. Then

$$S_z^k f=\prod_{j=1}^{r}\Delta^1(\lambda_j\circ z)f.$$

Applying Lemma 2 from [4] we thus obtain the formula

$$\|S_z^k f\|_p\leqslant\sum_{\emptyset\neq F\subset r^*}\|\Delta^r(v_F)f\|_p,\qquad r^*=\{1,\ldots,r\},\ v_F=\sum_{j\in F}j^{-1}\lambda_j\circ z. \tag{5.1}$$

The next formulas follow from the results of [5].

If $l\in\mathbf{Z}^n_+$, $|l|=r$, $X\in\mathscr{S}_r$, then

$$D^l B_X=(-1)^r S^l_{-e}B_{X(l)},\qquad X(l)=X\backslash E_l. \tag{5.2}$$

If $A\in\mathscr{P}_i$, $X\in\mathscr{S}_m$, $r\in\mathbf{Z}$, $0\leqslant r\leqslant m$, then $p_h^X r_h^A\colon W_p^r(\mathbf{R}^n)\to W_p^r(\mathbf{R}^n)$ and

$$\exists F_1>0\ \ \forall h\in H\ \ \forall f\in W_p^r(\mathbf{R}^n)\qquad |p_h^X r_h^A f|_p^{(r)}\leqslant F_1|f|_p^{(r)}; \tag{5.3}$$

if, moreover, (4.3) holds then

$$\exists F_2>0\ \ \forall h\in H\ \ \forall f\in W_p^{r+1}(\mathbf{R}^n)\qquad \|f-p_h^X r_h^A f\|_p\leqslant F_2|h|^{r+1}|f|_p^{(r+1)}. \tag{5.4}$$

It can easily be shown that if $f\in W_p^r(\mathbf{R}^n)$ then

$$\Delta_z^r f=r!\sum_{|k|=r}\frac{z^k}{k!}\int_{[0,1]^r}D^k f(\cdot+z|y|)\,dy;$$

the application of Hölder's inequality yields the estimate

$$\forall z\in\mathbf{R}^n\qquad \|\Delta_z^r f\|_p\leqslant n^r|z|^r|f|_p^{(r)}. \tag{5.5}$$

Further, if $z\in\mathbf{R}^n$, $r\in\mathbf{N}$, then

$$\forall f\in L_p(\mathbf{R}^n)\quad \|\Delta_z^r f\|_p\leqslant 2^r\|f\|_p,\qquad \forall f\in B^s_{pq}(\mathbf{R}^n)\quad |\Delta_z^r f|^{(s)}_{p,q}\leqslant 2^r|f|^{(s)}_{p,q}. \tag{5.6}$$

The second part of this section is connected with the real interpolation method. For the purpose of the paper, it suffices to consider the interpolation of two Banach spaces A, B only in the case when B is continuously embedded into A. Let us introduce the following definition (cf. [7], Sect. 2.4): if $0<\theta<1$, $1\leqslant q\leqslant\infty$, then the space $(A,B)_{\theta,q}$ is the set of all functions from A for which the number $\|f\|_{(A,B)_{\theta,q}}$ defined by

$$\|f\|_{(A,B)_{\theta,q}}=\Big\{\int_0^\infty t^{-1-\theta q}K(t,f;A,B)^q\,dt\Big\}^{1/q}\quad\text{if } q<\infty, \tag{5.7}$$

$$\|f\|_{(A,B)_{\theta,\infty}}=\sup\{t^{-\theta}K(t,f;A,B)\colon t\in\mathbf{R}_+\},$$

is finite; $\|\cdot\|_{(A,B)_{\theta,q}}$ is the norm in $(A,B)_{\theta,q}$ (K is defined in (2.4)).

