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On generalized canonical commutation relations

by

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Abstract. We prove that the generalized Heisenberg canonical commutation relation implies the corresponding generalized Weyl canonical commutation relation if the momentum operator possesses a suitable set of analytic vectors.

1. Introduction. Let P and Q be two selfadjoint operators in a Hilbert space H with domains $D(P)$ and $D(Q)$, respectively, and let c be a real number. Suppose that P, Q and c satisfy the *Weyl canonical commutation relation*:

$$\text{(WCCR)} \qquad e^{isP}e^{itQ} = e^{icst}e^{itQ}e^{isP} \qquad (s, t \in \mathbf{R}).$$

Then, as a simple argument shows, there exist various linear subspaces D of $D(P) \cap D(Q)$ such that $P(D) \cup Q(D) \subset D$, the restrictions of P and Q to D are essentially selfadjoint, and

$$\text{(HCCR)} \qquad PQf - QPf = icf$$

for any $f \in D$. The latter identity is called the *Heisenberg canonical commutation relation.*

A question significant for applications in physics is under what conditions on D, (HCCR) implies (WCCR) (cf. [P]). The present paper deals with this problem in a more general setting in which the operators P and Q act in a Banach space and the commutator of P and Q is an arbitrary (not necessarily scalar) bounded operator commuting, in a natural sense, with P and Q. Such a situation was considered also in [J-M]. The main result of this paper asserts that if all elements of D are analytic vectors for P (or for Q), then the implication in question holds. That it holds if D is a set of analytic vectors for both P and Q is a corollary to some results on integrability of Lie algebra representations (cf. [N, R, S]).

2. An auxiliary result. Let E be a Banach space and A a linear operator in E. We recall that an element x of $\bigcap_{n=1}^{\infty} D(A^n)$ is called an *analytic vector* for A if

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n f\| < \infty$$

for some $t > 0$.

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THEOREM 2.1. *Let E be a Banach space, and let A and B be two linear closable operators with common domain D such that $A(D)\cup B(D)\subset D$. Suppose that each element of D is an analytic vector for A, and the closure $\bar{A}$ of A is the generator of a strongly continuous one-parameter group $(e^{t\bar{A}})_{t\in\mathbf{R}}$. Suppose, moreover, that there exists a linear bounded operator T in E such that for each $f\in D$,*

$$BAf-ABf=Tf,\qquad ATf-TAf=0. \tag{2.1}$$

Then, for each $t\in\mathbf{R}$,

$$e^{t\bar{A}}(D(\bar{B}))\subset D(\bar{B}) \tag{2.2}$$

and, for each $f\in D(\bar{B})$ and each $t\in\mathbf{R}$,

$$\bar{B}e^{t\bar{A}}f=e^{t\bar{A}}(\bar{B}+tT)f. \tag{2.3}$$

Proof. We first prove that for each $t\in\mathbf{R}$, T commutes with $e^{t\bar{A}}$.

Given $f\in D$, let $r_0(f)$ be the radius of convergence of the series $\sum_{n=0}^{\infty}(t^n/n!)\|A^nf\|$. Then for $|t|<r_0(f)$ we have

$$e^{t\bar{A}}f=\sum_{n=0}^{\infty}\frac{t^n}{n!}A^nf.$$

Indeed, both sides are local solutions of the Cauchy problem $du(t)/dt=\bar{A}u(t)$, $u(0)=f$. Since $\bar{A}$ is the generator of a strongly continuous group the solution is unique.

Since, by the hypothesis, $r_0(f)$ is positive, we see that $e^{t\bar{A}}$ and T commute if $|t|$ is sufficiently small. Now the general case follows upon applying, for each $t\in\mathbf{R}$, the identity $e^{t\bar{A}}=(e^{(t/n)\bar{A}})^n$ with $n\in\mathbf{N}$ sufficiently large.

