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An axiomatic definition of the entropy of a $\mathbf{Z}^d$-action on a Lebesgue space

by

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Abstract. We introduce the concept of a principal factor for a $\mathbf{Z}^d$-action and we use a characterization of these factors to obtain an axiomatic definition of the entropy of a $\mathbf{Z}^d$-action, $d \geqslant 2$.

Introduction. The notion of the entropy of a $\mathbf{Z}^d$-action on a Lebesgue probability space has been introduced by A. N. Kolmogorov in [9] for $d = 1$ and then generalized by several authors ([1], [7], [11], [12]) to arbitrary $d \geqslant 1$.

In this paper we give an axiomatic definition of the entropy of a $\mathbf{Z}^d$-action for every $d \geqslant 2$. Our result is an analogue of the result of V. A. Rokhlin ([14]).

To obtain our result we first prove the existence of relatively perfect partitions for a given $\mathbf{Z}^d$-action. Next we introduce a concept of a principal factor and, using the above result, we give a characterization of principal factors by means of entropy. This characterization and the generalized Sinai theorem ([8]) allow us to obtain, by the use of the Rokhlin idea ([14]), our axiomatic definition of entropy.

Our result is an example of a result of ergodic theory obtained by a relativization method also used by other authors (see [2], [3], [10], [16], [17]).

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§ 1. **Preliminaries.** Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space, let $\mathscr{M}$ be the set of all measurable partitions of X and let $\mathscr{Z}$ be the subset of $\mathscr{M}$ consisting of partitions with finite entropy.

We denote by ε the measurable partition of X into single points and by ν the trivial measurable partition whose only element is X.

Let $\prec$ denote the lexicographical ordering of the group $\mathbf{Z}^d$, $d \geqslant 2$. Let $e^i \in \mathbf{Z}^d$ be the ith standard unit vector. We put

$$\mathbf{Z}^d_n = \{g = (m_1, \ldots, m_d) \in \mathbf{Z}^d;\ m_1 = \ldots = m_n = 0\}, \quad 1 \leqslant n \leqslant d,$$
$$\mathbf{Z}^d_- = \{g \in \mathbf{Z}^d;\ g \prec 0\}.$$

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Let Φ be a $\mathbf{Z}^d$-action on the space $(X, \mathscr{B}, \mu)$, i.e. Φ is a homomorphism of the group $\mathbf{Z}^d$ into the group of all measure-preserving automorphisms of $(X, \mathscr{B}, \mu)$, and set $\Phi_n = \Phi|\mathbf{Z}^d_n$, $1 \leqslant n \leqslant d$. Let $T_i = \Phi^{e^i}$, $1 \leqslant i \leqslant d$. It is clear that $\Phi^g = T_1^{m_1}\dots T_d^{m_d}$ where $g = (m_1, \dots, m_d) \in \mathbf{Z}^d$.

For a given set $A \subset \mathbf{Z}^d$ and a partition $P \in \mathscr{M}$ we define

$$P(A) = \bigvee_{g \in A} \Phi^g P.$$

Let

$$P_\Phi = P(\mathbf{Z}^d), \quad P^- = P^-_\Phi = P(\mathbf{Z}^d_-), \quad P^n = P(\mathbf{Z}^d_n),\ 1 \leqslant n \leqslant d.$$

Thus the partition P^n is the join of all partitions $T_{n+1}^{i_{n+1}}\dots T_d^{i_d}P$ where $i_k \in \mathscr{Z}$, $n+1 \leqslant k \leqslant d$. In particular, $P^d = P$.

Now, let $\sigma \in \mathscr{M}$ be *totally invariant*, i.e. $\Phi^g\sigma = \sigma$, $g \in \mathbf{Z}^d$, and let Φ_σ be the factor of Φ defined by σ. For a given $P \in \mathscr{M}$ we put

$$\hat{P} = \bigwedge_{n=0}^{\infty} \Big(\bigvee_{k=1}^{d} T_k^{-n}(P^k)^-_{T_k} \vee \sigma \Big).$$

In other words, $\hat{P}$ is the tail partition given by P and σ.

