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# Boundedness of classical operators on classical Lorentz spaces

by

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**Abstract.** The classical Lorentz space $\Lambda_p(v)$ consists of those measurable functions $f$ on $\mathbf{R}^n$ such that $(\int_0^\infty f^*(x)^p v(x)\,dx)^{1/p} < \infty$. We characterize when a variety of classical operators, including Hilbert and Riesz transforms, fractional integrals and maximal functions, are bounded from one Lorentz space, $\Lambda_p(v)$, to another, $\Lambda_q(w)$. In addition, we give a simple and explicit description of the dual of $\Lambda_p(v)$ and determine when $\Lambda_p(v)$ is a Banach space.

**§ 1. Introduction.** For $1 \leqslant p < \infty$ and $v(x)$ a nonnegative function on $(0, \infty)$, the classical Lorentz spaces $\Lambda_p(v)$ on $\mathbf{R}^n$, introduced and studied by G. Lorentz in [7] for the intervals $(0, l)$, $0 < l \leqslant \infty$, are given by

$$\Lambda_p(v) = \{f \text{ measurable on } \mathbf{R}^n\colon (\int_0^\infty f^*(x)^p v(x)\,dx)^{1/p} < \infty\},$$

where $f^*(x) = \inf\{\lambda\colon |\{t \in \mathbf{R}^n\colon |f(t)| > \lambda\}| \leqslant x\}$ is the nonincreasing rearrangement of $f$ on $(0, \infty)$ with respect to Lebesgue measure on $\mathbf{R}^n$ ($|E|$ denotes the Lebesgue measure of a set $E$). M. Ariño and B. Muckenhoupt observed in [2] that the Hardy–Littlewood maximal operator $M$, defined by

$$Mf(x) = \sup\{|Q|^{-1} \int_Q |f(y)|\,dy\colon Q \text{ is a cube in } \mathbf{R}^n \text{ containing } x\},$$

is bounded from $\Lambda_p(v)$ to $\Lambda_p(w)$ if and only if

$$(1.1)\qquad (\int_0^\infty (x^{-1} \int_0^\infty f(t)\,dt)^q w(x)\,dx)^{1/q} \leqslant C(\int_0^\infty f(x)^p v(x)\,dx)^{1/p}$$

for all nonnegative and nonincreasing functions $f$ on $(0, \infty)$. Indeed, this follows immediately from the rearrangement inequality for the maximal function ([6], [12] and [15])

$$(1.2)\qquad (Mf)^*(x) \leqslant C_1 x^{-1} \int_0^x f^*(t)\,dt \leqslant C_2 (Mf)^*(x), \qquad x > 0,$$

coupled with the fact that every nonincreasing function $f^*$ on $(0, \infty)$ occurs as

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the rearrangement of some function $\tilde{f}$ on $\mathbf{R}^n$, e.g. $\tilde{f}(y) = f^*(A|y|^n)$ where $A$ is the volume of $S^{n-1}$. Moreover, Ariño and Muckenhoupt showed that in the case $1 < p = q < \infty$ and $w(x) = v(x)$, (1.1) holds for all nonnegative and nonincreasing $f$ if and only if

$$(1.3)\qquad \Big(\int_r^\infty x^{-q} w(x)\,dx\Big)^{1/q} \leqslant \frac{B}{r}\Big(\int_0^r v(x)\,dx\Big)^{1/p}, \quad \text{for all } r > 0.$$

In this paper, we characterize when a variety of classical operators, including the Hardy–Littlewood maximal function, fractional integrals and Hilbert and Riesz transforms, are bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$, for arbitrary weights $w$, $v$ and indices $p$, $q$. We begin by outlining our approach. Without the restriction to nonincreasing $f$, inequality (1.1) has been studied extensively and in the case $1 < p \leqslant q < \infty$, holds for all nonnegative $f$ if and only if ([1], [3], [5], [11], [13] and [14])

$$(1.4)\qquad \Big(\int_r^\infty x^{-q} w(x)\,dx\Big)^{1/q}\Big(\int_0^r v(x)^{1-p'}\,dx\Big)^{1/p'} \leqslant B, \quad \text{for all } r > 0,$$

where $1/p + 1/p' = 1$. In the case $1 < q < p < \infty$, (1.1) holds for all nonnegative $f$ if and only if ([10]; p. 47)

$$(1.5)\qquad \Big\{\int_0^\infty \Big[\Big(\int_x^\infty t^{-q} w(t)\,dt\Big)^{1/q}\Big(\int_0^x v^{1-p'}\Big)^{1/q'}\Big]^r v(x)^{1-p'}\,dx\Big\}^{1/r} < \infty,$$

where $1/r = 1/q - 1/p$.

A key step in obtaining these characterizations of (1.1) for nonnegative $f$ is Hölder's inequality and its converse in the form:

$$(1.6)\qquad \sup_{f \text{ nonnegative}} \frac{\int_0^\infty f(x) g(x)\,dx}{\Big(\int_0^\infty f(x)^p v(x)\,dx\Big)^{1/p}} = \Big(\int_0^\infty g(x)^{p'} v(x)^{1-p'}\,dx\Big)^{1/p'},$$

for all $v$, $g \geqslant 0$. A basic result of this paper is the analogue of (1.6) when the supremum is restricted to nonnegative and nonincreasing functions $f$ — a characterization of (1.1) for nonincreasing $f$ then follows by duality. In the setting of [8], the following theorem provides a simple and explicit equivalent norm for the dual space of $\Lambda_p(v)$, improving on the necessarily more complicated expression for the dual norm itself, found by I. Halperin ([8]; Theorem 3.6.5); the case $p = 1$ is Theorem 6 of [7]. As usual, we adhere to the convention that $0 \cdot \infty = 0/0 = \infty/\infty = 0$.