The Besov spaces can be considered as an interpolation of Sobolev spaces. Namely, it follows from formulas (2.5.7/12) (2.5.7/13) in [7] that if $m\in\mathbf{N}$, $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$ and $0<\theta<1$ then

$$B^{\theta m}_{pq}(\mathbf{R})=(L_p(\mathbf{R}^n),\ W_p^m(\mathbf{R}^n))_{\theta,q},\qquad B^{\theta m}_{\infty q}(\mathbf{R}^n)=(C(\mathbf{R}^n),\ C^m(\mathbf{R}^n))_{\theta,q} \tag{5.8}$$

(the second formula is written in [7] only for $q=\infty$, but the proof for $q<\infty$ is analogous). In the same way it can be proved that if $s>0$, $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$, $1\leqslant q_1\leqslant\infty$, $0<\theta<1$, then

$$B^{\theta s}_{pq}(\mathbf{R}^n)=(L_p(\mathbf{R}^n),\ B^s_{pq_1}(\mathbf{R}^n))_{\theta,q},\qquad B^{\theta s}_{\infty q}(\mathbf{R}^n)=(C(\mathbf{R}^n),\ B^s_{\infty q_1}(\mathbf{R}^n))_{\theta,q}. \tag{5.9}$$

Now, let $s>0$, $1\leqslant p\leqslant\infty$, $1\leqslant q<q_1\leqslant\infty$. Then it follows from Theorems 6.2.4 and 6.3.1 in [2] and from (2.3) that $B^s_{pq}(\mathbf{R}^n)\subset B^s_{pq_1}(\mathbf{R}^n)$ and there exists a constant F_3 such that

$$\forall f\in B^s_{pq}(\mathbf{R}^n)\qquad |f|^{(s)}_{p,q_1}\leqslant F_3|f|^{(s)}_{p,q}. \tag{5.10}$$

Finally, we prove the following inequality.

If $B\subset A$, $0<\theta<1$, $1\leqslant q\leqslant\infty$, then there is a number F_4 such that

$$\forall f\in B\ \ \forall\varepsilon>0\qquad \|f\|_{(A,B)_{\theta,q}}\leqslant F_4(\varepsilon^{1-\theta}\|f\|_B+\varepsilon^{-\theta}\|f\|_A). \tag{5.11}$$

Proof. Let $f\in B$ and let $\varepsilon>0$ be a fixed number. Setting $g=f$ and $g=0$ in formula (2.4) we obtain the inequalities

$$K(t,f;A,B)\leqslant t\|f\|_B,\qquad K(t,f;A,B)\leqslant\|f\|_A.$$

Now, using the first estimate for $t<\varepsilon$, and the second for $t>\varepsilon$, we deduce from (5.7) that if $q<\infty$ then

$$\|f\|_{(A,B)_{\theta,q}} \leqslant \{\|f\|_B^q \int_0^\varepsilon t^{-1+(1-\theta)q}dt + \|f\|_A^q \int_\varepsilon^\infty t^{-1-\theta q}dt\}^{1/q}.$$

Thus, inequality (5.11) holds with $F_4 = \max\big(((1-\theta)q)^{-1/q}, (\theta q)^{-1/q}\big)$. The proof in the case $q=\infty$ is analogous.

6. Proofs

Proof of Lemma 1. The first inequality in (2.5) can be obtained from estimate (5.5), $M_3 = n^r$. Further, formulas (5.8) and (5.11) yield the inequality

$$\|\Delta_z^r f\|_{p,q}^{(s)} \leqslant F_4(\varepsilon^{1-s/r}\|\Delta_z^r f\|_p^{(r)} + \varepsilon^{-s/r}\|\Delta_z^r f\|_p).$$

Using (5.5) and the estimate

$$\|\Delta_z^r f\|_p^{(r)} \leqslant N_1 |f|_p^{(r)} \quad \text{if } |z| \leqslant 1,$$

which follows from (5.6) and (5.5), and taking $\varepsilon = |z|^r$, we obtain the second inequality from (2.5).

Now, let $f \in B_{pq}^d(\mathbf{R}^n)$. Applying (5.10) and (2.3), we obtain the sequence of inequalities

$$|f|_{p,q}^{(d)} \geqslant F_3^{-1}|f|_{p,\infty}^{(d)} \geqslant F_3^{-1}M_1\Omega_{p,\infty}^{r,d}(f).$$

Since for every $t>0$, $\omega_p^r(t,f) \leqslant t^d\Omega_{p,\infty}^{r,d}(f)$, the first estimate in (2.6) holds with $M_5 = F_3(d,p,q,\infty)/M_1(r,d,p,\infty)$. The second inequality can be obtained from the first one by the application of (5.9), (5.11) and (5.5).