Now we recall the concepts of the relative entropy and the relative Pinsker partition ([6]).

The *relative entropy* $h(\Phi|\sigma)$ of the action Φ with respect to σ is defined by the formula

$$h(\Phi|\sigma) = \sup\{h(P, \Phi|\sigma);\ P \in \mathscr{Z}\} \quad \text{where}$$

$$h(P, \Phi|\sigma) = H(P|P^-_\Phi \vee \sigma), \qquad P \in \mathscr{M}.$$

The partition

$$\pi(\Phi|\sigma) = \bigvee_{P \in \mathscr{N}} P \quad \text{where} \quad \mathscr{N} = \{P \in \mathscr{Z};\ h(P, \Phi|\sigma) = 0\}$$

is said to be the *relative Pinsker partition* of the action Φ with respect to σ.

It is easy to observe that in the case $\sigma = \nu$ the entropy $h(\Phi|\sigma)$ and the partition $\pi(\Phi|\sigma)$ reduce to the usual entropy $h(\Phi)$ and the Pinsker partition $\pi(\Phi)$ of Φ ([1]).

Now, let us note some properties of the relative entropy and the relative Pinsker partition used in the sequel.

Let $P, Q, R \in \mathscr{Z}$ be arbitrary.

(A) $h(P \vee Q, \Phi|\sigma) = h(P, \Phi|\sigma) + H(Q|Q^-_\Phi \vee P_\Phi \vee \sigma)$.

(B) *If* $P \leqslant Q$ *then*

$$\lim_{n\to\infty} H(P|Q^-_\Phi \vee T_1^{-n}R^-_\Phi \vee \sigma) = H(P|Q^-_\Phi \vee \sigma).$$

(C) $\hat{P} \leqslant \pi(\Phi|\sigma)$.

Let G_r be a subgroup of $\mathbf{Z}^d$ of finite index r and let Φ^r be the restriction of Φ to G_r, $r \geqslant 1$.

(D) $h(P, \Phi^r|\sigma) \leqslant r \cdot h(P, \Phi|\sigma)$.

(E) *If* $P \in \mathscr{Z}$ *is such that* $P_\Phi \vee \sigma = \varepsilon$ *then* $h(P, \Phi|\sigma) = h(\Phi|\sigma)$.

(F) $h(\Phi) = h(\Phi_\sigma) + h(\Phi|\sigma)$.

(G) *For any totally invariant partitions* σ_i, $i = 1, 2$,

$$h\big(P, \Phi|\pi(\Phi|\sigma_1) \vee \sigma_2\big) = h(P, \Phi|\sigma_1 \vee \sigma_2).$$

The proofs of (A)–(E) run in the same manner as in the case $\sigma = \nu$ (cf. [1]). (F) and (G) are proved in [5] and [4] respectively for $\mathbf{Z}^1$-actions. The proofs for arbitrary $d \geqslant 1$ are the same.

(H) *If* $P, Q \in \mathscr{Z}$ *and* $h(P, \Phi|\sigma) = 0$ *then* $H(Q|P) \geqslant H(Q|\hat{Q})$.

Proof. For $n \in \mathbf{N}$ let Φ^n be the restriction of Φ to the subgroup $\{(nm_1, \dots, nm_d);\ (m_1, \dots, m_d) \in \mathbf{Z}^d\}$. Let $P \in \mathscr{Z}$ and $h(P, \Phi|\sigma) = 0$. It follows from (D) that $h(P, \Phi^n|\sigma) = 0$. Set

$$P_n = P_{\Phi^n}, \qquad P_n^- = P^-_{\Phi^n}.$$

Let $Q \in \mathscr{Z}$ be arbitrary. Using the property (A) we have

$$\begin{aligned} h(P \vee Q, \Phi^n|\sigma) &= h(P, \Phi^n|\sigma) + H(Q|Q_n^- \vee P_n \vee \sigma) \\ &= h(Q, \Phi^n|\sigma) + H(P|P_n^- \vee Q_n \vee \sigma). \end{aligned}$$