Given $f\in D$, let

$$r_1(f)=\min\{r_0(f),r_0(Bf)\}.$$

We shall prove the following

FACT 1. *For each $f\in D$ and each $m\in\mathbf{N}$, we have*

$$r_0(A^mf)=r_0(f),\qquad r_1(A^mf)\geqslant r_1(f).$$

Proof. The first statement is obvious. To prove the second, note that

$$BA^mf-A^mBf=mTA^{m-1}f. \tag{2.4}$$

Thus

$$r_1(A^mf)\geqslant\min\{r_0(f),\,r_0(A^mBf),\,r_0(mTA^{m-1}f)\}.$$

But, clearly, $r_0(A^mBf)=r_0(Bf)$. Moreover, $\|TA^{m-1}f\|\leqslant\|T\|\,\|A^{m-1}f\|$, and so $r_0(mTA^{m-1}f)\geqslant r_0(f)$, completing the proof.

Regard $D(\bar{B})$ as being the completion of D under the norm $\|f\|_B=\|f\|+\|Bf\|$ $(f\in D)$.

Now we prove the following

FACT 2. *For each $f\in D$, if $|t|<r_1(f)$, then the series $\sum_{n=0}^{\infty}(t^n/n!)A^nf$ is absolutely convergent in $D(\bar{B})$ to $e^{t\bar{A}}f$ and, moreover,*

$$\bar{B}e^{t\bar{A}}f=e^{t\bar{A}}(B+tT)f.$$

Proof. Let $f\in D$ and $|t|<r_1(f)$. Then, in view of (2.4), for any $N\in\mathbf{N}$,

$$\Big\|B\sum_{n=0}^{N}\frac{t^n}{n!}A^nf\Big\|\leqslant\sum_{n=0}^{N}\frac{|t|^n}{n!}\|A^nBf\|+\sum_{n=1}^{N}\frac{|t|^n}{(n-1)!}\|T\|\,\|A^{n-1}f\|.$$

If we let $N\to\infty$, then the first series on the right-hand side converges whenever $|t|<r_0(Bf)$ and the second converges whenever $|t|<r_0(f)$. Hence both series converge provided $|t|<r_1(f)$. Noting that

$$\begin{aligned}\bar{B}e^{t\bar{A}}f&=\sum_{n=0}^{\infty}\frac{t^n}{n!}A^nBf+T\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!}A^{n-1}f\\&=e^{t\bar{A}}Bf+tTe^{t\bar{A}}f=e^{t\bar{A}}(B+tT)f\qquad(|t|<r_1(f))\end{aligned}$$

completes the proof.

Given $f\in D$, let $R(f)$ be the greatest element of $\mathbf{R}\cup\{+\infty\}$ such that if $|t|<R(f)$ and $m\in\mathbf{N}$, then

$$e^{t\bar{A}}A^mf\in D(\bar{B}), \tag{2.5}$$

$$\bar{B}e^{t\bar{A}}A^mf=e^{t\bar{A}}(B+tT)A^mf. \tag{2.6}$$

Facts 1 and 2 ensure that $R(f)\geqslant r_1(f)>0$. Let $|s|<r_1(f)$, $|t|<R(f)$, and $m\in\mathbf{N}$. Since $e^{t\bar{A}}A^mf$ is an analytic vector for $\bar{A}$ such that the radius of convergence of the corresponding series is not less than $r_0(f)$, it follows that

$$e^{(s+t)\bar{A}}A^mf=\sum_{n=0}^{\infty}\frac{s^n}{n!}\bar{A}^ne^{t\bar{A}}A^mf.$$

Moreover, for each $N\in\mathbf{N}$,

$$\sum_{n=0}^{N}\frac{s^n}{n!}A^ne^{t\bar{A}}A^mf=\sum_{n=0}^{N}\frac{s^n}{n!}e^{t\bar{A}}A^{n+m}f$$

is an element of $D(\bar{B})$ and, in view of (2.6),

$$\bar{B}\sum_{n=0}^{N}\frac{s^n}{n!}e^{t\bar{A}}A^{n+m}f=\sum_{n=0}^{N}\frac{s^n}{n!}e^{t\bar{A}}BA^{n+m}f+te^{t\bar{A}}T\sum_{n=0}^{N}\frac{s^n}{n!}A^{n+m}f.$$

