An axiomatic definition of the entropy of a $Z^d$-action on a Lebesgue space

by

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Abstract. We introduce the concept of a principal factor for a $Z^d$-action and we use a characterization of these factors to obtain an axiomatic definition of the entropy of a $Z^d$-action, $d \geq 2$.

Introduction. The notion of the entropy of a $Z^d$-action on a Lebesgue probability space has been introduced by A. N. Kolmogorov in [9] for $d = 1$ and then generalized by several authors ([1], [7], [11], [12]) to arbitrary $d \geq 1$.

In this paper we give an axiomatic definition of the entropy of a $Z^d$-action for every $d \geq 2$. Our result is an analogue of the result of V. A. Rokhlin ([14]).

To obtain our result we first prove the existence of relatively perfect partitions for a given $Z^d$-action. Next we introduce a concept of a principal factor and, using the above result, we give a characterization of principal factors by means of entropy. This characterization and the generalized Sinai theorem ([8]) allow us to obtain, by the use of the Rokhlin idea ([14]), our axiomatic definition of entropy.

Our result is an example of a result of ergodic theory obtained by a relativization method also used by other authors (see [2], [3], [10], [16], [17]).

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§ 1. Preliminaries. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space, let $\mathcal{A}$ be the set of all measurable partitions of $X$ and let $\mathcal{F}$ be the subset of $\mathcal{A}$ consisting of partitions with finite entropy.

We denote by $x$ the measurable partition of $X$ into single points and by $v$ the trivial measurable partition whose only element is $X$.

Let $< \in Z^d$ denote the lexicographical ordering of the group $Z^d$, $d \geq 2$. Let $e \in Z^d$ be the 0th standard unit vector. We put

$$Z_0^d = \{ g = (m_1, \ldots, m_d) \in Z^d; m_1 = \ldots = m_d = 0 \}, \quad 1 \leq n \leq d,$$

$$Z^d = \{ g \in Z^d; g < 0 \}.$$
Let $\Phi$ be a $\mathbb{Z}^d$-action on the space $(X, \mathcal{S}, \mu)$, i.e. $\Phi$ is a homomorphism of the group $\mathbb{Z}^d$ into the group of all measure-preserving automorphisms of $(X, \mathcal{S}, \mu)$, and set $\Phi_n = \Phi|_{\mathbb{Z}^d_n}$, $1 \leq n \leq d$. Let $T_i = \Phi^i$, $1 \leq i \leq d$. It is clear that $\Phi^i = T_{m_1}^1 \cdots T_{m_d}^i$ where $g = (m_1, \ldots, m_d) \in \mathbb{Z}^d$.

For a given set $A \subset \mathbb{Z}^d$ and a partition $P \in \mathcal{M}$ we define

$$P(A) = \bigvee_{\phi \in P} \phi(A).$$

Let

$$P_\Phi = P(\mathbb{Z}^d), \quad P^- = P_{\Phi}^{-\infty} = P(\mathbb{Z}^d_0), \quad P^n = P(\mathbb{Z}^d_n), \quad 1 \leq n \leq d.$$ 

Thus the partition $P^\Phi$ is the join of all partitions $T_{m_1}^1 \cdots T_{m_d}^n$ where $i_k \in \mathbb{Z}^d$, $n+1 \leq k \leq d$. In particular, $P^0 = P$.

Now, let $\sigma \in \mathcal{M}$ be totally invariant, i.e. $\Phi^\sigma = \sigma$, $\sigma \in \mathbb{Z}^d$, and let $\Phi_\sigma$ be the factor of $\Phi$ defined by $\sigma$. For a given $P \in \mathcal{M}$ we put

$$\hat{P} = \bigvee_{n=0}^{\infty} (\bigvee_{k=1}^{\infty} T_k^{-n}(P^\Phi)_{\Phi_\sigma} \cup \sigma).$$

In other words, $\hat{P}$ is the tail partition given by $P$ and $\sigma$.

Now we recall the concepts of the relative entropy and the relative Pinsker partition ([6], [1]).