Hence $h(Q, \Phi^n|\sigma) = H(Q|Q_n^- \vee P_n \vee \sigma)$ and so

$$\begin{aligned} H(Q|P) &\geqslant H(Q|Q_n^- \vee P_n \vee \sigma) = H(Q|Q_n^- \vee \sigma) \\ &\geqslant H\Big(Q\Big|\bigvee_{k=1}^{d} T_k^{-n+1}(Q^k)^-_{T_k}\Big), \qquad n \geqslant 1. \end{aligned}$$

Therefore taking the limit as $n \to \infty$ in the last inequality we obtain $H(Q|P) \geqslant H(Q|\hat{Q})$, which completes the proof.

§ 2. Relatively perfect partitions. First we recall some notions from [4].

A partition $\zeta \in \mathscr{M}$ is said to be *invariant* if $\Phi^g\zeta \leqslant \zeta$ for every $g \prec 0$. It is easy to check that this is equivalent to $T_i^{-1}\zeta^i \leqslant \zeta$, $1 \leqslant i \leqslant d$.

A partition $\zeta \in \mathscr{M}$ is called *strongly invariant* if it is invariant and if

$$\bigvee_{g \in A} \Phi^g\zeta = \bigwedge_{g \in B} \Phi^g\zeta$$

where the sets A, B form a partition of $\mathbf{Z}^d$ such that $g \prec h$, $g \in A$, $h \in B$, A does not contain a greatest element and B does not contain a smallest element. It is easy to show that this condition is equivalent to

$$\bigwedge_{n=0}^{\infty} T_k^{-n}\zeta^k = T_{k-1}^{-1}\zeta^{k-1}, \qquad 2 \leqslant k \leqslant d.$$

A partition $\zeta \in \mathscr{M}$ is called *exhaustive* if it is strongly invariant and $\zeta_\Phi = \varepsilon$.

The results of Lemmas 1, 2 and of Theorem 1 together with some remarks about their proofs in the case $\sigma = \nu$ have been announced in [4]. However, especially in the case of Lemma 1, the complete proofs even for $\sigma = \nu$ need additional considerations. For this reason and for completeness we give the proofs with some shortenings.

LEMMA 1. *If a partition $\zeta \in \mathscr{M}$ is exhaustive and $\zeta \geqslant \sigma$ then*

$$\bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1 \geqslant \pi(\Phi|\sigma).$$

Proof. Let $P_i, Q_i \in \mathscr{Z}$ and let r_i be a positive integer such that

$$P_1 \leqslant \pi(\Phi|\sigma), \quad P_i \leqslant \hat{Q}_{i-1}, \quad 2 \leqslant i \leqslant d,$$
$$Q_j \leqslant T_j^{r_j}\zeta^j, \quad 1 \leqslant j \leqslant d.$$

The fact that $\zeta \geqslant \sigma$ and the strong invariance of ζ imply

$$\hat{Q}_d \leqslant \bigwedge_{n=0}^{\infty} T_d^{-n}\zeta = T_{d-1}^{-1}\zeta^{d-1}. \tag{1}$$

Since $P_d \leqslant \hat{Q}_{d-1} \leqslant T_{d-1}^{r_{d-1}}\zeta^{d-1}$, the property (C) implies $h(P_d, \Phi|\sigma) = 0$ and so

$$T_{d-1}^{-r_{d-1}}P_d \leqslant \zeta^{d-1}, \quad h(T_{d-1}^{-r_{d-1}}P_d, \Phi|\sigma) = 0. \tag{2}$$

By (1), (2) and (H) we get

$$H(Q_d|T_{d-1}^{-r_{d-1}}P_d) \geqslant H(Q_d|\hat{Q}_d) \geqslant H(Q_d|T_{d-1}^{-1}\zeta^{d-1}). \tag{3}$$

Since $Q_d \leqslant T_d^{r_d}\zeta^d = T_d^{r_d}\zeta$ we see that the inequalities (3) are satisfied for Q_d running over a dense subset of the set

$$\{P \in \mathscr{Z};\ P \leqslant \zeta_{T_d} = \zeta^{d-1}\}.$$

Therefore they are satisfied for all partitions in this set. If we take in (3), in particular, $Q_d = T_{d-1}^{-r_{d-1}}P_d$ we obtain $P_d \leqslant T_{d-1}^{r_{d-1}-1}\zeta^{d-1}$.