If we let $N\to\infty$, then the second sum on the right-hand side converges to

$$te^{t\bar{A}}Te^{s\bar{A}}A^mf=te^{(t+s)\bar{A}}TA^mf.$$

In view of (2.4), the first sum is equal to

$$e^{t\bar{A}}\sum_{n=0}^{N}\frac{s^n}{n!}A^nBA^mf+e^{t\bar{A}}\sum_{n=0}^{N}\frac{s^n}{(n-1)!}TA^{n-1}A^mf$$

and converges, as $N\to\infty$, to

$$e^{(s+t)\bar{A}}BA^mf+se^{t\bar{A}}Te^{s\bar{A}}A^mf=e^{(s+t)\bar{A}}(B+sT)A^mf.$$

We thus see that (2.5) and (2.6) hold with t replaced by $s+t$. Hence $R(f)\geqslant R(f)+r_1(f)$, and so $R(f)=+\infty$. Letting $m=0$ in (2.6), we obtain for any $f\in D$ and any $t\in\mathbf{R}$,

$$\bar{B}e^{t\bar{A}}f=e^{t\bar{A}}(B+tT)f.$$

Moreover,

$$\|e^{t\bar{A}}f\|_{\bar{B}}\leqslant\|f\|+\|e^{t\bar{A}}Bf\|+|t|\,\|T\|\,\|e^{t\bar{A}}f\|\leqslant C\|e^{t\bar{A}}\|\,\|f\|_B,$$

where C is a constant depending on t, which implies (2.2) and (2.3).

The proof is complete.

3. The main result. We now state our main result.

THEOREM 3.1. *Let P and Q be the generators of strongly continuous groups e^{tP} and e^{tQ} respectively, acting in a Banach space E. Let D be a dense linear subspace of $D(P)\cap D(Q)$ such that $P(D)\cup Q(D)\subset D$ and such that the closures of the restrictions of P and Q to D are equal to P and Q respectively. Suppose that each element of D is an analytic vector for P. Suppose, moreover, that there exists a linear bounded operator S in E such that for each $f\in D$,*

(3.1) $$QPf-PQf=Pf,\quad PSf-SPf=0,\quad QSf-SQf=0.$$

Then for all $t,s\in\mathbf{R}$ we have

(3.2) $$e^{tQ}e^{sP}=e^{stS}e^{sP}e^{tQ}.$$

Proof. Applying Theorem 2.1 to $A=P$, $B=Q$, $T=S$ and next to $A=S$, $B=Q$, $T=0$ (notice that in the latter case each element of E is an analytic vector for A), we obtain for any $s,t\in\mathbf{R}$,

(3.3) $$e^{sP}D(Q)=e^{tS}D(Q)=D(Q)$$

and for any $s,t\in\mathbf{R}$ and any $f\in D(Q)$,

(3.4) $$Qe^{sP}f=e^{sP}(Q+sS)f,\quad Qe^{tS}f=e^{tS}Qf.$$

Given $s\in\mathbf{R}$ and $f\in D(Q)$, define a function $G\colon\mathbf{R}\to E$ by setting

$$G(t)=e^{-sP}e^{-tQ}e^{stS}e^{sP}e^{tQ}f\quad(t\in\mathbf{R}).$$

In view of (3.3) and (3.4), G is differentiable and $G'=0$. Hence $G(t)=G(0)$ for each $t\in\mathbf{R}$, which concludes the proof.

4. The spectrum of the commutator. If P and Q in Theorem 3.1 are bounded operators then by (3.1) (in this case trivially equivalent to (3.2)) and by the Kleinecke–Shirokov theorem ([6]) the spectrum of S is equal to $\{0\}$. We shall show that in the case of unbounded operators P and Q satisfying (3.2) (with bounded S) the spectrum of S can be any bounded closed subset of $i\mathbf{R}$.