The relative entropy $h(\Phi|\sigma)$ of the action $\Phi$ with respect to $\sigma$ is defined by the formula

$$h(\Phi|\sigma) = \sup \{h(P, \Phi|\sigma); P \in \mathcal{Z} \}$$

where

$$h(P, \Phi|\sigma) = H(P, \Phi_\sigma) \cup \sigma, \quad P \in \mathcal{M}.$$ 

The partition

$$\pi(\Phi|\sigma) = \bigvee_{P \in \mathcal{Z}} P \quad \text{where} \quad \mathcal{Z} = \{P \in \mathcal{Z}; h(P, \Phi|\sigma) = 0\}$$

is said to be the relative Pinsker partition of the action $\Phi$ with respect to $\sigma$.

It is easy to observe that in the case $\sigma = \nu$ the entropy $h(\Phi|\sigma)$ and the partition $\pi(\Phi|\sigma)$ reduce to the usual entropy $h(\Phi)$ and the Pinsker partition $\pi(\Phi)$ of $\Phi$ ([1]), [12].

Now, let us note some properties of the relative entropy and the relative Pinsker partition used in the sequel.

Let $P, Q, R \in \mathcal{Z}$ be arbitrary.

(A) $h(P \vee Q, \Phi|\sigma) = h(P, \Phi|\sigma) + H(Q|\Phi) \vee \Phi_\sigma \vee \sigma$.

(B) If $P \ll Q$ then

$$\lim_{n \to \infty} H(P|\Phi^n \vee T_{m_1}^{-n}R_{\Phi} \cup \sigma) = H(P|\Phi) \cup \sigma).$$

(C) $\hat{P} \ll \pi(\Phi|\sigma)$.

Let $G$ be a subgroup of $\mathbb{Z}^d$ of finite index $r$ and let $\Phi^r$ be the restriction of $\Phi$ to $G$, $r \geq 1$.

(D) if $h(P, \Phi^r|\sigma) \ll r h(P, \Phi|\sigma)$.

(E) If $P \in \mathcal{Z}$ is such that $\Phi \circ \sigma = \sigma$ then $h(P, \Phi|\sigma) = h(\Phi|\sigma)$. (F) $h(\Phi) = h(\Phi_\sigma) + h(\Phi|\sigma)$.

(G) For any totally invariant partitions $\sigma_i$, $i = 1, 2$,

$$h(P, \Phi|\sigma_i \cup \sigma_j) = h(P, \Phi|\sigma_i \cup \sigma_j).$$

The proofs of (A)-(E) run in the same manner as in the case $\sigma = \nu$ (cf. [12]). (F) and (G) are proved in [5] and [4] respectively for $\mathbb{Z}^d$-actions. The proofs for arbitrary $d \geq 1$ are the same.

(H) If $P, Q \in \mathcal{Z}$ and $h(P, \Phi|\sigma) = 0$ then $H(Q|P) \geq H(Q|\hat{Q})$.

Proof. For $n \in \mathbb{N}$ let $\Phi^n$ be the restriction of $\Phi$ to the subgroup $\{(m_1, \ldots, m_n); (m_1, \ldots, m_n) \in \mathbb{Z}^n\}$. Let $P \in \mathcal{Z}$ and $h(P, \Phi|\sigma) = 0$. It follows from (D) that $h(P, \Phi^n|\sigma) = 0$. Set

$$P_n = P_{\Phi^n}, \quad P_n^\Phi = P_{\Phi^n}^\Phi.$$ 

Let $Q \in \mathcal{Z}$ be arbitrary. Using the property (A) we have

$$h(P \vee Q, \Phi^n|\sigma) = h(P, \Phi^n|\sigma) + H(Q|\Phi_{\Phi^n} \vee P_{\Phi^n} \vee \sigma)$$

$$= h(Q, \Phi^n|\sigma) + H(P|P_{\Phi^n} \vee Q_{\Phi^n} \vee \sigma).$$

Hence

$$H(Q|P) \geq H(Q|P_{\Phi^n} \vee P_{\Phi^n} \vee \sigma) = H(Q|Q_{\Phi^n} \vee \sigma)$$

$$\geq H(Q) \bigvee_{k=1}^{\infty} T_k^{-n+1}(P_{\Phi^n})_{\Phi_{\sigma}}, \quad n \geq 1.$$ 

Therefore taking the limit as $n \to \infty$ in the last inequality we obtain $H(Q|P) \geq H(Q|\hat{Q})$, which completes the proof.

§ 2. Relatively perfect partitions. First we recall some notions from [4].

A partition $\zeta \in \mathcal{M}$ is said to be invariant if $\Phi^n \zeta \ll \zeta$ for every $g \ll 0$. It is easy to check that this is equivalent to $T_{m_1}^{-n} \zeta \ll \zeta$, $1 \leq n \leq d$.