THEOREM 1. *Suppose* $1 < p < \infty$ *and that* $v(x)$ *and* $g(x)$ *are nonnegative measurable functions on* $[0, \infty)$ *with* $v$ *locally integrable. Then*



$$(1.7)\qquad \sup_{\substack{f \text{ nonnegative}\\ \text{and nonincreasing}}} \frac{\int_0^\infty f(x) g(x)\,dx}{\Big(\int_0^\infty f(x)^p v(x)\,dx\Big)^{1/p}} \approx \left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'} v(x)\,dx\right)^{1/p'}$$

$$\approx \Big(\int_0^\infty \Big(\int_0^x g\Big)^{p'-1}\Big(\int_0^x v\Big)^{1-p'} g(x)\,dx\Big)^{1/p'}$$

$$\approx \left(\int_0^\infty \Big(\int_0^x g\Big)^{p'} \frac{v(x)}{\big(\int_0^x v\big)^{p'}}\,dx\right)^{1/p'} + \frac{\big(\int_0^\infty g\big)}{\big(\int_0^\infty v\big)^{1/p}},$$

*where the symbol* $\approx$ *in* (1.7) *means that the ratio of left and right hand sides is bounded between two positive constants depending only on* $p$ (*and not on* $v$ *or* $g$).

Remark. Let $\Gamma_p(v)$ be defined in the same way as $\Lambda_p(v)$, but with the nonincreasing rearrangement $f^*$ of $f$ replaced by the average nonincreasing rearrangement, $f^{**}(x) = x^{-1}\int_0^x f^*(t)\,dt$:

$$\Gamma_p(v) = \Big\{f \text{ measurable on } \mathbf{R}^n\colon \Big(\int_0^\infty f^{**}(x)^p v(x)\,dx\Big)^{1/p} < \infty\Big\}.$$

If $\int_0^\infty v = \infty$, so that the second summand on the right side of (1.7) is zero, then Theorem 1 together with the inequality $\big|\int_{\mathbf{R}^n} fg\big| \leqslant \int_0^\infty f^* g^*$ shows that the dual of $\Lambda_p(v)$ can be identified with $\Gamma_{p'}(\tilde{v})$, where $\tilde{v}(x) = \big(x^{-1}\int_0^x v\big)^{-p'} v(x)$, under the pairing $(f, g) = \int_{\mathbf{R}^n} f(y) g(y)\,dy$, $f \in \Lambda_p(v)$, $g \in \Gamma_{p'}(\tilde{v})$.

Theorem 1 provides a duality principle which permits a weighted inequality for nonnegative and nonincreasing functions to be replaced by an equivalent inequality for nonnegative functions. For example, if $Tf(x) = \int_0^\infty K(x, y) f(y)\,dy$, where $K(x, y) \geqslant 0$, then (1.6) and (1.7) show that

$$(1.8)\qquad \Big(\int_0^\infty Tf(x)^q w(x)\,dx\Big)^{1/q} \leqslant C\Big(\int_0^\infty f(x)^p v(x)\,dx\Big)^{1/p}$$

for all nonnegative and nonincreasing functions $f$ if and only if

$$(1.9)\qquad \left(\int_0^\infty \Big(\int_0^x T^* g\Big)^{p'} \frac{v(x)}{\big(\int_0^x v\big)^{p'}}\,dx\right)^{1/p'} + \frac{\big(\int_0^\infty T^* g\big)}{\big(\int_0^\infty v\big)^{1/p}} \leqslant C\Big(\int_0^\infty g(x)^{q'} w(x)^{1-q'}\,dx\Big)^{1/q'}$$

for all nonnegative functions $g$. Here $T^*$ denotes the adjoint of $T$ given by $T^* g(y) = \int K(x, y) g(x)\,dx$. In the case $Tf(x) = x^{-1}\int_0^x f(y)\,dy$,

$$\int_0^\infty T^*g(y)\,dy = \int_0^x\int_y^\infty t^{-1}g(t)\,dt\,dy = \int_0^x g(t)\,dt + x\int_x^\infty t^{-1}g(t)\,dt$$
$$= Pg(x) + xP^*(t^{-1}g(t))(x),$$

where $Pf(x) = \int_0^x f(t)\,dt$ is the Hardy operator, and (1.9) can be characterized using the known characterizations of weighted inequalities for the Hardy operator $P$ and its adjoint $P^*$ (p. 47 of [10]; [11]). This leads to the following result.

THEOREM 2. *Suppose that $w(x)$ and $v(x)$ are nonnegative measurable functions on $(0, \infty)$. If $1 < p \leqslant q < \infty$, then $M$ is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$, or equivalently (1.1) holds for all nonnegative and nonincreasing functions $f$, if and only if both of the following conditions hold:*

(1.10) $$\Big(\int_0^r w(x)\,dx\Big)^{1/q} \leqslant A\Big(\int_0^r v(x)\,dx\Big)^{1/p}, \quad \textit{for all } r > 0;$$

(1.11) $$\Big(\int_r^\infty x^{-q}w(x)\,dx\Big)^{1/q}\Big(\int_0^r \Big(x^{-1}\int_0^x v\Big)^{-p'} v(x)\,dx\Big)^{1/p'} \leqslant B, \quad \textit{for all } r > 0.$$

*Moreover, if $C$ is the best constant in (1.1), then $C \approx A+B$. If $1 < q < p < \infty$, then $M$ is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$, or equivalently (1.1) holds for all nonnegative and nonincreasing functions $f$, if and only if both of the following conditions hold:*

(1.12) $$\Big(\int_0^\infty \Big[\Big(\int_0^x w(t)dt\Big)^{1/p}\Big(\int_0^x v(t)\,dt\Big)^{-1/p}\Big]^r w(x)\,dx\Big)^{1/r} = A < \infty;$$