Repeating this procedure we get

$$P_d \leqslant T_{d-1}^{r_{d-1}-n}\zeta^{d-1} \quad \text{for all } n \geqslant 1.$$

Using again the fact that ζ is strongly invariant we have

$$P_d \leqslant \bigwedge_{n=0}^{\infty} T_{d-1}^{-n}\zeta^{d-1} = T_{d-2}^{-1}\zeta^{d-2}.$$

Thus we have shown that $P_d \leqslant T_{d-2}^{-1}\zeta^{d-2}$ for $P_d \leqslant \hat{Q}_{d-1}$, i.e.

$$\hat{Q}_{d-1} \leqslant T_{d-2}^{-1}\zeta^{d-2}. \tag{4}$$

Repeating $d-3$ times the considerations above we get $\hat{Q}_2 \leqslant T_1^{-1}\zeta^1$, $h(P_2, \Phi|\sigma) = 0$ and so

$$H(Q_2|T_1^{-r_1}P_2) \geqslant H(Q_2|\hat{Q}_2) \geqslant H(Q_2|T_1^{-1}\zeta^1).$$

Since $Q_2 \leqslant T_2^{r_2}\zeta^2$ the last inequality is satisfied for all Q_2 in a dense subset of $\{P \in \mathscr{Z};\ P \leqslant \zeta^1\}$. Hence, as above, $P_2 \leqslant \bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1$, i.e. $\hat{Q}_1 \leqslant \bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1$. Thus using the equality $h(P_1, \Phi|\sigma) = 0$ we get

$$H(Q_1|P_1) \geqslant H(Q_1|\hat{Q}_1) \geqslant H(Q_1|\bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1). \tag{5}$$

Since ζ is exhaustive, applying again the density argument we may take $Q_1 = P_1$ in (5). Hence we obtain

$$P_1 \leqslant \bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1,$$

which completes the proof.

LEMMA 2. *There exists a measurable partition $\eta \geqslant \sigma$ which is invariant, generating and such that*

(a) $\bigwedge_{n=0}^{\infty} T_1^{-n}\pi(\Phi_1|\eta^1) \leqslant \pi(\Phi|\sigma)$,
(b) $h(\Phi|\sigma) = H(\eta|\eta^-) = H(\eta|T_d^{-1}\eta)$.

Proof. Let $(P_k) \in \mathscr{Z}$ be a sequence such that $P_k \nearrow \varepsilon$. Using the property (B) we may construct, similarly to [15], a strictly increasing sequence (n_k) of positive integers such that

$$H(Q_r|Q_r^- \vee \sigma) - H(Q_r|Q_{r+s+1}^- \vee \sigma) < 1/r, \quad s \geqslant 0, \tag{6}$$

where $Q_r = \bigvee_{k=1}^{r} T_1^{-n_k}P_k$ and $Q_r^- = (Q_r)_{\Phi}^-$, $r \geqslant 1$. We put

$$Q = \bigvee_{r=1}^{\infty} Q_r, \quad \eta = Q \vee Q^- \vee \sigma.$$

It is clear that the partition η is invariant, generating, i.e. $\eta_\Phi = \varepsilon$, and $\eta \geqslant \sigma$.