A partition $\zeta \in \mathcal{M}$ is called strongly invariant if it is invariant and if

$$\bigvee_{\nu \in A} \Phi^n \zeta = \bigvee_{\nu \in B} \Phi^n \zeta$$

where the sets $A, B$ form a partition of $\mathbb{Z}^d$ such that $g \ll h, g \in A, h \in B$, $A$ does not contain a greatest element and $B$ does not contain a smallest element. It is easy to show that this condition is equivalent to

$$\bigvee_{n=0}^{\infty} T_k^{-n} \zeta \ll T_{m_1}^{-1} \zeta \ll 1, \quad 2 \leq k \leq d.$$
A partition $\mathcal{A}$ is called exhaustive if it is strongly invariant and $\zeta_\sigma = \zeta_\varphi$. The results of Lemmas 1, 2 and of Theorem 1 together with some remarks about their proofs in the case $\sigma = \varphi$ have been announced in [4]. However, especially in the case of Lemma 1, the complete proofs even for $\sigma = \varphi$ need additional considerations. For this reason and for completeness we give the proofs with some shortening.

**Lemma 1.** If a partition $\mathcal{A}$ is exhaustive and $\zeta_\sigma \geq \sigma$ then

$$\bigwedge_{n=0}^{\sigma} T_{1}^{-n} \zeta_1 \geq \pi(\Phi(\sigma)).$$

**Proof.** Let $P_1, Q \in \mathcal{Z}$ and let $r_1$ be a positive integer such that

$$P_1 \leq \pi(\Phi(\sigma)), \quad P_1 \leq \zeta_{1-i}, \quad 2 \leq i \leq d,$$

$$Q_1 \leq T_{r_1+i}, \quad 1 \leq j \leq d.$$

The fact that $\zeta_\sigma \geq \sigma$ and the strong invariance of $\zeta$ imply

$$Q_{d} \leq \bigwedge_{n=0}^{\infty} T_{d}^{-n} \zeta = T_{d}^{-1} \zeta^{-1}.$$  

Since $P_d \leq Q_{d-1} \leq T_{d-1}^{-1} \zeta^{-1}$, the property (C) implies $h(P_d, \Phi(\sigma)) = 0$ and so

$$T_{d}^{-1} P_d \leq \zeta^{-1}, \quad h(T_{d}^{-1} P_d, \Phi(\sigma)) = 0.$$  

By (1), (2), and (H) we get

$$H(Q_{d} \mid T_{d}^{-1} P_d) \geq H(Q_{d} \mid Q_{d}) \geq H(Q_{d} \mid T_{d}^{-1} \zeta^{-1}).$$

Since $Q_{d} \leq T_{d}^{-1} \zeta^{d}$, we see that the inequalities (3) are satisfied for $Q_1$ running over a dense subset of the set

$$\{P \in \mathcal{Z}; P \leq \zeta_{d} \leq \zeta^{-1}\}.$$  

Therefore they are satisfied for all partitions in this set. If we take in (3), in particular, $Q_{d} = T_{d}^{-1} P_d$ we obtain $P_d \leq T_{d}^{-1} \zeta^{-1}$.

Repeating this procedure we get

$$P_d \leq T_{d}^{-1} \zeta^{-1} \leq T_{d}^{-n} \zeta^{-1} \leq T_{d}^{-1} \zeta^{-1}$$

for all $n \geq 1$.

Using again the fact that $\zeta$ is strongly invariant we have

$$P_d \leq \bigwedge_{n=0}^{\infty} T_{d}^{-n} \zeta^{-1} = T_{d}^{-1} \zeta^{-2}.$$  

Thus we have shown that $P_d \leq T_{d}^{-1} \zeta^{-2}$ for $P_d \leq \zeta_{d-1}$, i.e.

$$\zeta_{d-1} \leq T_{d}^{-1} \zeta^{-2}.$$  

Repeating $d-3$ times the considerations above we get $\zeta_2 \leq T_{1}^{-1} \zeta_1$, $h(P_2, \Phi(\sigma)) = 0$ and so

$$H(Q_2 \mid T_{1}^{-1} P_2) \geq H(Q_2 \mid Q_2) \geq H(Q_2 \mid T_{1}^{-1} \zeta_1).$$