(1.13) $$\Big(\int_0^\infty \Big[\Big(\int_x^\infty t^{-q}w(t)\,dt\Big)^{1/p}\Big(\int_0^x \Big(t^{-1}\int_0^t v\Big)^{-p'} v(t)\,dt\Big)^{1/p'}\Big]^r \Big(x^{-1}\int_0^x v\Big)^{-p'} v(x)\,dx\Big)^{1/r}$$
$$= B < \infty,$$

*where $1/r = 1/q - 1/p$. Moreover, if $C$ is the best constant in (1.1), then $C \approx A+B$.*

Remarks. (i) For $1 < p \leqslant q < \infty$, $\Lambda_p(v) \subset \Lambda_q(w)$ if and only if (1.10) holds. Indeed, (1.8) holds for the identity operator if and only if (1.9) holds with $\int_0^x T^*g = \int_0^x g = Pg(x)$. By [11] and Hölder's inequality, this latter holds if and only if

$$\Big(\int_r^\infty \Big(\int_0^x v\Big)^{-p'} v(x)\,dx\Big)^{1/p'}\Big(\int_0^r w\Big)^{1/q} \leqslant A, \quad r > 0, \quad \text{and} \quad \Big(\int_0^\infty w\Big)^{1/q} \leqslant C\Big(\int_0^\infty v\Big)^{1/p},$$

which together are equivalent to (1.10) (see the proof of Theorem 2 in § 3 for details). For $1 < q < p < \infty$, similar reasoning using ([10], p. 47) shows that $\Lambda_p(v) \subset \Lambda_q(w)$ if and only if (1.12) holds.

(ii) Neither (1.10) nor (1.11) alone is sufficient for (1.1) to hold for nonincreasing $f$. For example, if $w(x) = x^{q-1}$ and $v(x) = x^{p-1}$, then (1.10) holds but not (1.11), while if $w(x) = x^q(x+1)^{-q-1}$ and $v(x) = \chi_{(0,1)}(x)$, then (1.11) holds but not (1.10).



(iii) Conditions (1.3) and (1.10) together are not in general sufficient for (1.1) to hold for nonincreasing $f$. Let $v(x) = x^{p-1}$. If $x^{-q}w(x)$ is respectively integrable, bounded by $x^{-1}$, and not identically zero, then respectively (1.3) holds, (1.10) holds and (1.11) fails.

(iv) In the case $1 < p = q < \infty$ and $w(x) = v(x)$ considered in [2], condition (1.10) becomes vacuous while (1.11) follows easily from (1.3) using Lemma 2.1 of [2].

(v) There are analogues of Theorem 2, as well as the other results in this paper, for Lorentz spaces on the circle $\mathbf{T}$ where rearrangements are taken with respect to Haar measure. Integrals on $(0, \infty)$ are replaced by integrals on $(0, 1)$ in (1.7), (1.8) and (1.9), with corresponding changes elsewhere.

In Theorem 2, we used Theorem 1 to characterize when the maximal function is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$. Similar results can be obtained from Theorem 1 for any operator $S$ on $\mathbf{R}^n$ for which a pair of rearrangement inequalities of the following form hold for some positive linear operator $T$:

(1.14) $$(Sf)^*(x) \leqslant C_1 T(f^*)(x), \quad x > 0,$$

for all $f$ with $T(f^*)$ finite on $(0, \infty)$, and conversely, for $f^*$ nonnegative and nonincreasing on $[0, \infty)$ and say, bounded with compact support, there is a function $\tilde{f}$ on $\mathbf{R}^n$ with $\tilde{f}^* \leqslant f^*$ and

$$T(f^*)(x) \leqslant C_2(S\tilde{f})^*(x), \quad x > 0.$$

Such sharp rearrangement inequalities are known for many classical operators, including Riesz transforms and fractional integrals (see below—a notable exception seems to be the Fourier transform for which none of the known inequalities of the form (1.14) are sharp), where the operator $T$ is sufficiently simple that (1.9) can be characterized. We illustrate the idea with the Hilbert transform $H$, given by the principle value integral

$$Hf(x) = \lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} \int_{\varepsilon<|y|<\eta} \frac{f(x-y)}{y}\,dy,$$

which satisfies the rearrangement inequality

(1.15) $$(Hf)^*(x) \leqslant C_1\Big[x^{-1}\int_0^x f^*(t)\,dt + \int_x^\infty t^{-1}f^*(t)\,dt\Big] \leqslant C_2(Hf^*)^*(x).$$

The first inequality in (1.15) is proved in [4] while the second follows upon considering $Hf^*(y)$ for $y < 0$. From (1.15), we deduce that $H$ is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ if and only if (1.8) holds for all nonincreasing $f$ with

$$Tf(x) = x^{-1}\int_0^x f(t)\,dt + \int_x^\infty t^{-1}f(t)\,dt.$$

By Theorem 1, this holds if and only if (1.9) holds for all nonnegative $g$ with

$$(1.16)\quad \int_0^x T^*g(y)\,dy = \int_0^x \Big[y^{-1}\int_0^y g(t)\,dt + \int_y^\infty t^{-1}g(t)\,dt\Big]\,dy, \qquad \text{since } T^* = T,$$

$$= \int_0^x \ln(x/t)g(t)\,dt + \int_0^x g(t)\,dt + x\int_x^\infty t^{-1}g(t)\,dt$$

$$= T_\varphi g(x) + Pg(x) + xP^*(t^{-1}g(t))(x),$$

where $T_\varphi g(x) = \int_0^x \varphi(t/x)g(t)\,dt$ and $\varphi(t) = \ln_+(1/t)$. Using the weighted inequality for $T_\varphi$ in [9] together with Theorem 2, we obtain