Now we check (a). Since $Q_r^- \vee \sigma \nearrow Q^- \vee \sigma = \eta^-$ the inequalities (6) give

$$H(Q_r|Q_r^- \vee \sigma) - H(Q_r|\eta^-) < 1/r, \quad r \geqslant 1. \tag{7}$$

Let $P \in \mathscr{Z}$ and $P \leqslant \bigwedge_{n=0}^{\infty} T_1^{-n}\pi(\Phi_1|\eta^1)$. The last partition is of course totally invariant with respect to Φ and so

$$P_\Phi \leqslant \bigwedge_{n=0}^{\infty} \pi(\Phi_1|T_1^{-n}\eta^1) \leqslant \pi(\Phi_1|(\eta^1)_{T_1}^-).$$

The property (A) implies

$$h(P, \Phi|\sigma) = h(Q_r, \Phi|\sigma) - H(Q_r|Q_r^- \vee P_\Phi \vee \sigma) + H(P|P^- \vee (Q_r)_\Phi \vee \sigma). \tag{8}$$

The use of (G) to the action Φ_1 gives

$$\begin{aligned} H(Q_r|Q_r^- \vee P_\Phi \vee \sigma) &\geqslant H\big(Q_r|Q_r^- \vee \pi(\Phi_1|(\eta^1)_{T_1}^-) \vee \sigma\big) \\ &= h\big(Q_r, \Phi_1\big|(Q_r^1)_{T_1}^- \vee \pi(\Phi_1|(\eta^1)_{T_1}^- \vee \sigma)\big) \\ &= h\big(Q_r, \Phi_1\big|(Q_r^1)_{T_1}^- \vee (\eta^1)_{T_1}^- \vee \sigma\big) \\ &\geqslant H(Q_r|Q_r^- \vee \eta^- \vee \sigma) = H(Q_r|\eta^-), \qquad r \geqslant 1. \end{aligned}$$

Applying this result and (7) in (8) and then taking the limit as $r \to \infty$ we get $h(P, \Phi|\sigma) = 0$, i.e. $P \leqslant \pi(\Phi|\sigma)$. This proves (a).

In order to check (b) observe that

$$P_r \leqslant T_1^{n_r} Q_r, \qquad Q_r \leqslant T_1^{-n_1} P_r \vee \ldots \vee T_1^{-n_r} P_r.$$

Therefore $(P_r)_\Phi = (Q_r)_\Phi$ and so $h(P_r, \Phi|\sigma) = h(Q_r, \Phi|\sigma)$. Hence, using the fact that $P_r \nearrow \varepsilon$ and $Q_r \nearrow Q$ we get

$$\lim_{r\to\infty} H(Q_r|Q_r^- \vee \sigma) = h(\Phi|\sigma),$$

$$\lim_{r\to\infty} H(Q_r|\eta^-) = H(Q|Q^- \vee \sigma) = H(\eta|\eta^-).$$

Then, taking in (7) the limit as $r \to \infty$ we obtain (b) and the proof is complete.

DEFINITION. A partition $\zeta \in \mathcal{M}$ is said to be *relatively perfect* with respect to σ if ζ is exhaustive, $\zeta \geqslant \sigma$ and

(i) $\bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1 = \pi(\Phi|\sigma)$,
(ii) $h(\Phi|\sigma) = H(\zeta|\zeta^-)$.

It is clear that a partition relatively perfect with respect to $\sigma = \nu$ is perfect (cf. [4]).

THEOREM 1. *For every positive integer d, every $\mathbf{Z}^d$-action on $(X, \mathscr{B}, \mu)$ and every totally invariant measurable partition of X there exists a relatively perfect partition with respect to this totally invariant partition.*

Proof. We prove the theorem by induction on d. The proof for $d = 1$ is similar to that of the Rokhlin–Sinai theorem (cf. [15]).