Since $Q_2 \leq T_{1}^{-1} \zeta_1$ the last inequality is satisfied for all $Q_2$ in a dense subset of $\{P \in \mathcal{Z}; P \leq \zeta_1\}$. Hence, as above, $P_2 \leq \bigwedge_{n=0}^{\infty} T_{1}^{-n} \zeta_1$, i.e.

$$\zeta_1 \leq \bigwedge_{n=0}^{\infty} T_{1}^{-n} \zeta_1.$$  

Thus using the equality $h(P_1, \Phi(\sigma)) = 0$ we get

$$H(Q_1 \mid P_1) \geq H(Q_1 \mid Q_1) \geq H(Q_1 \mid \bigwedge_{n=0}^{\infty} T_{1}^{-n} \zeta_1).$$

Since $\zeta$ is exhaustive, applying again the density argument we may take $Q_1 = P_1$ in (5). Hence we obtain

$$P_1 \leq \bigwedge_{n=0}^{\infty} T_{1}^{-n} \zeta_1,$$

which completes the proof.

**Lemma 2.** There exists a measurable partition $\eta \geq \sigma$ which is invariant, generating and such that

(a) $\bigwedge_{n=0}^{\infty} T_{1}^{-n} \pi(\Phi(\eta)) \leq \pi(\Phi(\sigma))$,

(b) $h(\Phi(\sigma) = H(\eta \mid \eta)) = H(\eta \mid T_{1}^{-1} \eta)$.

**Proof.** Let $(P_k) \in \mathcal{Z}$ be a sequence such that $P_k \leq \sigma$. Using the property (B) we may construct, similarly to [15], a strictly increasing sequence $(n_k)$ of positive integers such that

$$H(Q_n \mid Q_n^{-} \sigma \vee \sigma) - H(Q_n \mid Q_n^{+} \sigma \vee \sigma) < 1/r, \quad s \geq 0,$$

where $Q_n = \bigvee_{i=1}^{n} T_{1}^{-i} P_k$ and $Q_n^{-} = (Q_n)^{-}$. We put

$$Q = \bigvee_{n=0}^{\infty} Q_n, \quad \eta = Q \setminus Q^{-} \sigma.$$

It is clear that the partition $\eta$ is invariant, generating, i.e. $\eta_\sigma = \eta$, and $\eta \geq \sigma$.

Now we check (a). Since $Q_n^{-} \sigma \vee \sigma \setminus \eta \leq \eta^{-}$ the inequalities (6) give

$$H(Q_n \mid Q_n^{-} \sigma \vee \sigma) - H(Q_n \mid \eta^{-} \sigma) < 1/r, \quad r \geq 1.$$  

Let $P \in \mathcal{Z}$ and $P \leq \bigwedge_{n=0}^{\infty} T_{1}^{-n} \pi(\Phi(\eta))$. The last partition is of course totally invariant with respect to $\Phi$ and so

$$P \leq \bigwedge_{n=0}^{\infty} \pi(\Phi(\eta))^{-} \eta^{-} \leq \pi(\Phi(\eta))^{-} \eta^{-} \eta^{-} \sigma.$$

The property (A) implies

$$h(P, \Phi(\sigma)) = h(Q_n, \Phi(\sigma)) - H(Q_n \mid Q_n^{-} \sigma \vee \sigma) + H(P \mid Q_n^{-} \sigma \vee \sigma)$$
The use of (G) to the action $\Phi_1$ gives
\[ H(Q, Q_r \vee P_\sigma \vee \sigma) \geq H(Q, Q_r \vee \pi(\Phi_1(\eta_r^1) \vee \sigma)) \]
\[ = h(Q, \Phi_1(Q || Q_r^1) \vee \pi(\Phi_1(\eta_r^1) \vee \sigma)) \]
\[ = h(Q, \Phi_1(Q || Q_r^1) \vee \eta_r^1 \vee \sigma) \geq H(Q, Q_r \vee \eta_r^1 \vee \sigma) = H(Q, \eta_r^1), \quad r \geq 1. \]

Applying this result and (7) in (8) and then taking the limit as $r \to \infty$ we get $h(P, \Phi|\sigma) = 0$, i.e. $P \leq \pi(\Phi|\sigma)$. This proves (a).