THEOREM 3. *Suppose $1 < p \leqslant q < \infty$ and that $w(x)$ and $v(x)$ are nonnegative measurable functions on $(0, \infty)$. Then the Hilbert transform is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ if and only if $\int_0^\infty v(x)\,dx = \infty$, (1.10) and (1.11) hold, and the following two conditions hold:*

$$(1.17)\quad \Big(\int_0^r (\ln(r/x))^q w(x)\,dx\Big)^{1/q} \leqslant A\Big(\int_0^r v(x)\,dx\Big)^{1/p}, \quad \text{for all } r > 0,$$

$$(1.18)\quad \Big(\int_0^r w(x)\,dx\Big)^{1/q}\Big(\int_r^\infty (\ln(x/r))^{p'}\Big(\int_0^x v\Big)^{-p'}v(x)\,dx\Big)^{1/p'} \leqslant B, \quad \text{for all } r > 0.$$

Note that, unlike the maximal function, the Hilbert transform is not necessarily bounded on $\Lambda_p(v)$ if (1.3) holds with $p = q$ and $w = v$. Indeed, when $p = q$ and $w = v$, (1.3) can hold with $\int_0^\infty v < \infty$, for example $v(x) = \chi_{(0,1)}(x)$.

Since the Riesz transforms $R_j$, $1 \leqslant j \leqslant n$, and even more general singular integrals, satisfy essentially the same rearrangement inequality as does the Hilbert transform:

$$(1.19)\qquad (R_j f)^*(x) \leqslant C_1 Tf^*(x) \leqslant C_2(R_j\tilde{f})^*(x), \qquad x > 0,$$

where $T$ is as above, $\tilde{f}(y) = f^*(A|y|^n)\chi_{(0,\infty)}(y_j)$ and $A$ is the volume of $S^{n-1}$, it follows that the Riesz transforms are bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ if and only if $\int_0^\infty v = \infty$ and (1.10), (1.11), (1.17), and (1.18) hold. The first inequality in (1.19) is in Theorem 16.12 of [4] while the second follows upon considering $R_j\tilde{f}$ for $y_j < 0$.

Similarly one can characterize when the Riesz potentials $I_\alpha$, $0 < \alpha < n$, are bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ using the rearrangement inequality

$$(1.20)\quad (I_\alpha f)^*(x) \leqslant C_1\Big[x^{\alpha/n-1}\int_0^x f^*(t)\,dt + \int_x^\infty t^{\alpha/n-1}f^*(t)\,dt\Big]$$

$$\leqslant C_2(I_\alpha\tilde{f})^*(x), \qquad x > 0,$$

where $\tilde{f}(y) = f^*(A|y|^n)$. The first inequality in (1.20) is Corollary 17.3 of [4] and the second inequality is an easy exercise.



Finally, Theorem 1 together with the result of Ariño and Muckenhoupt in [2] can be used to determine when $\Lambda_p(v)$ is a Banach space, i.e. when there exists a norm $\|\ \|$ on $\Lambda_p(v)$ and positive constants $C_1$ and $C_2$ such that

$$(1.21)\quad C_1\|f\| \leqslant \Big(\int_0^\infty f^*(x)^p v(x)\,dx\Big)^{1/p} \leqslant C_2\|f\|, \quad \text{for all } f \text{ in } \Lambda_p(v).$$

In Theorem 1 of [7], G. Lorentz showed that the middle expression in (1.21) satisfies the triangle inequality if and only if $v$ is nonincreasing, but the next theorem shows that $\Lambda_p(v)$ is a Banach space for a much wider class of weights $v$, namely those satisfying (1.3) with $p = q$ and $w(x) = v(x)$. We remark that $\Gamma_p(v)$, defined following Theorem 1, is a Banach space since $(f+g)^{**} \leqslant f^{**} + g^{**}$.

THEOREM 4. *Suppose $1 < p < \infty$ and $v(x)$ is a nonnegative measurable function on $[0, \infty)$. Then the following four conditions are equivalent:*

$$(1.22)\qquad \Lambda_p(v) \text{ is a Banach space},$$

$$(1.23)\quad \Lambda_p(v) = \Gamma_p(v) \text{ and (1.21) holds with } \|f\| = \Big(\int_0^\infty f^{**}(x)^p v(x)\,dx\Big)^{1/p},$$

$$(1.24)\quad \Big(\int_0^r v(x)\,dx\Big)^{1/p}\Big(\int_0^r \Big(x^{-1}\int_0^x v\Big)^{1-p'}dx\Big)^{1/p'} \leqslant Cr, \quad \text{for all } r > 0,$$

$$(1.25)\quad \Big(\int_r^\infty x^{-p}v(x)\,dx\Big)^{1/p} \leqslant \frac{B}{r}\Big(\int_0^r v(x)\,dx\Big)^{1/p}, \quad \text{for all } r > 0.$$

Theorem 1 is proved in § 2, Theorems 2 and 3 are proved in § 3 and Theorem 4 is proved in § 4. The letter $C$ will be used to denote a positive constant that may change from line to line but will remain independent of the appropriate quantities.

**§ 2. Proof of Theorem 1.** By virtue of the monotone convergence theorem, we may assume that $g$ is supported in a compact subset of $(0, \infty)$ with $\int_0^\infty g(x)\,dx = 1$, and then by considering $v + \chi_{(0,\delta)}$ in place of $v$ for small $\delta$, that $0 < \int_0^x v(t)\,dt < \infty$ for all $x > 0$. Set

$$\varphi(x) = \left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'-1}, \qquad 0 < x < \infty.$$

Then $\varphi$ is bounded and nonincreasing on $(0, \infty)$ and an integration by parts shows that

$$\int_0^\infty \varphi(x)^p v(x)\,dx = \varphi(x)^p\int_0^x v\Big|_0^\infty - \int_0^\infty p\varphi(x)^{p-1}\varphi'(x)\Big(\int_0^x v\Big)dx$$

$$= p'\int_0^\infty \varphi(x)g(x)\,dx.$$

Thus the supremum on the left side of (1.7) is at least

$$(2.1)\quad \frac{1}{p'}\Big(\int_0^\infty \varphi(x)^p v(x)\,dx\Big)^{1/p'} = \frac{1}{p'}\left(\int_0^\infty\left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'} v(x)\,dx\right)^{1/p'}$$

$$= \left(\frac{1}{p'}\right)^{1/p}\Big(\int_0^\infty \varphi(x)g(x)\,dx\Big)^{1/p'} = \left(\frac{1}{p'}\right)^{1/p}\left(\int_0^\infty\left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'-1} g(x)\,dx\right)^{1/p'}.$$

