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A density theorem for F-spaces

by

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Abstract. The main result of this paper expressed in terms of representation theory states that any algebraically irreducible representation T of an algebra A in the algebra of all continuous endomorphisms of an F-space is totally irreducible provided the only intertwining operators for T are the scalar multiples of the identity operator. We apply this result for characterizing strongly generating sets for the algebra of all continuous endomorphisms of a B_0-space.

§ 1. Definitions and notation. An *F-space*, or a *space of type F*, is a completely metrizable topological linear space. The topology of an F-space X can be given by means of an *F-norm*, i.e. a functional $\|\cdot\|$ satisfying the following conditions (see [1] or [5]):

(i) $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|x+y\| \leqslant \|x\| + \|y\|$,
(iii) $\lim \|x_n\| = 0$ implies $\lim \|\lambda x_n\| = 0$ for all λ,
(iv) $\lim |\lambda_n| = 0$ implies $\lim \|\lambda_n x\| = 0$ for all x.

Here x, x_n, y denote arbitrary elements of X and λ, λ_n arbitrary (real or complex) scalars. The distance in X is given by $\|x-y\|$ and the space X is complete in this metric. For F-spaces the closed graph theorem holds true: If T is a linear map of one F-space into another and its graph is closed in the product of these spaces, then T is continuous (see [1]). A locally convex space of type F is called a B_0-*space*. For an F-space X denote by $L(X)$ the algebra of all its continuous endomorphisms. While for B_0-spaces this algebra always has a rich structure, for some F-spaces it can be very poor. There are (infinite-dimensional) so-called *rigid spaces* of type F in which the only continuous endomorphisms are the scalar multiples of the identity operator (see [4], [8], or [5], p. 210). In particular, a rigid space cannot have a nontrivial continuous linear functional, while for B_0-spaces there always exists a separating family of such functionals.

The present paper is a by-product of our efforts at characterizing strongly generating sets for $L(X)$. It turned out (many thanks to Pavla Vrbová for

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calling my attention to this fact) that such a characterization is intimately related to the so-called density theorems in representation theory, and that it immediately follows from the known facts at least in the case when X is a complex Banach space (see § 3). However, the method we use works as well in a more general setting when X is an F-space. In § 2 we prove our density theorem, in § 3 we interpret it as a result in representation theory and in § 4 we use it for characterizing strongly generating sets for $L(X)$ when X is a B_0-space.

§ 2. The density theorem. Let X be a real or complex space of type F and let S be a subset of $L(X)$. The *commutant* S' of S is defined as

$$S' = \{T \in L(X):\ TR = RT \text{ for all } R \text{ in } S\}.$$

We say that S *has a trivial commutant* if S' consists of scalar multiples of the identity operator on X. A subalgebra $A \subset L(X)$ is said to be *transitive* if for each nonzero element $x \in X$ the orbit $\mathcal{O}(A, x) = \{Tx \in X:\ T \in A\}$ coincides with X. Note that if X is rigid then there is no transitive subalgebra in $L(X)$. For $x_1, \ldots, x_n$ in X define the *multiple orbit* of A as

$$\mathcal{O}(A, x_1, \ldots, x_n) = \{(Tx_1, \ldots, Tx_n) \in X^n:\ T \in A\}.$$

Observe that X^n is also an F-space under the norm $\|(x_1, \ldots, x_n)\| = \max\{\|x_i\|: 1 \leqslant i \leqslant n\}$.

We now formulate our main result.

THEOREM 1. *Let X be a real or complex space of type F and let A be a transitive subalgebra of $L(X)$ with trivial commutant. Then for any n-tuple $(x_1, \ldots, x_n)$ of linearly independent elements in X the orbit $\mathcal{O}(A, x_1, \ldots, x_n)$ is dense in X^n.*

For the proof of this theorem we shall need the following

LEMMA. *Let X be a real or complex space of type F and let A be a nonvoid subset of $L(X)$. Let $M = (m_{i,j})$ be an $n \times n$ matrix with scalar entries and nonzero determinant. Suppose that $x_1, \ldots, x_n$ are linearly inedpendent in X and put $y_i = \sum_{j=1}^{n} m_{i,j} x_j$. Then $\mathcal{O}(A, x_1, \ldots, x_n)$ is dense in X^n if and only if so is $\mathcal{O}(A, y_1, \ldots, y_n)$.*

The proof is straightforward and is left to the reader.