In order to check (b) observe that
\[ P_r \leq T^{-n} Q_r, \quad Q_r \leq T^{-n} P_r \vee ... \vee T^{-n} P_r. \]

Therefore $(P_r)_\sigma = (Q_r)_\sigma$ and so $h(P_r, \Phi|\sigma) = h(Q_r, \Phi|\sigma)$. Hence, using the fact that $P_r \sim \sigma$ and $Q_r \sim Q$ we get
\[ \lim_{r \to \infty} H(Q, Q_r \vee \sigma) = h(\Phi|\sigma), \]
\[ \lim_{r \to \infty} H(Q, \eta_r^1) = H(Q, \eta_r^1 \vee \sigma) = H(\eta, \eta_r^1). \]

Then, taking in (7) the limit as $r \to \infty$ we obtain (b) and the proof is complete.

**Definition.** A partition $\zeta \in \mathcal{M}$ is said to be relatively perfect with respect to $\sigma$ if $\zeta$ is exhaustive, $\zeta \geq \sigma$ and
\[ (i) \int_{\tau_n=0}^{\infty} T^{-n} \zeta^1 = \pi(\Phi|\sigma), \]
\[ (ii) h(\Phi|\sigma) = H(\zeta). \]

It is clear that a partition relatively perfect with respect to $\sigma = \nu$ is perfect (cf. [43]).

**Theorem 1.** For every positive integer $d$, every $Z^d$-action on $(X, \mathcal{B}, \mu)$ and every totally invariant measurable partition of $X$ there exists a relatively perfect partition with respect to this totally invariant partition.

**Proof.** We prove the theorem by induction on $d$. The proof for $d = 1$ is similar to that of the Rokhlin–Sinai theorem (cf. [15]).

Suppose our theorem is valid for $d-1$. Let $\Phi$ be an arbitrary $Z^d$-action and $\sigma \in \mathcal{M}$ an arbitrary totally invariant partition of $X$. By Lemma 2 there exists a partition $\eta \in \mathcal{M}$ which is invariant, generating, $\eta \geq \sigma$ and such that
\[ \int_{\tau_n=0}^{\infty} T^{-n} \pi(\Phi|\eta^1) \leq \pi(\Phi|\sigma), \]
\[ h(\Phi|\sigma) = H(\eta) = H(\eta || T^{-1} \eta). \]

Now we apply the induction assumption to the space $X/\pi(\Phi|\eta^1)$, to the action $\Phi_1$ and to the totally invariant (with respect to $\Phi_1$) partition $T^{-1} \pi(\Phi|\eta^1)$.

Hence there exists a partition $\zeta \in \mathcal{M}$ with the following properties:
\[ T^{-1} \pi(\Phi|\eta^1) \leq \zeta \leq \pi(\Phi|\eta^1), \]
\[ \zeta \text{ is strongly invariant with respect to } \Phi_1, \]
\[ \zeta^1 = \pi(\Phi|\eta^1), \]
\[ \int_{\tau_n=0}^{\infty} T^{-n} \zeta^1 = T^{-1} \pi(\Phi|\eta^1), \]
\[ h(\Phi_1 | T^{-1} \pi(\Phi|\eta^1)) = H(\zeta; \zeta) = H(\zeta || T^{-1} \zeta). \]

It follows from (11) that $\zeta \geq \sigma$ and $T^{-1} \zeta \leq \zeta$. Therefore using (12) we see that $\zeta$ is invariant with respect to $\Phi$. The strong invariance of $\zeta$ readily follows from (12)–(14). Now we check that
\[ \int_{\tau_n=0}^{\infty} T^{-n} \zeta^1 = \int_{\tau_n=0}^{\infty} T^{-n} \pi(\Phi|\eta^1) \leq \pi(\Phi|\sigma), \]

Since $\zeta \geq \sigma$ and $\zeta$ is exhaustive Lemma 1 gives $\int_{\tau_n=0}^{\infty} T^{-n} \zeta^1 \geq \pi(\Phi|\sigma)$. On the other hand, (9) and (13) imply
\[ \int_{\tau_n=0}^{\infty} T^{-n} \pi(\Phi|\eta^1) \leq \pi(\Phi|\sigma), \]

and so (16) holds.

It remains to show the equality
\[ h(\Phi|\sigma) = H(\zeta || \zeta) = H(\zeta || T^{-1} \zeta). \]

It is clear that it is sufficient to show the inequality $h(\Phi|\sigma) \leq H(\zeta || T^{-1} \zeta)$.