Conversely, for $f(x)$ nonnegative and nonincreasing,

$$(2.2)\quad \int_0^\infty f(x)g(x)\,dx = \int_0^\infty f(x)\frac{g(x)}{\int_0^x v}\int_0^x v(t)\,dt\,dx$$

$$= \int_0^\infty\left(\int_t^\infty \frac{f(x)g(x)}{\int_0^x v}\,dx\right)v(t)\,dt$$

$$\leqslant \int_0^\infty f(t)\left(\int_t^\infty \frac{g(x)}{\int_0^x v}\,dx\right)v(t)\,dt, \quad \text{since } f \text{ nonincreasing},$$

$$\leqslant \Big(\int_0^\infty f(t)^p v(t)\,dt\Big)^{1/p}\left(\int_0^\infty\left(\int_t^\infty \frac{g(x)}{\int_0^x v}\,dx\right)^{p'} v(t)\,dt\right)^{1/p'}$$

by Hölder's inequality. The first equivalence in (1.7) follows from (2.1) and (2.2).

Now let $x_j$ satisfy $\int_0^{x_j} g(x)\,dx = 2^j$, $-\infty < j \leqslant 0$. Then

$$(2.3)\quad \int_0^\infty\left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'-1} g(x)\,dx = \sum_{j=-\infty}^{0}\int_{x_{j-1}}^{x_j}\left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'-1} g(x)\,dx$$

$$\geqslant \sum_{j=-\infty}^{-1}\Big(\int_{x_j}^{x_{j+1}} g\Big)^{p'-1}\Big(\int_0^{x_{j+1}} v\Big)^{1-p'}\Big(\int_{x_{j-1}}^{x_j} g\Big)$$

$$\geqslant C\sum_{j=-\infty}^{-2}\Big(\int_0^{x_{j+2}} g\Big)^{p'-1}\Big(\int_0^{x_{j+1}} v\Big)^{1-p'}\Big(\int_{x_{j+1}}^{x_{j+2}} g\Big), \quad \text{since } \int_0^{x_j} g = 2^j,$$

$$\geqslant C\int_0^\infty\Big(\int_0^x g\Big)^{p'-1}\Big(\int_0^x v\Big)^{1-p'} g(x)\,dx.$$

On the other hand, if $\int_0^{x_j} v(x)\,dx = 2^j$ for all $j$ with $2^j < \int_0^\infty v(x)\,dx$ (and $x_N = \infty$ if $N$ is the largest integer with $2^{N-1} < \int_0^\infty v(x)\,dx$), then



$$(2.4)\quad \int_0^\infty\left(\int_x^\infty \frac{g(t)}{\int_0^t v}\,dt\right)^{p'} v(x)\,dx \leqslant \sum_j\Big(\int_{x_j}^{x_{j+1}} v(x)\,dx\Big)\Big(\sum_{k\geqslant j}\Big(\int_{x_k}^{x_{k+1}} g\Big)\Big(\int_0^{x_k} v\Big)^{-1}\Big)^{p'}$$

$$\leqslant C\sum_j 2^j\Big(\sum_{k\geqslant j} 2^{-k}\int_{x_k}^{x_{k+1}} g\Big)^{p'}$$

$$\leqslant C\sum_j 2^j\Big(2^{-j}\int_{x_j}^{x_{j+1}} g\Big)^{p'}$$

$$= C\sum_j\Big(\tfrac{1}{2}\int_0^{x_{j+1}} v\Big)^{1-p'}\int_{x_j}^{x_{j+1}} p'\Big(\int_{x_j}^{x} g\Big)^{p'-1} g(x)\,dx$$

$$\leqslant C\sum_j\int_{x_j}^{x_{j+1}}\Big(\int_0^x g\Big)^{p'-1}\Big(\int_0^x v\Big)^{1-p'} g(x)\,dx$$

$$= C\int_0^\infty\Big(\int_0^x g\Big)^{p'-1}\Big(\int_0^x v\Big)^{1-p'} g(x)\,dx,$$

where we have used the inequality

$$\sum_j 2^j\Big(\sum_{k\geqslant j} a_k\Big)^{p'} = \sum_j 2^j\Big(\sum_{k\geqslant j} 2^{(j-k)/(pp')}2^{(k-j)/(pp')}a_k\Big)^{p'}$$

$$\leqslant \sum_j 2^j\Big(\sum_{k\geqslant j} 2^{(j-k)/p'}\Big)^{p'-1}\Big(\sum_{k\geqslant j} 2^{(k-j)/p}a_k^{p'}\Big)$$

$$\leqslant C\sum_j 2^j\Big(\sum_{k\geqslant j} 2^{(k-j)/p}a_k^{p'}\Big)$$

$$= C\sum_k 2^{k/p}a_k^{p'}\sum_{j\leqslant k} 2^{j/p'} \leqslant C\sum_k 2^k a_k^{p'},$$

with $a_k = 2^{-k}\int_{x_k}^{x_{k+1}} g$. The second equivalence in (1.7) follows from (2.1), (2.3) and (2.4).