Suppose our theorem is valid for $d-1$. Let Φ be an arbitrary $\mathbf{Z}^d$-action and $\sigma \in \mathcal{M}$ an arbitrary totally invariant partition of X. By Lemma 2 there exists a partition $\eta \in \mathcal{M}$ which is invariant, generating, $\eta \geqslant \sigma$ and such that

$$\bigwedge_{n=0}^{\infty} T_1^{-n}\pi(\Phi_1|\eta^1) \leqslant \pi(\Phi|\sigma), \tag{9}$$

$$h(\Phi|\sigma) = H(\eta|\eta^-) = H(\eta|T_d^{-1}\eta). \tag{10}$$

Now we apply the induction assumption to the space $X/\pi(\Phi_1|\eta^1)$, to the action Φ_1 and to the totally invariant (with respect to Φ_1) partition $T_1^{-1}\pi(\Phi_1|\eta^1)$.

Hence there exists a partition $\zeta \in \mathcal{M}$ with the following properties:

$$T_1^{-1}\pi(\Phi_1|\eta^1) \leqslant \zeta \leqslant \pi(\Phi_1|\eta^1), \tag{11}$$

(12) ζ is strongly invariant with respect to Φ_1,

$$\zeta^1 = \pi(\Phi_1|\eta^1), \tag{13}$$

$$\bigwedge_{n=0}^{\infty} T_2^{-n}\zeta^2 = T_1^{-1}\pi(\Phi_1|\eta^1), \tag{14}$$

$$h(\Phi_1|T_1^{-1}\pi(\Phi_1|\eta^1)) = H(\zeta|\zeta_{\Phi_1}^-) = H(\zeta|T_d^{-1}\zeta). \tag{15}$$

It follows from (11) that $\zeta \geqslant \sigma$ and $T_1^{-1}\zeta^1 \leqslant \zeta$. Therefore using (12) we see that ζ is invariant with respect to Φ. The strong invariance of ζ readily follows from (12)–(14). Now we check that

$$\bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1 = \pi(\Phi|\sigma). \tag{16}$$

Since $\zeta \geqslant \sigma$ and ζ is exhaustive Lemma 1 gives $\bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1 \geqslant \pi(\Phi|\sigma)$. On the other hand, (9) and (13) imply

$$\bigwedge_{n=0}^{\infty} T_1^{-n}\zeta^1 = \bigwedge_{n=0}^{\infty} T_1^{-n}\pi(\Phi_1|\eta^1) \leqslant \pi(\Phi|\sigma),$$

and so (16) holds.

It remains to show the equality

$$h(\Phi|\sigma) = H(\zeta|\zeta_\Phi^-) = H(\zeta|T_d^{-1}\zeta).$$

It is clear that it is sufficient to show the inequality $h(\Phi|\sigma) \leqslant H(\zeta|T_d^{-1}\zeta)$.

First we show

$$h\big(\Phi_1\big|T_1^{-1}\pi(\Phi_1|\eta^1)\big) \geqslant H(\eta|T_d^{-1}\eta). \tag{17}$$

Let $P \in \mathscr{Z}$ and $P \leqslant \eta$. The invariance of η and (G) give

$$\begin{aligned} h\big(\Phi_1\big|T_1^{-1}\pi(\Phi_1|\eta^1)\big) &\geqslant h\big(P, \Phi_1\big|T_1^{-1}\pi(\Phi_1|\eta^1)\big) \\ &= H\big(P\big|P_{\Phi_1}^- \vee T_1^{-1}\pi(\Phi_1|\eta^1)\big) \\ &\geqslant H(P|\eta_{\Phi_1}^- \vee T_1^{-1}\eta^1) \\ &= H(P|T_d^{-1}\eta). \end{aligned}$$

Let $(P_k) \subset \mathscr{Z}$ be such that $P_k \nearrow \eta$. Replacing P by P_k, $k \geqslant 1$, in the last inequality and taking the limit as $k \to \infty$ we get (17).

Now, combining (10), (15) and (17) we have

$$h(\Phi|\sigma) = H(\eta|T_d^{-1}\eta) \leqslant h\big(\Phi_1|T_1^{-1}\pi(\Phi_1|\eta^1)\big) = H(\zeta|T_d^{-1}\zeta),$$

which completes the proof.