First we show
\[ h(\Phi_1 | T^{-1} \pi(\Phi|\eta^1)) \geq H(\eta || T^{-1} \eta). \]

Let $P \in \mathcal{Z}$ and $P \leq \eta$. The invariance of $\eta$ and (G) give
\[ h(\Phi_1 | T^{-1} \pi(\Phi|\eta^1)) \geq h(P, \Phi_1 | T^{-1} \pi(\Phi|\eta^1)) \]
\[ = h(P, \Phi|\eta^1) \geq h(P, \Phi|\eta^1) \]
\[ = h(P, \Phi|\eta^1). \]

Let $(P_k) \in \mathcal{Z}$ be such that $P_k \geq \eta$. Replacing $P$ by $P_k$, $k > 1$, in the last inequality and taking the limit as $k \to \infty$ we get (17).

Now, combining (10), (15) and (17) we have
\[ h(\Phi|\sigma) = H(\eta || T^{-1} \eta) \leq h(\Phi_1 | T^{-1} \pi(\Phi|\eta^1)) = H(\zeta || T^{-1} \zeta), \]

which completes the proof.
§ 3. Principal partitions and an axiomatic definition of entropy

Definition. A totally invariant partition $\sigma \in M$ is said to be principal if every strongly invariant partition $\zeta \in M$ is also totally invariant.

A factor $\Psi$ of the action $\Phi$ is said to be principal if every totally invariant partition $\sigma$ such that $\sigma$ and $\Phi_{\sigma}$ are isomorphic, is principal.

Theorem 2. If an action $\Psi$ is a principal factor of $\Phi$ then $h(\Phi) = h(\Psi)$. In the case $h(\Phi) < \infty$ the converse theorem is also true.

Proof. Let $\Psi$ be a principal factor of $\Phi$ and let $\sigma \in M$. Then $\Psi_{\sigma}$ is also principal and so $h(\Psi_{\sigma}) = h(\Phi_{\sigma}) = h(\Phi) = h(\Psi)$. Now suppose $h(\Phi) < \infty$. Let $\Psi$ be a factor of $\Phi$ such that $h(\Psi) = h(\Phi)$ and let $\sigma$ be a totally invariant partition such that $\Phi_{\sigma}$ and $\Psi_{\sigma}$ are isomorphic. Using again (F) we have

$$h(\Psi_{\sigma}) = h(\Phi_{\sigma}) - h(\Psi).$$

Let $\zeta \geq \sigma$ be a strongly invariant partition. Since $h(\Psi_{\sigma}) = h(\Phi_{\sigma}) = 0$ and $\zeta$ is invariant we have $T_{\Phi}^{-1}\zeta = \zeta = \zeta$. This equality and the strong invariance of $\zeta$ imply that $\Psi$ is totally invariant, which means that $\Psi$ is principal.

Corollary. If $h(\Phi) < \infty$ and $\Phi_{\sigma}$ is an ergodic factor such that every element of $\sigma$ is a finite set then $\Phi_{\sigma}$ is principal.

Proof. It follows from our assumption and the Rohlin theorem ($[13]$) that there exists a finite partition $P$ such that $P \supseteq \sigma$. Using the property (E) we have $h(\Phi_{\sigma}) = h(\Phi_P) = h(\Phi) = h(\Psi)$ and so, applying (F) and Theorem 2, we get the result.

The following example shows that Theorem 2 fails to be true if we replace strong invariance by invariance in the definition of a principal factor.

Example. Let $(Y, \mathcal{F}, \lambda, \phi)$ be an arbitrary dynamical system with $h(\phi) = 0$. Let $R = (\mathcal{C}, \mathcal{D}, \mu, T)$ be a Bernoulli dynamical system with state space $(Y, \mathcal{F}, \lambda)$. Let $\Phi$ be the $Z^2$-action on $X$ defined by the formula

$$\Phi = T_1S_0, \quad g = (i, j) \in Z^2,$$

where $(S_0x)(n) = (\phi x)(n), n \in Z$. It is known (cf. [1]) that $h(\Phi) = h(\phi) = 0$. Let $\sigma$ be a partition such that $h(\Phi) = h(\Phi_{\sigma})$. We want to show that there exists an invariant partition $\zeta$ of $X$ which is not totally invariant. Let $Q = \{ Q_0, Q_1 \}$ be a nontrivial partition of $Y$ and let