Proof of Theorem 1. We proceed by induction on n. For $n = 1$ the conclusion follows from the definition of a transitive algebra. Suppose now that for all k, $1 \leqslant k \leqslant n$, and for each choice of k linearly independent elements $x_1, \ldots, x_k$ in X the orbit $\mathcal{O}(A, x_1, \ldots, x_k)$ is dense in X^k. We have to show that for every linearly independent $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ the orbit $\mathcal{O}(A, x_1, \ldots, x_{n+1})$ is dense in X^{n+1}. We shall be done if we show that for each i_0, $1 \leqslant i_0 \leqslant n+1$, and for each z in X there is a sequence $T_k^{(i_0,z)}$ of operators in

A such that

$$\lim_k T_k^{(i_0,z)} x_i = 0 \text{ for } i \neq i_0 \text{ and } \lim_k T_k^{(i_0,z)} x_{i_0} = z. \tag{1}$$

Indeed, for a given $(n+1)$-tuple $(z_1, \ldots, z_{n+1})$ in X^{n+1} we define $T_k = \sum_{i=1}^{n+1} T_k^{(i,z_i)}$ and we see that $\lim_k T_k x_i = z_i$, $i = 1, \ldots, n+1$, which proves that $\mathcal{O}(A, x_1, \ldots x_{n+1})$ is dense in X^{n+1} since $z_1, \ldots, z_{n+1}$ were chosen arbitrarily.

Conversely, if $\mathcal{O}(A, x_1, \ldots, x_{n+1})$ is dense in X^{n+1}, then there exist in A operators $T_k^{(i_0,z}$ satisfying (1) for every z in X and i_0, $1 \leqslant i_0 \leqslant n+1$. Thus, by the inductive assumption, we have (1) with an arbitrary k-tuple $(y_1, \ldots, y_k)$ of linearly independent elements, $1 \leqslant k \leqslant n$, in place of the $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$.

Observe that (1) is equivalent to the following.

For each i_0, $1 \leqslant i_0 \leqslant n+1$, there is a sequence $(T_k^{(i_0)})$ of elements of A and an element v in X, $v \neq 0$, such that

$$\lim_k T_k^{(i_0)} x_i = 0 \text{ for } i \neq i_0 \text{ and } \lim_k T_k^{(i_0)} x_{i_0} = v. \tag{2}$$

Indeed, (1) implies (2), and if we have (2) and we are given an element z in X, then by transitivity there is an operator $R^{(z)}$ in A with $R^{(z)} v = z$. Setting $T_k^{(i_0,z)} = R^{(z)} T_k^{(i_0)}$ we obtain operators satisfying (1).

For the $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ there are two possibilities:

(a_1) There is an element R_0 in A such that for some i_1, i_2 with $1 \leqslant i_1, i_2 \leqslant n+1$, $R_0 x_{i_1} = 0$ and $R_0 x_{i_2} \neq 0$.

(a_2) For any R in A either $Rx_i = 0$ for all i, or $Rx_i \neq 0$ for all i, $1 \leqslant i \leqslant n+1$.

Consider first the case (a_2). We define an operator Q from X to X^n by

$$QRx_1 = (Rx_2, \ldots, Rx_{n+1}), \qquad R \in A.$$

This is a well-defined linear operator, since $Rx_1 = Sx_1$, $S \in A$, implies $Rx_i = Sx_i$ for all remaining indices i. Moreover, by transitivity of A, Q is defined on the whole of X. Consider again two cases:

(b_1) Q is closed, and so continuous ([1], p. 54).

(b_2) Q is not closed.

In the case (b_1) all operators Q_i given by $Q_i R x_1 = R x_{i+1}$ are continuous for $i = 1, \ldots, n$, as compositions of Q and the continuous projections of X^n onto the coordinate spaces. Take for example Q_1. For T in A we have $TQ_1 R x_1 = TRx_2 = Q_1 TRx_1$, and so Q_1 is in A'. By the assumption $Q_1 = \lambda I$ for some scalar λ, where I is the identity operator on X. Thus $\lambda R x_1 = Rx_2$, or $R(\lambda x_1 - x_2) = 0$ for all R in A. This is impossible because $\lambda x_1 - x_2 \neq 0$ and A is transitive. This excludes the case (b_1). In the case (b_2) there is a sequence (R_k) of operators in A such that $\lim R_k x_1 = y_1$, the limits $\lim R_k x_i = y_i$ exist, $i = 2, \ldots, n+1$, but $Qy_1 \neq (y_2, \ldots, y_{n+1})$. Since y_1 can be written as $\tilde{R}_0 x_1$ for some $\tilde{R}_0$ in A, we have $Q\tilde{R}_0 x_1 = (\tilde{R}_0 x_2, \ldots, \tilde{R}_0 x_{n+1})$. Setting $S_k = R_k - \tilde{R}_0$ we obtain the following situation:

(3) $\lim_k S_k x_i = z_i$ exists, $1 \leqslant i \leqslant n+1$, $z_{i_0} = 0$ for some i_0 and $\sum_{i=1}^{n+1} \|z_i\| \neq 0$,

provided (a_2) holds true. (3) is also true in the case (a_1) because we can use as (S_k) the constant sequence $S_k = R_0$. Thus (3) is true in cases (a_1) and (a_2).

The limits in (3) exist for many sequences (S_k) and among these, there are ones with the maximal possible number of the elements z_i in (3) equal to zero. After a suitable renumbering of indices we have the following situation. There is a sequence (T_k) in A such that the following limits exist:

(4) $\lim_k T_k x_i = z_i$, $z_i = 0$ for $1 \leqslant i \leqslant s$ and $z_i \neq 0$ for $s+1 \leqslant i \leqslant n+1$.

Here $1 \leqslant s \leqslant n$ and s is maximal in the sense that if for some sequence (T_k') in A the limits $\lim T_k' x_i = z_i'$ exist, $1 \leqslant i \leqslant n+1$, and more than s of the z_i' are equal to zero then they are all zeroes.

From now on we fix a sequence (T_k) as above, so the $(n+1)$-tuple $(z_1, \ldots, z_{n+1})$ is also fixed. Consider again two cases:

(c_1) $s = n$,
(c_2) $s < n$.

The case (c_1) implies (2) and hence (1) with $i_0 = n+1$. Take now an arbitrary i_0, $1 \leqslant i_0 \leqslant n$. By the inductive assumption the formula (1) holds true for the n-tuple $(x_1, \ldots, x_n)$ and for a given y_0 in X there is a sequence (S_k) in A with $\lim_k S_k x_i = 0$ for $i \neq i_0$, $1 \leqslant i \leqslant n$, and $\lim_k S_k x_{i_0} = y_0$. By passing to a subsequence if necessary, we can assume

$$\|S_m x_i\| \leqslant \frac{1}{2m} \quad \text{for } i \neq i_0,\ 1 \leqslant i \leqslant n, \qquad \|S_m x_{i_0} - y_0\| \leqslant \frac{1}{2m}.$$

Put $y_{n+1} = y_{n+1}(m) = S_m x_{n+1}$. Because we have (1) for $i_0 = n+1$ we can find an operator R_m in A such that

$$\|R_m x_i\| \leqslant \frac{1}{2m} \quad \text{for } 1 \leqslant i \leqslant n, \qquad \|R_m x_{n+1} + y_{n+1}\| \leqslant \frac{1}{2m}.$$

Setting $U_m = S_m + R_m$ we have $U_m \in A$, $\|U_m x_i\| \leqslant 1/m$ for $i \neq i_0$, $1 \leqslant i \leqslant n+1$, and $\|U_m x_{i_0} - y_0\| \leqslant 1/m$. Thus we have (1) for the $(n+1)$-tuple $(x_1, \ldots, x_{n+1})$ so that in the case (c_1) the conclusion follows.

Consider now the case (c_2) and define $p = \dim \operatorname{span}(z_1, \ldots, z_{n+1})$. Consider again two cases:

(d_1) $p > 1$,
(d_2) $p = 1$.

In the case (d_1) choose, among $z_{s+1}, \ldots, z_{n+1}$, p linearly independent elements $z_{i_1}, \ldots, z_{i_p}$, $s < i_k \leqslant n+1$. Since $1 < p \leqslant n$, by the inductive assumption and (2) there is a sequence (V_k) in A such that $\lim_k V_k z_{i_l} = v_{i_l}$, $v_{i_l} = 0$ for $1 \leqslant l < p$ and $v_{i_p} \neq 0$. Because all remaining z_i are linear combinations of $z_{i_1}, \ldots, z_{i_p}$, the limits $\lim_k V_k z_i = v_i$, $1 \leqslant i \leqslant n+1$, all exist, and $v_i = 0$ for $1 \leqslant i \leqslant s$ and for $i = i_l$ with $1 \leqslant l < p$, in particular $v_{i_1} = 0$. As before, passing to a subsequence if necessary, we can assume

$$\|V_m z_i - v_i\| \leqslant 1/m, \quad i = 1, \ldots, n+1.$$

Consider now the operators T_k of (4). For any natural r we have

(5) $\|V_m T_r x_i - v_i\| \leqslant \|V_m T_r x_i - V_m z_i\| + \|V_m z_i - v_i\| \leqslant \|V_m T_r x_i - V_m z_i\| + 1/m$,

for $1 \leqslant i \leqslant n+1$. Since V_m is a continuous operator and $\lim_r T_r x_i = z_i$, there is a natural number r_m such that