Finally, integration by parts yields

$$(2.5)\quad \int_0^\infty\Big(\int_0^x g\Big)^{p'-1}\Big(\int_0^x v\Big)^{1-p'} g(x)\,dx = \frac{1}{p'}\Big(\int_0^x g\Big)^{p'}\Big(\int_0^x v\Big)^{1-p'}\Big|_0^\infty + \frac{1}{p}\int_0^\infty\Big(\int_0^x g\Big)^{p'}\Big(\int_0^x v\Big)^{-p'} v(x)\,dx$$

$$= \frac{1}{p}\int_0^\infty\Big(\int_0^x g\Big)^{p'}\frac{v(x)}{\big(\int_0^x v\big)^{p'}}\,dx + \frac{1}{p'}\Big(\int_0^\infty g\Big)^{p'}\Big(\int_0^\infty v\Big)^{1-p'},$$

where the second summand above is zero if $\int_0^\infty v = \infty$ (recall that $\int_0^x g = 1$ for large $x$). This proves the third equivalence in (1.7) and completes the proof of Theorem 1.

**§ 3. Classical Lorentz space inequalities.** We begin by proving Theorem 2. As mentioned in §1, (1.2) shows that $M$ is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ if and only if (1.8) holds for nonincreasing $f$ with $Tf(x) = x^{-1}\int_0^x f(t)\,dt$; and by

Theorem 1, if and only if (1.9) holds for nonnegative $g$ with $\int_0^x T^*g = Pg(x)+xP^*(t^{-1}g(t))(x)$. Suppose $1<p\leqslant q<\infty$. From [11], $P$ is bounded from $L^{q'}(w^{1-q'})$ to $L^{p'}\big((\int_0^x v)^{-p'}v(x)\big)$ if and only if

$$(3.1)\qquad \Big(\int_r^\infty\Big(\int_0^x v\Big)^{-p'}v(x)\,dx\Big)^{1/p'}\Big(\int_0^r w(x)\,dx\Big)^{1/q}\leqslant A,\quad \text{for all } r>0,$$

and (with $g(t)$ replaced by $tg(t)$) $P^*$ is bounded from $L^{q'}(x^{q'}w(x)^{1-q'})$ to $L^{p'}\big(x^{p'}(\int_0^x v)^{-p'}v(x)\big)$ if and only if

$$(3.2)\qquad \Big(\int_0^r x^{p'}\Big(\int_0^x v\Big)^{-p'}v(x)\,dx\Big)^{1/p'}\Big(\int_r^\infty x^{-q}w(x)\,dx\Big)^{1/q}\leqslant B,\quad \text{for all } r>0.$$

Also,

$$\int_0^\infty T^*g=\int_0^\infty g\leqslant C\Big(\int_0^\infty v\Big)^{1/p}\Big(\int_0^\infty g^{q'}w^{1-q'}\Big)^{1/q'}$$

for all nonnegative $g$ if and only if

$$(3.3)\qquad \Big(\int_0^\infty w\Big)^{1/q}\leqslant C\Big(\int_0^\infty v\Big)^{1/p}.$$

Performing the integration in the first factor on the left side of (3.1) yields

$$(3.4)\qquad (p'-1)\int_r^\infty\Big(\int_0^x v\Big)^{-p'}v(x)\,dx=\Big(\int_0^r v(x)\,dx\Big)^{1-p'}-\Big(\int_0^\infty v(x)\,dx\Big)^{1-p'},$$

and it follows that (3.1) and (3.3) together are equivalent to (1.10). Since (3.2) is (1.11), we conclude that (1.9) is equivalent to (1.10) and (1.11), as required.

In the case $1<q<p<\infty$, $P$ is bounded from $L^{q'}(w^{1-q'})$ to $L^{p'}\big((\int_0^x v)^{-p'}v(x)\big)$ if and only if ([10], p. 47)

$$(3.5)\qquad \Big(\int_0^\infty\Big[\Big(\int_0^x w\Big)^{1/p}\Big(\int_x^\infty\Big(\int_0^t v\Big)^{-p'}v(t)\,dt\Big)^{1/p'}\Big]^r w(x)\,dx\Big)^{1/r}=A<\infty,$$

where $1/r=1/q-1/p$. Again, from (3.4), it follows that (3.5) and (3.3) together are equivalent to (1.12) — indeed, the left side of (1.12) is, by (3.4), at most a multiple of the left side of (3.5) plus

$$\Big(\int_0^\infty\Big[\Big(\int_0^x w\Big)^{1/p}\Big(\int_0^\infty v\Big)^{-1/p}\Big]^r w(x)\,dx\Big)^{1/r}=\Big(\int_0^\infty v\Big)^{-1/p}\Big(\frac{1}{r/p+1}\Big(\int_0^x w\Big)^{r/p+1}\Big|_0^\infty\Big)^{1/r}$$
$$=C\Big(\int_0^\infty v\Big)^{-1/p}\Big(\int_0^\infty w\Big)^{1/q}$$

where $1/p+1/r=1/q$. Since $P^*$ is bounded from $L^{q'}(x^{q'}w(x)^{1-q'})$ to $L^{p'}\big(x^{p'}(\int_0^x v)^{-p'}v(x)\big)$ if and only if (1.13) holds ([10], p. 47), we conclude that (1.9) is equivalent to (1.12) and (1.13) and this completes the proof of Theorem 2.



We now turn to the proof of Theorem 3. As mentioned in § 1, (1.15) shows that $H$ is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ if and only if (1.8) holds for nonincreasing $f$ with $Tf(x)=x^{-1}\int_0^x f(t)\,dt+\int_x^\infty t^{-1}f(t)\,dt$; and by Theorem 1, if and only if (1.9) holds for nonnegative $g$ with $\int_0^x T^*g=T_\varphi g(x)+Pg(x)+xP^*(t^{-1}g(t))(x)$, where $\varphi(t)=\ln_+(1/t)$ and $T_\varphi g(x)=\int\varphi(t/x)g(t)\,dt$. Since $\varphi$ satisfies the inequality ((1.6) of [9]) $\varphi(ab)\leqslant D(\varphi(a)+\varphi(b))$ for $0<a,b<1$ (with $D=1$), the theorem in [9] shows that $T_\varphi$ is bounded from $L^{q'}(w^{1-q'})$ to $L^{p'}\big((\int_0^x v)^{-p'}v(x)\big)$ if and only if both of the following conditions hold:

$$(3.6)\qquad \Big(\int_r^\infty\Big(\int_0^x v\Big)^{-p'}v(x)\,dx\Big)^{1/p'}\Big(\int_0^r\varphi(x/r)^q w(x)\,dx\Big)\leqslant A,\quad \text{for all } r>0,$$

$$(3.7)\qquad \Big(\int_r^\infty\varphi(r/x)^{p'}\Big(\int_0^x v\Big)^{-p'}v(x)\,dx\Big)^{1/p'}\Big(\int_0^r w(x)\,dx\Big)^{1/q}\leqslant B,\quad \text{for all } r>0.$$