Thus, Lemma 1 is proved.

Proof of Lemma 2. Let us start with proving the first inequality in (2.7). Let $f \in B_{pq}^s(\mathbf{R}^n)$ and $g \in W_p^r(\mathbf{R}^n)$. The inequality can be obtained from the triangle inequality $\lambda_{p,q}^{r,s}(t,f) \leqslant \lambda_{p,q}^{r,s}(t,f-g) + \lambda_{p,q}^{r,s}(t,g)$ and estimates (5.6), (2.6) and (2.5); the constant M_7 equals $1/\max(2^r + M_5, M_3 + M_4)$.

Now, we have to prove the second inequality in (2.7). As in the proof of Lemma 3 in [4], for every t a function g satisfying the inequality

$$\|f-g\|_{p,q}^{(s)} + t^{r-s}|g|_p^{(r)} \leqslant M_8\lambda_{p,q}^{r,s}(t,f) \tag{6.1}$$

will be constructed. The construction is done with the use of multivariate splines.

Let X be an arbitrary element of $\mathscr{S}_r$ (see (3.2)) and let B_X be the corresponding box spline. We use the notation

$$N = N_X = \operatorname{supp} B_X, \quad v = \sup\{|x|:\ x \in N\}, \quad v_{1,X} = \operatorname{vol} N_X.$$

Let the functions $\mathscr{M}_\varepsilon^X f$, $\mathscr{N}_\varrho f$ $(\varepsilon, \varrho > 0)$ be defined by the formulas

$$\mathscr{M}_\varepsilon^X f = \int_N B_X(y)f(\cdot+\varepsilon y)\,dy, \qquad \mathscr{N}_\varrho f = -\sum_{i=1}^r (-1)^i\binom{r}{i}\mathscr{M}_{i\varrho}^X f. \tag{6.2}$$

(Observe that if $X = E_{re}$ then $\mathscr{N}_\varrho f$ is the function g_s (the Steklov means) used in the proof of Lemma 3 in [4].)

The following formulas can easily be checked:

$$\forall i \in \mathbf{N},\ \forall z \in \mathbf{R}^n \quad \Delta_z^i(\mathscr{M}_\varepsilon^X f) = \mathscr{M}_\varepsilon^X(\Delta_z^i f), \tag{6.3}$$

$$f - \mathscr{N}_\varrho f = (-1)^r \int_N B_X(y)\Delta_{\varrho y}^r f(\cdot)\,dy \tag{6.4}$$

(the difference operator Δ_z^i is defined by (2.1)). Applying Hölder's inequality to formulas (6.2), (6.4) and using (6.3) we obtain

$$\|\mathscr{M}_\varepsilon^X f\|_p \leqslant N_2\|f\|_p, \quad |\mathscr{M}_\varepsilon^X f|_{p,q}^{(s)} \leqslant N_2|f|_{p,q}^{(s)}, \quad N_2(X) = \|B_X\|_{p'}v_{1,X}^{1/p}, \tag{6.5}$$

$$\|f - \mathscr{N}_\varrho f\|_{p,q}^{(s)} \leqslant N_2\omega_{p,q}^{r,s}(v\varrho, f). \tag{6.6}$$

Let us now estimate $|\mathscr{N}_\varrho f|_p^{(r)}$. Let $l \in \mathbf{Z}_+^n$, $|l| = r$. It can be deduced from definition (6.2) and formula (5.2) that

$$D^l\mathscr{M}_\varepsilon^X f = \varepsilon^{-r}\int_N D^l B_X(y)f(\cdot+\varepsilon y)\,dy = \varepsilon^{-r}\mathscr{M}_\varepsilon^{X(l)}(S_{\varepsilon e}^l f),$$

where $X(l) = X\setminus E_l$. Hence

$$D^l\mathscr{N}_\varrho f = -\sum_{i=1}^r (-1)^i\binom{r}{i}(i\varrho)^{-r}\mathscr{M}_{i\varrho}^{X(l)}(S_{i\varrho e}^l f).$$