(6) $\|V_m T_{r_m} x_i - V_m z_i\| \leqslant 1/m, \quad 1 \leqslant i \leqslant n+1.$

Setting $T_m^{(1)} = V_m T_{r_m}$ we have $T_m^{(1)} \in A$ for $m = 1, 2, \ldots$, and by (5) and (6) we obtain $\lim_m T_m^{(1)} x_i = v_i$, $1 \leqslant i \leqslant n+1$, and $v_i = 0$ for $1 \leqslant i \leqslant s$ and for $i = i_1$. Since $i_1 > s$ and $v_{i_p} \neq 0$ we have the situation as in (4), but with more than s limit elements v_i equal to zero. This contradicts the definition of s and shows that the case (d_1) cannot happen.

Thus we have (d_2), which means that there are scalars λ_i, $s < i \leqslant n$, such that $z_i = \lambda_i z_{n+1}$. In this case we replace $(x_1, \ldots, x_{n+1})$ by $(y_1, \ldots, y_{n+1})$, where $y_i = x_i$ for $1 \leqslant i \leqslant s$ and for $i = n+1$, and $y_i = x_i - \lambda_i x_{n+1}$ for $s < i \leqslant n$. The elements $y_1, \ldots, y_{n+1}$ are linearly independent and the matrix M which sends $(x_1, \ldots, x_{n+1})$ to $(y_1, \ldots, y_{n+1})$ has a nonzero determinant. We also have $\lim_k T_k y_i = 0$ for $1 \leqslant i \leqslant n$ and $\lim_k T_k y_{n+1} \neq 0$. Thus we have the situation as in (c_1) with $x_1, \ldots, x_{n+1}$ replaced by $y_1, \ldots, y_{n+1}$ and in that case we have shown that $\mathcal{O}(A, y_1, \ldots, y_{n+1})$ is dense in X^{n+1}. Applying the lemma we see that $\mathcal{O}(A, x_1, \ldots, x_{n+1})$ is dense in X^{n+1}. The conclusion of the theorem follows.

The above theorem fails without the assumption that the commutant A' is trivial. In [7] Waelbroeck constructed an infinite-dimensional F-algebra (an *F-algebra* being an F-space with a jointly continuous multiplication) which is a field, i.e. each its nonzero element is invertible. If X is such an algebra and T_a, $a \in X$, is the operator given by $T_a x = ax$, then $T_a \in L(X)$, and the algebra A of all such operators is transitive. On the other hand, for any two elements $x_1, x_2 \in X$ the orbit $\mathcal{O}(A, x_1, x_2)$ cannot be dense in X^2. Indeed, if $T_{a_n} x_1 \to v$, i.e. $a_n x_1 \to v$, and $x_1 \neq 0$ then $a_n \to v x_1^{-1}$, and so $T_{a_n} x_2 = a_n x_2 \to v x_1^{-1} x_2$ and the limit $\lim T_{a_n} x_2$ is uniquely determined by v. Also the assumption that the considered space X is complete was essential in our proof. We do not know whether the theorem is true without assuming the completeness of X or without assuming its metrizability. We do not know either whether a stronger conclusion is possible, namely that the orbits $\mathcal{O}(A, x_1, \ldots, x_n)$ are equal to X^n for $x_1, \ldots, x_n$ linearly independent. Nor do we know whether the condition of transitivity can be replaced by the weaker assumption of density of the single orbits $\mathcal{O}(A, x)$, $x \neq 0$.