$$P = \{ P_0, P_1 \}$$

be the partition of $X$ given by

$$P_i = \{ x \in X; x(i) \in Q_i \}, \quad i = 0, 1.$$

The partition $\zeta \supseteq P \supseteq P_0$ is of course invariant and, since $h(\phi) = 0$, we have $S^{-1}\zeta = \zeta$. However, $\zeta$ is not totally invariant. Indeed, if $\zeta$ is totally invariant then $T^{-1}\zeta = \zeta$, i.e., $P_0 \supseteq (P_0)_T = (P_0)_T$. It follows from the definition of $\mu$ that the partitions $P_0$ and $(P_0)_T$ are independent. Therefore $P_0$ is a trivial partition, contradicting the nontriviality of $Q$.

We denote the set of all ergodic $Z^2$-actions on Lebesgue probability spaces by $\operatorname{Act}Z^2$. Let $\Phi_{\sigma}$ be the Bernoulli $Z^2$-action defined by the vector $(1, \frac{1}{2})$. Applying the generalized Sinai theorem concerning Bernoulli factors (cf. $[6]$) and Theorem 2 we may prove, using the Rohlin idea ($[14]$), the following

Corollary. Let $H: \operatorname{Act}Z^2 \to [0, \infty]$ be a function such that $H(\Phi) = \log 2$ for all $\Phi \in \operatorname{Act}Z^2$. The following conditions are satisfied:

(i) if $\Psi$ is a factor of $\Phi$ then $H(\Phi) = H(\Psi)$,
(ii) if $\Psi$ is a principal factor of $\Phi$ then $H(\Phi) = H(\Psi)$,
(iii) $H(\Phi \times \Psi) = H(\Phi) + H(\Psi)$.

Then $H(\Phi) = h(\Phi)$, $\Phi \in \operatorname{Act}Z^2$.

References

Boundedness of classical operators on classical Lorentz spaces

by

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Abstract. The classical Lorentz space $A_p(x)$ consists of those measurable functions $f$ on $\mathbb{R}^n$ such that $\left( \int_0^\infty (f^*(x))^{p'} w(x) \, dx \right)^{1/p} < \infty$. We characterize when a variety of classical operators, including Hilbert and Riesz transforms, fractional integrals and maximal functions, are bounded from one Lorentz space, $A_p(x)$, to another, $A_q(w)$. In addition, we give a simple and explicit description of the dual of $A_p(x)$ and determine when $A_p(x)$ is a Banach space.

§ 1. Introduction. For $1 < p < \infty$ and $v(x)$ a nonnegative function on $(0, \infty)$, the classical Lorentz spaces $A_p(x)$ on $\mathbb{R}^n$, introduced and studied by G. Lorentz in [7] for the intervals $(0,1)$, is defined by

$$A_p(v) = \{ f \text{ measurable on } \mathbb{R}^n : \left( \int_0^\infty (f^*(x))^{p'} v(x) \, dx \right)^{1/p} < \infty \}.$$ 

where $f^*(x) = \inf \{ \lambda : \int_{|t| < \lambda} |f(t)| \, dx \leq x \}$ is the nonincreasing rearrangement of $f$ on $(0, \infty)$ with respect to Lebesgue measure on $\mathbb{R}^n$. Let $E$ denote the Lebesgue measure of a set $E$. M. Arinó and B. Muckenhoupt observed in [2] that the Hardy–Littlewood maximal operator $M$, defined by

$$Mf(x) = \sup_{Q} \frac{1}{|Q|} \int_{Q} |f(y)| \, dy ; Q \text{ is a cube in } \mathbb{R}^n \text{ containing } x,$$

is bounded from $A_p(v)$ to $A_p(w)$ if and only if

$$\left( \int_0^\infty \left( \int_0^x f(t) \, dt \right)^{p'} w(x) \, dx \right)^{1/p} \leq C \left( \int_0^\infty \left( \int_0^\infty f^*(x) v(x) \, dx \right)^{1/p} \right)^{1/p}$$

for all nonnegative and nonincreasing functions $f$ on $(0, \infty)$. Indeed, this follows immediately from the rearrangement inequality for the maximal function ([6], [12] and [15])

$$\left( Mf \right)^*(x) \leq C_1 x^{-1} \int_0^x f^*(t) \, dt \leq C_2 \left( Mf \right)^*(x), \quad x > 0,$$

coupled with the fact that every nonincreasing function $f^*$ on $(0, \infty)$ occurs as


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