Moreover,

$$\infty=\int_0^\infty T^*g\leqslant C\Big(\int_0^\infty v\Big)^{1/p}\Big(\int_0^\infty g^{q'}w^{1-q'}\Big)^{1/q'}$$

for all nonnegative $g$ if and only if $\int_0^\infty v=\infty$. Now the proof of Theorem 2 shows that if $\int_0^\infty v=\infty$, then the map sending $g(x)$ to $Pg(x)+xP^*(t^{-1}g(t))(x)$ is bounded from $L^{q'}(w^{1-q'})$ to $L^{p'}\big((\int_0^x v)^{-p'}v(x)\big)$ if and only if (1.10) and (1.11) hold. Also, if we perform the integration in the first factor of (3.6) using $\int_0^\infty v=\infty$, we obtain (1.17). Since (3.7) is (1.18), we conclude from all of the above that (1.9) is equivalent to $\int_0^\infty v=\infty$, (1.10), (1.11), (1.17) and (1.18). This completes the proof of Theorem 3.

§ 4. **Proof of Theorem 4.** We prove (1.22)⇒(1.24)⇒(1.25)⇒(1.23)⇒(1.22). Suppose first that (1.22) holds, i.e. there is a norm $\|\ \|$ on $\Lambda_p(v)$ and positive constants $C_1$ and $C_2$ such that (1.21) holds. Let $f(x)$ be nonincreasing on $[0,\infty)$ and such that $(\int_0^\infty f^p v)^{1/p}<\infty$. Let $g(x)=\chi_{[0,r)}(x)$ and set $\tilde f(y)=f(A|y|^n)$ and $\tilde g(y)=g(A|y|^n)$ so that $\tilde f^*=f$ and $\tilde g^*=g$. Then

$$(\tilde f*\tilde g)(y)\geqslant C\int_0^r f(x)\,dx,\quad \text{for } |y|\leqslant(A^{-1}r)^{1/n},$$

i.e. for $y$ in a set of measure $r$. We thus obtain

$$C\Big(\int_0^r f\Big)\Big(\int_0^r v\Big)^{1/p}\leqslant\Big(\int_0^\infty(\tilde f*\tilde g)^{*p}v\Big)^{1/p}\leqslant C_2\|\tilde f*\tilde g\|,\quad \text{by (1.21)},$$
$$\leqslant C_2\int_{\mathbf{R}^n}\tilde g(t)\|\tilde f_t\|\,dt,\quad \text{where } \tilde f_t(x)=\tilde f(x-t),\text{ since }\|\ \|\text{ is a norm},$$
$$\leqslant C_1^{-1}C_2 r\Big(\int_0^\infty f^p v\Big)^{1/p},$$

by (1.21) again and since $(\tilde f_t)^*=\tilde f^*=f$ and $\int_{\mathbf{R}^n}\tilde g(t)\,dt=r$. Thus

$$(4.1)\quad \Big(\int_0^r v\Big)^{1/p}\frac{\int_0^\infty f(x)g(x)\,dx}{\big(\int_0^\infty f(x)^p v(x)\,dx\big)^{1/p}} \leqslant Cr, \quad \text{for all } f \geqslant 0 \text{ and nonincreasing.}$$

Taking the supremum over $f$ nonnegative and nonincreasing in (4.1) yields (1.24) by Theorem 1 since

$$\Big(\int_0^\infty\Big(\int_0^x g\Big)^{p'-1}\Big(\int_0^x v\Big)^{1-p'} g(x)\,dx\Big)^{1/p'} = \Big(\int_0^r\Big(x^{-1}\int_0^x v\Big)^{1-p'} dx\Big)^{1/p'}.$$

Now suppose (1.24) holds. We use the argument of M. Ariño and B. Muckenhoupt in the proof of Lemma 2.1 of [2]. Let $A_k = \big(2^{-kp}\int_0^{2^k} v\big)^{1-p'}$ for $k$ in $\mathbf{Z}$. Then

$$(4.2)\quad \sum_{k=-\infty}^{m} A_k - \sum_{k=-\infty}^{m-1} A_k = A_m = \Big(2^{-mp}\int_0^{2^m} v\Big)^{1-p'}$$
$$\geqslant C^{-p'}\int_0^{2^m}\Big(x^{-p}\int_0^x v\Big)^{1-p'}\frac{dx}{x}, \quad \text{by (1.24) with } r = 2^m,$$
$$= C^{-p'}\sum_{k=-\infty}^{m}\int_{2^{k-1}}^{2^k}\Big(x^{-p}\int_0^x v\Big)^{1-p'}\frac{dx}{x} \geqslant C^{-1}\sum_{k=-\infty}^{m} A_k.$$