EXAMPLE 4.1. Let M be any closed bounded subset of $\mathbf{R}$ and let μ be a finite measure with support M. Let H be the Hilbert space $L^2(M\times\mathbf{R},\mu\times l_1)$, where l_1 is the one-dimensional Lebesgue measure. Let D be the linear subset of H consisting of all functions $f(\lambda,x)$ such that for all $m,n\in\mathbf{N}$

$$\sup_{\substack{\lambda\in M\\x\in\mathbf{R}}}\left|x^m\frac{\partial^nf}{\partial x^n}(\lambda,x)\right|<\infty.$$

D is obviously dense in H. For $f\in D$ let

$$(Pf)(\lambda,x)=i\lambda xf(\lambda,x),\quad (Qf)(\lambda,x)=\frac{\partial f}{\partial x}(\lambda,x),$$

and let $U(t)$ and $V(s)$ be the unitary strongly continuous groups defined by

$$U(t)f(\lambda,x)=e^{it\lambda x}f(\lambda,x),\quad V(t)f(\lambda,x)=f(\lambda,x+t).$$

For each $t\in\mathbf{R}$ we have $U(t)D\subset D$, $V(t)D\subset D$, and for each $t\in\mathbf{R}$ and each $f\in D$,

$$\frac{dU(t)f}{dt}=PU(t)f,\quad \frac{dV(t)f}{dt}=QV(t)f.$$

Thus by the de Leeuw theorem (cf. [L]) iP and iQ are essentially selfadjoint and $U(t)=e^{t\bar{P}}$ and $V(t)=e^{t\bar{Q}}$. Moreover, for $f\in D$,

$$QPf-PQf=Sf,\quad PSf-SPf=0,\quad QSf-SQf=0,$$

where S is the bounded operator defined by

$$(Sf)(\lambda,x)=i\lambda f(\lambda,x),$$

and for $t,s\in\mathbf{R}$ we have $e^{t\bar{Q}}e^{s\bar{P}}=e^{stS}e^{s\bar{P}}e^{t\bar{Q}}$. It is clear that the spectrum of S is equal to iM.

Now we shall show the following

THEOREM 4.2. *If P, Q, S are the generators of strongly continuous groups e^{tP}, e^{tQ}, e^{tS} respectively, satisfying (3.2), then the spectrum of S is contained in $i\mathbf{R}$.*

Proof. By the Hille–Yosida theorem (cf. [H-P]) for each strongly continuous group e^{tS} with generator S there exist $M>0$ and $\omega>0$ such that

(4.1) $$\|e^{tS}\|\leqslant Me^{|t|\omega};$$

moreover,

(4.2) if the group e^{tS} satisfies (4.1) then the spectrum of S is contained in $\{\lambda\colon|\mathrm{Re}\lambda|\leqslant\omega\}$.

Let $\|e^{tP}\| \leqslant Me^{|t|\omega}$, $\|e^{tQ}\| \leqslant M'e^{|t|\omega'}$. By (3.2) we have

$$\|e^{tS}\| \leqslant M^2 M'^2 \exp(2\sqrt{|t|}(\omega+\omega')).$$

Hence for any $\varepsilon > 0$ we can choose $M > 0$ such that $\|e^{tS}\| \leqslant Me^{|t|\varepsilon}$. In virtue of (4.2) the spectrum of S is contained in $i\mathbf{R}$, which concludes the proof.

In the Kleinecke–Shirokov theorem one assumes that S commutes only with P (or only with Q). The following simple example demonstrates that in the case of unbounded operators such a generalization of Theorem 4.2 is impossible.

EXAMPLE 4.3. Let $H = L^2(\mathbf{R})$. Let $D = C_0^\infty(\mathbf{R})$ and let g be any function from D. Let

$$(Pf)(x) = g(x)f(x) \quad (f \in H),$$
$$(Qf)(x) = f'(x) \quad (f \in D)$$
$$(Sf)(x) = g'(x)f(x) \quad (f \in H).$$

For $f \in D$ we have

$$QPf - PQf = Sf, \quad PSf - SPf = 0.$$

P and S are bounded and iQ is essentially selfadjoint on D. The spectrum of S is equal to the image of the function g'. Since g was an arbitrary function from $C_0^\infty(\mathbf{R})$ the spectrum of S can contain any complex number. This example together with Theorem 4.2 also shows that all three equalities in (3.1) are essential in Theorem 3.1.

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