§ 3. Principal partitions and an axiomatic definition of entropy

DEFINITION. A totally invariant partition $\sigma \in \mathcal{M}$ is said to be *principal* if every strongly invariant partition $\zeta \in \mathcal{M}$ with $\zeta \geqslant \sigma$ is totally invariant.

A factor Ψ of the action Φ is said to be *principal* if every totally invariant partition σ such that Ψ and Φ_σ are isomorphic, is principal.

THEOREM 2. *If an action Ψ is a principal factor of Φ then $h(\Phi) = h(\Psi)$. In the case $h(\Phi) < \infty$ the converse theorem is also true.*

Proof. Let Ψ be a principal factor of Φ and let $\sigma \in \mathcal{M}$ be a totally invariant partition such that Ψ and Φ_σ are isomorphic. It follows from Theorem 1 that there exists a partition $\zeta \in \mathcal{M}$ with $\zeta \geqslant \sigma$ which is strongly invariant and $h(\Phi|\sigma) = H(\zeta|\zeta^-)$. By our assumption ζ is totally invariant and so $h(\Phi|\sigma) = 0$. The property (F) implies

$$h(\Phi) = h(\Phi_\sigma) + h(\Phi|\sigma) = h(\Phi_\sigma) = h(\Psi).$$

Now, suppose $h(\Phi) < \infty$. Let Ψ be a factor of Φ such that $h(\Psi) = h(\Phi)$ and let σ be a totally invariant partition such that Φ_σ and Ψ are isomorphic. Using again (F) we have

$$h(\Phi|\sigma) = h(\Phi) - h(\Psi) = 0.$$

Let $\zeta \geqslant \sigma$ be a strongly invariant partition. Since $H(\zeta|\zeta^-) \leqslant h(\Phi|\sigma) = 0$ and ζ is invariant we have $T_d^{-1}\zeta = \zeta^- = \zeta$. This equality and the strong invariance of ζ imply ζ is totally invariant, which means that Ψ is principal.

COROLLARY. *If $h(\Phi) < \infty$ and Φ_σ is an ergodic factor such that every element of σ is a finite set then Φ_σ is principal.*

Proof. It follows from our assumption and the Rokhlin theorem ([13]) that there exists a finite partition P such that $P \vee \sigma = \varepsilon$. Using the property (E) we have $h(\Phi|\sigma) = h(P, \Phi|\sigma) = 0$ and so, applying (F) and Theorem 2, we get the result.

The following example shows that Theorem 2 fails to be true if we replace strong invariance by invariance in the definition of a principal factor.

EXAMPLE. Let $(Y, \mathcal{F}, \lambda, \varphi)$ be an arbitrary dynamical system with $h(\varphi) = 0$. Let $(X, \mathcal{B}, \mu, T)$ be a Bernoulli dynamical system with state space $(Y, \mathcal{F}, \lambda)$. Let Φ be the $\mathbf{Z}^2$-action on X defined by the formula

$$\Phi^g = T^i S_\varphi^j, \qquad g = (i, j) \in \mathbf{Z}^2,$$

where $(S_\varphi x)(n) = (\varphi x)(n)$, $n \in \mathbf{Z}$. It is known (cf. [1]) that $h(\Phi) = h(\varphi) = 0$. Let $\sigma = \nu$. It is clear that $h(\Phi) = h(\Phi_\sigma)$.

We want to show that there exists an invariant partition ζ of X which is not totally invariant. Let $Q = \{Q_0, Q_1\}$ be a nontrivial partition of Y and let

$P = \{P_0, P_1\}$ be the partition of X given by

$$P_i = \{x \in X;\ x(0) \in Q_i\}, \qquad i = 0, 1.$$

The partition $\zeta = P \vee P_\Phi^-$ is of course invariant and, since $h(\varphi) = 0$, we have $S^{-1}\zeta = \zeta$. However, ζ is not totally invariant. Indeed, if ζ is totally invariant then $T^{-1}\zeta = \zeta$, i.e. $P_S \vee (P_S)_T^- = (P_S)_T^-$. It follows from the definition of μ that the partitions P_S and $(P_S)_T^-$ are independent. Therefore P is a trivial partition, contradicting the nontriviality of Q.