§ 3. **The density theorem in representation theory.** For a real or complex linear space X denote by $\mathscr{L}(X)$ the algebra of all its endomorphisms. Let A be an algebra over the same field of scalars as X. A *representation* of A is a homomorphism T, $a \to T_a$, of A into $\mathscr{L}(X)$. A representation T is said to be (algebraically) *irreducible* if the only invariant subspaces for T (i.e. subspaces $X_0 \subset X$ such that $T_a X_0 \subset X_0$ for all a in A) are (0) and X. For an irreducible representation T of A on X put

$$\mathscr{D} = \{R \in \mathscr{L}(X)\colon RT_a = T_a R \text{ for all } a \text{ in } A\}.$$

Schur's lemma (see [2], p. 121) says that $\mathscr{D}$ is a division algebra and the Jacobson density theorem says that for $\mathscr{D}$-independent elements $x_1, \ldots, x_n$ in X (i.e. $\sum_{i=1}^n D_i x_i = 0$, $D_i \in \mathscr{D}$, implies $D_i = 0$ for $i = 1, \ldots, n$) the orbits

$$\mathcal{O}(T, x_1, \ldots, x_n) = \{(T_a x_1, \ldots, T_a x_n) \in X^n\colon a \in A\}$$

coincide with X^n (see [2], p. 123, Theorem 10).

In the case when X is a topological vector space (t.v.s.) a representation T of A on X is said to be a *t.v.s.-representation* if all operators T_a are continuous. Such a representation is said to be *algebraically irreducible* if it is irreducible in the sense described above, and it is called *irreducible* if there are no proper (i.e. different from (0) and X) closed T-invariant subspaces of X. In the case when X is a normed space one can put a submultiplicative homogeneous norm on the division algebra $\mathscr{D}$ of the Schur lemma. Thus by the Gelfand–Mazur theorem $\mathscr{D} = \boldsymbol{C}$ if X is a complex normed space. In this case the Jacobson density theorem implies that an algebraically irreducible representation T is *algebraically totally irreducible*, i.e. the orbits $\mathcal{O}(T, x_1, \ldots, x_n)$ coincide with X^n for all n-tuples $x_1, \ldots, x_n$ of linearly independent elements. We say that T is *totally irreducible* if the orbits $\mathcal{O}(T, x_1, \ldots, x_n)$ are dense in X^n for linearly independent elements $x_1, \ldots, x_n$. This property is weaker than algebraic total irreducibility. For a t.v.s.-representation T of A on X call an operator R in $L(X)$ a *T-intertwining operator* if $RT_a x = T_a Rx$ for all x in X and all a in A.

One of important problems of representation theory (cf. [3], p. 329, problem II) is the question whether an irreducible t.v.s.-representation T of an algebra A, whose only T-intertwining operators are scalar multiples of the identity operator, is necessarily totally irreducible.

Our Theorem 1, formulated as in the abstract, can be treated as a partial answer to this question (in the case when X is an F-space and in addition T is algebraically irreducible).

§ 4. **An application to strong generation.** Let X be a real or complex space of type F. The strong operator topology on $L(X)$ is given by a basis of neighbourhoods of T_0 in $L(X)$ of the form

(7) $$V(T_0; x_1, \ldots x_n, \varepsilon) = \{T \in L(X)\colon \|T_0 x_i - Tx_i\| < \varepsilon,\ i = 1, \ldots, n\},$$

where $x_1, \ldots, x_n \in X$, $\varepsilon > 0$, and $\|\cdot\|$ is an F-norm generating the topology of X. Without loss of generality we can assume that the x_i in (7) are linearly independent. The strong topology is the topology of pointwise convergence of nets of operators and it is nonmetrizable if X is infinite-dimensional and locally convex (for rigid spaces it is, of course, metrizable).

We say that a subset S of $L(X)$ *strongly generates* $L(X)$ if the smallest subalgebra $A(S)$ of $L(X)$ which contains S and is closed in the strong operator topology coincides with $L(X)$. We prove the following easy

PROPOSITION. *Let X be a real or complex space of type F. Suppose that S is a subset of $L(X)$ such that for any n-tuple $(x_1, \ldots, x_n)$ of linearly independent elements of X the orbit $\mathcal{O}(A(S), x_1, \ldots, x_n)$ is dense in X^n. Then S strongly generates $L(X)$. Conversely, suppose that S strongly generates $L(X)$. Then $\mathcal{O}(A(S), x_1, \ldots, x_n) = X^n$ for any n-tuple $(x_1, \ldots, x_n)$ of linearly independent elements of X provided X is in addition a locally convex space (a B_0-space).*

Proof. If for every n-tuple $(x_1, \ldots, x_n)$ of linearly independent elements of X the orbit $\mathcal{O}(A(S), x_1, \ldots, x_n)$ is dense in X^n, then choosing an arbitrary operator T_0 in $L(X)$ we see immediately that any neighbourhood of T_0 of the form (7) contains an element T of $A(S)$, and since $A(S)$ is closed in the strong operator topology, we have $T_0 \in A(S)$.