Applying inequality (6.5) we deduce that

$$\|D^l\mathscr{N}_\varrho f\|_p \leqslant \varrho^{-r}N_2(X(l))\sum_{i=1}^r \binom{r}{i}i^{-r}\|S_{i\varrho e}^l f\|_p. \tag{6.7}$$

Now, we can use formula (5.1) with $z = i\varrho e$. For every $F \subset r^*$ we have

$$|v_F| = \Big|\sum_{j\in F} j^{-1}\lambda_j i\varrho\Big| \leqslant i\varrho\sum_{j=1}^r j^{-1} \leqslant r(1+\ln r)\varrho,$$

and hence

$$\|S_{i\varrho e}^l f\|_p \leqslant 2^r\omega_p^r(r(1+\ln r)\varrho, f).$$

Substituting this inequality into (6.7) we obtain

$$|\mathscr{N}_\varrho f|_p^{(r)} \leqslant \varrho^{-r}N_3\omega_p^r(r(1+\ln r)\varrho, f), \tag{6.8}$$

$$N_3 = 2^r\sum_{|l|=r} N_2(X(l))\sum_{i=1}^r \binom{r}{i}i^{-r}.$$

Thus, if we take $\varrho = t/\max(v, r(1+\ln r))$ and $g = \mathscr{N}_\varrho f$, we deduce using (6.6) and (6.8) that (6.1) is satisfied with $M_8 = N_2 + N_3\max(v, r(1+\ln r))^r$.

Therefore, both inequalities in (2.7) are proved. Let us now turn to formula (2.8). Let z be an arbitrary vector from $\mathbf{R}^n$ and let $|z| < t$. Then, by (2.3),

$$\|\Delta_z^r f\|_{p,q}^{(s)} \leqslant \|\Delta_z^r f\|_p + M_2\Big\{\int_0^\infty \tau^{-1-sq}\omega_p^r(\tau, \Delta_z^r f)^q d\tau\Big\}^{1/q}. \tag{6.9}$$

According to (2.2), $\omega_p^r(\tau, \Delta_z^r f) = \sup\{\|\Delta_w^r \Delta_z^r f\|_p\colon |w| < \tau\}$. Now, if $\tau < t$, then we use (5.6) for Δ_z^r, and in the opposite case for Δ_w^r. Thus,

$$\omega_p^r(\tau, \Delta_z^r f) \leqslant 2^r \omega_p^r(\tau, f) \quad \text{if } \tau < t,$$

$$\omega_p^r(\tau, \Delta_z^r f) \leqslant 2^r \|\Delta_z^r f\|_p \quad \text{if } \tau > t.$$

Putting these inequalities into (6.9), we obtain

$$\|\Delta_z^r f\|_{p,q}^{(s)} \leqslant \|\Delta_z^r f\|_p + 2^r M_2 \Big(\Big\{\int_0^t \tau^{-1-sq} \omega_p^r(\tau, f)^q \, d\tau\Big\}^{1/q} + (sq)^{-1/q} t^{-s} \|\Delta_z^r f\|_p\Big).$$

Hence,

$$\omega_{p,q}^{r,s}(t, f) \leqslant (1 + C_4 t^{-s}) \omega_p^r(t, f) + 2^r M_2 \Big\{\int_0^t \tau^{-1-sq} \omega_p^r(\tau, f)^q \, d\tau\Big\}^{1/q}. \tag{6.10}$$

Since for every $f \in B_{pq}^s(\mathbf{R}^n)$ the integral $\int_0^\infty \tau^{-1-sq} \omega_p^r(\tau, f)^q \, d\tau$ is finite, both terms on the right-hand side of (6.10) tend to 0 as t goes to 0. Therefore, the proof of Lemma 2 is finished.

From now on, the mesh size h is fixed, and hence the subscript h is suppressed.