Thus $\sum_{k=-\infty}^{m-1} A_k \leqslant \beta\sum_{k=-\infty}^{m} A_k$ where $\beta = 1 - C^{-1}$. Iterating this inequality and using (4.2) again yields

$$(4.3)\quad A_j \leqslant \sum_{k=-\infty}^{j} A_k \leqslant \beta^{m-j}\sum_{k=-\infty}^{m} A_k \leqslant C\beta^{m-j}A_m, \quad \text{for } -\infty < j \leqslant m < \infty,$$

where $C$ and $\beta$ are positive constants with $\beta < 1$. We now have

$$\int_{2^j}^\infty x^{-p}v(x)\,dx \leqslant C\sum_{m=j+1}^\infty 2^{-mp}\int_{2^{m-1}}^{2^m} v \leqslant C\sum_{m=j+1}^\infty A_m^{1-p}$$
$$\leqslant C\sum_{m=j+1}^\infty \beta^{(p-1)(m-j)}A_j^{1-p}, \quad \text{by (4.3)},$$
$$\leqslant CA_j^{1-p}, \quad \text{since } 0 < \beta^{p-1} < 1,$$
$$= C2^{-jp}\int_0^{2^j} v,$$

and this is (1.25) for $r = 2^j$, $j$ in $\mathbf{Z}$. Inequality (1.25) now follows easily for all $r > 0$.

Now suppose (1.25) holds. The result of Ariño and Muckenhoupt in [2] shows that (1.1), with $p = q$ and $w(x) = v(x)$, holds for $f$ nonnegative and nonincreasing. Thus (1.21) holds with $\|f\| = \big(\int_0^\infty f^{**}(x)^p v(x)\,dx\big)^{1/p}$ where $f^{**}(x) = x^{-1}\int_0^x f^*(t)\,dt$ (note that $f^* \leqslant f^{**}$ since $f^*$ is nonincreasing) and this proves (1.23).



Finally, (1.23) implies (1.22) since the inequality $(f+g)^{**} \leqslant f^{**} + g^{**}$ implies that $\|\ \ \|$ defines a norm on $\Gamma_p(v)$. This completes the proof of Theorem 4.

**Added in proof** (January 1990). In his paper *The Hilbert transform on rearrangement-invariant spaces* (Canad. J. Math. 19 (1967), 599–616), David Boyd included a nice characterization of when the Hilbert transform is bounded on $\Lambda_p(v)$, $1 < p < \infty$, in the case $v$ is nonincreasing. His techniques work just as well for general $v$ and show that in the case $w = v$ and $p = q$ of Theorem 3, conditions (1.10), (1.11), (1.17) and (1.18) can be replaced by the simpler conditions

$$(4.4)\quad \sup_{r>0}\frac{\int_0^{rs} v}{\int_0^r v} \leqslant Cs^{-\gamma p}, \quad \text{for all } 0 < s \leqslant 1 \text{ and some } 0 \leqslant \gamma < 1,$$

$$(4.5)\quad \sup_{r>0}\frac{\int_0^{rs} v}{\int_0^r v} \leqslant Cs^{-\gamma p}, \quad \text{for all } 1 \leqslant s \leqslant \infty \text{ and some } 0 < \gamma \leqslant 1.$$

Moreover, the same techniques show that the maximal operator is bounded on $\Lambda_p(v)$ if and only if (4.4) holds, yielding another proof of the result of Ariño and Muckenhoupt in [2]. However, in the case $\Lambda_p(v) \neq \Lambda_q(w)$, these techniques break down. For example, if $w(x) = x^{q-1}$ and $v(x) = x^{p-1}(-\log x)^p$ for $0 < x < \frac{1}{2}$ and vanish otherwise, then the maximal operator is bounded from $\Lambda_p(v)$ to $\Lambda_q(w)$ by Theorem 2, yet the integral

$$\int_0^1 h(s;\Lambda_p(v),\Lambda_q(w))\,ds = \int_0^1 \sup_{r>0}\frac{\big(\int_0^{rs} w\big)^{1/q}}{\big(\int_0^r v\big)^{1/p}}\,ds$$

in Theorem 3.1 of Boyd's paper fails to be finite.

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# A density theorem for $F$-spaces

by

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**Abstract.** The main result of this paper expressed in terms of representation theory states that any algebraically irreducible representation $T$ of an algebra $A$ in the algebra of all continuous endomorphisms of an $F$-space is totally irreducible provided the only intertwining operators for $T$ are the scalar multiples of the identity operator. We apply this result for characterizing strongly generating sets for the algebra of all continuous endomorphisms of a $B_0$-space.

**§ 1. Definitions and notation.** An *F-space*, or a *space of type F*, is a completely metrizable topological linear space. The topology of an $F$-space $X$ can be given by means of an *F-norm*, i.e. a functional $\|\cdot\|$ satisfying the following conditions (see [1] or [5]):

(i) $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$,
(ii) $\|x+y\| \leqslant \|x\| + \|y\|$,
(iii) $\lim\|x_n\| = 0$ implies $\lim\|\lambda x_n\| = 0$ for all $\lambda$,
(iv) $\lim|\lambda_n| = 0$ implies $\lim\|\lambda_n x\| = 0$ for all $x$.

Here $x$, $x_n$, $y$ denote arbitrary elements of $X$ and $\lambda$, $\lambda_n$ arbitrary (real or complex) scalars. The distance in $X$ is given by $\|x-y\|$ and the space $X$ is complete in this metric. For $F$-spaces the closed graph theorem holds true: If $T$ is a linear map of one $F$-space into another and its graph is closed in the product of these spaces, then $T$ is continuous (see [1]). A locally convex space of type $F$ is called a $B_0$*-space*. For an $F$-space $X$ denote by $L(X)$ the algebra of all its continuous endomorphisms. While for $B_0$-spaces this algebra always has a rich structure, for some $F$-spaces it can be very poor. There are (infinite-dimensional) so-called *rigid spaces* of type $F$ in which the only continuous endomorphisms are the scalar multiples of the identity operator (see [4], [8], or [5], p. 210). In particular, a rigid space cannot have a nontrivial continuous linear functional, while for $B_0$-spaces there always exists a separating family of such functionals.

The present paper is a by-product of our efforts at characterizing strongly generating sets for $L(X)$. It turned out (many thanks to Pavla Vrbová for

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