We denote the set of all ergodic $\mathbf{Z}^d$-actions on Lebesgue probability spaces by $\operatorname{Act}\mathbf{Z}^d$. Let Φ_b be the Bernoulli $\mathbf{Z}^d$-action defined by the vector $(\frac{1}{2}, \frac{1}{2})$.

Applying the generalized Sinai theorem concerning Bernoulli factors (cf. [8]) and Theorem 2 we may prove, using the Rokhlin idea ([14]), the following

COROLLARY. *Let $H\colon \operatorname{Act}\mathbf{Z}^d \to [0, \infty]$ be a function such that $H(\Phi_b) = \log 2$ and for all $\Phi, \Psi \in \operatorname{Act}\mathbf{Z}^d$ the following conditions are satisfied:*

(i) *if Ψ is a factor of Φ then $H(\Phi) \geqslant H(\Psi)$,*
(ii) *if Ψ is a principal factor of Φ then $H(\Phi) = H(\Psi)$,*
(iii) *$H(\Phi \times \Psi) = H(\Phi) + H(\Psi)$.*

Then $H(\Phi) = h(\Phi)$, $\Phi \in \operatorname{Act}\mathbf{Z}^d$.

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Boundedness of classical operators on classical Lorentz spaces

by

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Abstract. The classical Lorentz space $\Lambda_p(v)$ consists of those measurable functions f on $\mathbf{R}^n$ such that $(\int_0^\infty f^*(x)^p v(x)\,dx)^{1/p} < \infty$. We characterize when a variety of classical operators, including Hilbert and Riesz transforms, fractional integrals and maximal functions, are bounded from one Lorentz space, $\Lambda_p(v)$, to another, $\Lambda_q(w)$. In addition, we give a simple and explicit description of the dual of $\Lambda_p(v)$ and determine when $\Lambda_p(v)$ is a Banach space.

§ 1. Introduction. For $1 \leqslant p < \infty$ and $v(x)$ a nonnegative function on $(0, \infty)$, the classical Lorentz spaces $\Lambda_p(v)$ on $\mathbf{R}^n$, introduced and studied by G. Lorentz in [7] for the intervals $(0, l)$, $0 < l \leqslant \infty$, are given by

$$\Lambda_p(v) = \{f \text{ measurable on } \mathbf{R}^n\colon (\int_0^\infty f^*(x)^p v(x)\,dx)^{1/p} < \infty\},$$

where $f^*(x) = \inf\{\lambda\colon |\{t \in \mathbf{R}^n\colon |f(t)| > \lambda\}| \leqslant x\}$ is the nonincreasing rearrangement of f on $(0, \infty)$ with respect to Lebesgue measure on $\mathbf{R}^n$ ($|E|$ denotes the Lebesgue measure of a set E). M. Ariño and B. Muckenhoupt observed in [2] that the Hardy–Littlewood maximal operator M, defined by

$$Mf(x) = \sup\{|Q|^{-1}\int_Q |f(y)|\,dy\colon Q \text{ is a cube in } \mathbf{R}^n \text{ containing } x\},$$

is bounded from $\Lambda_p(v)$ to $\Lambda_p(w)$ if and only if

$$(1.1)\qquad (\int_0^\infty (x^{-1}\int_0^\infty f(t)\,dt)^q w(x)\,dx)^{1/q} \leqslant C(\int_0^\infty f(x)^p v(x)\,dx)^{1/p}$$

for all nonnegative and nonincreasing functions f on $(0, \infty)$. Indeed, this follows immediately from the rearrangement inequality for the maximal function ([6], [12] and [15])

$$(1.2)\qquad (Mf)^*(x) \leqslant C_1 x^{-1}\int_0^x f^*(t)\,dt \leqslant C_2 (Mf)^*(x), \quad x > 0,$$

coupled with the fact that every nonincreasing function f^* on $(0, \infty)$ occurs as

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