Conversely, if S strongly generates $L(X)$, i.e. $L(X) = A(S)$, then assuming X to be locally convex we can for given linearly independent elements $x_1, \ldots, x_n \in X$ find continuous linear functionals f_i on X such that $f_i(x_j) = \delta_{i,j}$ (the Kronecker symbol). Choosing arbitrarily $y_1, \ldots, y_n$ in X and setting $Tx = \sum_{i=1}^n f_i(x) y_i$ we obtain an element T in $A(S)$ with $Tx_i = y_i$, $i = 1, \ldots, n$. Thus $(y_1, \ldots, y_n) \in \mathcal{O}(A(S), x_1, \ldots, x_n)$, and so this orbit equals X^n. The conclusion follows.

We can now formulate our characterization theorem.

THEOREM 2. *Let X be a real or complex B_0-space. A subset $S \subset L(X)$ strongly generates $L(X)$ if and only if*

(i) *the commutant S' consists of scalar multiples of the identity operator, and*

(ii) *the algebra $A(S)$ strongly generated by S is transitive.*

Proof. One can easily see that the commutants of S and of $A(S)$ coincide. If S strongly generates $L(X)$, $L(X) = A(S)$, then clearly $A(S)$ is transitive and $S' = [A(S)]'$ consists of scalar multiples of the identity operator. On the other hand, if we have (i) and (ii), then $[A(S)]'$ consists of scalar multiples of the identity operator and by Theorem 1 all orbits $\mathcal{O}(A(S), x_1, \ldots, x_n)$ are dense in X^n, provided the x_i are linearly independent. The conclusion now follows from the proposition.

It would be desirable to replace condition (ii) by

(iii) $\mathrm{Lat}(S) = \{(0), X\}$,

where $\mathrm{Lat}(S)$ is the lattice of all closed subspaces of X which are invariant with respect to all operators T in S. One can see that Theorem 2 so modified would give a positive solution to the problem mentioned in § 3 in the case when X is a B_0-space.

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STUDIA MATHEMATICA, T. XCVI (1990)

Multifonctions analytiques polygonales

par

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Abstract. The spectrum of an analytic family of elements of a Banach algebra is an example of an analytic multivalued function. In this paper we study continuous analytic multivalued functions whose values are convex polygons. It is shown that if the number of vertices is bounded, then these vertices vary holomorphically away from branch points. Examples are given to show that this exceptional set can be quite large, and that the number of vertices may be unbounded. This shows that such a function cannot be seen as the convex hull of a finite analytic multivalued function, as was the case for segment-valued functions.

1. Introduction. La méthode sousharmonique s'est révélée au cours des dernières années une technique très puissante pour étudier la théorie spectrale dans les algèbres de Banach. Dans [8], Z. Słodkowski montre que cette théorie se laisse ramener à l'étude des *fonctions analytiques multiformes*, ou *multifonctions analytiques*, qu'on définit comme suit:

DÉFINITION 1. Soit $\lambda \to K(\lambda)$ une fonction définie sur un domaine D de $\mathbf{C}^n$, prenant ses valeurs dans $\mathscr{K}(\mathbf{C}^m)$, l'ensemble des compacts non vides de $\mathbf{C}^m$. On dira que la fonction K est *analytique* si elle satisfait les deux propriétés suivantes:

1. K est semi-continue supérieurement.

2. Pour tout ouvert $D_1 \subseteq D$, si V est un voisinage de $\mathrm{Gr}(K|_{D_1}) = \{(\lambda, z): \lambda \in D_1, z \in K(\lambda)\}$, et $\psi(\lambda, z)$ une fonction plurisousharmonique sur V, alors la fonction

$$\Psi(\lambda) = \sup\{\psi(\lambda, z):\ z \in K(\lambda)\}$$

est plurisousharmonique sur D_1.

Si $\lambda \mapsto f(\lambda)$ est une fonction holomorphe à valeurs dans une algèbre de Banach, la fonction multiforme $\lambda \mapsto \mathrm{Sp} f(\lambda)$ est analytique. La théorie des algèbres uniformes fournit d'autres exemples de fonctions analytiques multiformes [2].

Certains fonctions analytiques multiformes sont bien connues. Par exemple, les multifonctions analytiques finies sont précisément les fonctions algébroïdes, c'est-à-dire de la forme

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