Proof of Theorem 1. We have to prove estimates (4.1) and (4.2). First, according to Definition 1, we have

$$\|r^A f\|_{p,q}^{(s)} = \|p^Y r^A f\|_{p,q}^{(s)}.$$

Define the operator $T\colon L_1(\mathbf{R}^n)^{\mathrm{loc}} \to L_1(\mathbf{R}^n)^{\mathrm{loc}}$ by the formula $Tf = p^Y r^A f$. According to (5.3), if $f \in W_p^r(\mathbf{R}^n)$ then $Tf \in W_p^r(\mathbf{R}^n)$ and $\|Tf\|_p^{(r)} \leqslant F_1 \|f\|_p^{(r)}$ $(0 \leqslant r \leqslant m,\ r \in \mathbf{Z})$. Thus, applying (5.8) and Remark 2.4.1/3 from [7], we deduce that if $f \in B_{pq}^s(\mathbf{R}^n)$ then $Tf \in B_{pq}^s(\mathbf{R}^n)$ and $\|Tf\|_{p,q}^{(s)} \leqslant F_1 \|f\|_{p,q}^{(s)}$. Hence we have proved the first estimate of Theorem 1.

Note that the above estimates are true for every $Y \in \mathscr{S}_m$ and not only for $Y \in \mathscr{S}'_m$.

Now, let us estimate $\|p^X u\|_{p,q}^{(s)}$. It follows from Theorem 4 in [5] that, since $Y \in \mathscr{S}'_m$, there exists a function $B \in \mathscr{P}_i$ such that $r^B p^Y u = u$ for each $u \in m(\mathbf{R}_h^n)$. Hence, $p^X u = p^X r^B p^Y u$, and using the estimate just proved, we obtain

$$\|p^X r^B p^Y u\|_{p,q}^{(s)} \leqslant F_1(B, p) \|p^Y u\|_{p,q}^{(s)} = F_1(B, p) \|u\|_{p,q}^{(s)}.$$

Thus, $C_2 = F_1(B, p)$ and formula (4.1) is proved.

Inequality (4.2) can be proved by interchanging the roles of X and Y. This completes the proof of Theorem 1.

Proof of Theorem 2. In order to prove (4.4), take first an arbitrary function $g \in W_p^m(\mathbf{R}^n)$. By (5.3) and (5.4), we obtain

$$\|g - p^X r^A g\|_p \leqslant F_2 |h|^m |g|_p^{(m)}, \qquad \|g - p^X r^A g\|_p^{(m)} \leqslant (F_1 + mF_2) |g|_p^{(m)}.$$

Applying (5.8) and (5.11) with $\varepsilon = |h|^m$, we deduce that

$$\|g - p^X r^A g\|_{p,q}^{(s)} \leqslant N_4 |h|^{m-s} |g|_p^{(m)}, \qquad N_4 = F_4(F_1 + (m+1) F_2). \tag{6.11}$$

Let now $f \in B_{pq}^s(\mathbf{R}^n)$. Then

$$\|f - p^X r^A f\|_{p,q}^{(s)} \leqslant \|f - g\|_{p,q}^{(s)} + \|g - p^X r^A g\|_{p,q}^{(s)} + \|p^X r^A (g - f)\|_{p,q}^{(s)}.$$

The second term on the right-hand side can be estimated from (6.11), the third with the use of (4.1). Hence

$$\|f - p^X r^A f\|_{p,q}^{(s)} \leqslant N_5 (\|f - g\|_{p,q}^{(s)} + |h|^{m-s} |g|_p^{(m)}), \qquad N_5 = \max(1 + F_1 C_2, N_4).$$

The function g was taken arbitrarily, therefore we have the estimate

$$\|f - p^X r^A f\|_{p,q}^{(s)} \leqslant N_5 K_{p,q}^{r,s}(|h|^{m-s}, f),$$

and inequality (4.4) can be obtained by applying Lemma 2. This completes the proof of Theorem 2.

Acknowledgement. The author is thankful to Professor Zbigniew Ciesielski for his suggestions and advice concerning this paper.

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Received May 15, 1989 (2562)