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Weighted Lorentz norm inequalities for integral operators

by

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Abstract. Conditions depending on the kernel $K(x, y)$ are given for weight functions w and v so that the integral operator $Kf(x) = \int_{-\infty}^{x} K(x, y) f(y)\, dy$, where $K(x, y) \geq 0$ is defined on $\Delta = \{(x, y)\colon y < x\}$, is bounded from a Lorentz space $L^{r,s}((-\infty, \infty), v\,dx)$ into another Lorentz space $L^{p,q}((-\infty, \infty), w\,dx)$. In Theorem 1 the kernel $K(x, y)$ is supposed to be nonincreasing in x. In Theorem 2 the kernel is supposed to be nondecreasing in y. Dual results for the dual operators are given. Finally, it is shown that the stated conditions on the kernels are not always necessary.

1. Introduction. Our purpose is to find conditions that imply weighted Lorentz norm inequalities for the integral operators K and K^* defined by

$$Kf(x) = \int_{-\infty}^{x} K(x, y) f(y)\, dy, \tag{1.1}$$

$$K^* f(x) = \int_{x}^{\infty} K(y, x) f(y)\, dy, \tag{1.2}$$

where $K(x, y)$ is defined on $\Delta = \{(x, y) \in \mathbf{R}^2\colon y < x\}$ and it is nonnegative. Two kinds of kernels $K(x, y)$, either nonincreasing in x, or nondecreasing in y, are considered separately. In the last section we deal with the necessity of our conditions.

The Hardy operator $Tf(x) = \int_0^x f$, $x > 0$, and the modified Hardy operators $T_\eta f(x) = x^{-\eta} \int_0^x f$, with real η, are examples of the above operators. Several authors have obtained inequalities for weighted Lebesgue norms for these operators (cf. [2]–[4], [7], [9] and [10]). Our results compare with others in the literature as follows. If we restrict ourselves to the Hardy operator, the sufficient condition (1.3) of Theorem 1 is known to be also a necessary condition ([8]). The same is true for condition (1.5) of Theorem 2 when restricted to the modified Hardy operators. If we consider only Lebesgue norms our results are related to those of Andersen and Heinig [1] as follows. Our monotonicity conditions on $K(x, y)$ are more general than those in [1], while the weights considered by Andersen and Heinig are in a class larger than ours.

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Before we state our theorems we recall the usual conventions $0\cdot\infty = 0$, $1/p+1/p' = 1$. In what follows $K(x, y)$ is nonnegative and it is defined on $\Delta = \{(x, y)\in \mathbf{R}^2: y < x\}$. The characteristic function of the interval (a, b) is denoted by $\chi_{(a,b)}(\cdot)$. The norms used below are defined in Section 2.

THEOREM 1. *Let K be the integral operator defined by* (1.1), *where $K(x, y)$ is nonincreasing in x. Let $1 \leqslant p, q, s \leqslant \infty$, $1 \leqslant r < \infty$, $s = 1$ if $r = 1$, and $\max\{r, s\} \leqslant \min\{p, q\}$. If*

$$\sup_{a\in\mathbf{R}}\Big(\int_a^\infty w\Big)^{1/p}\|K(a, y)v^{-1}(y)\chi_{(-\infty,a)}(y)\|_{L^{r',s'}(v)} = B < \infty, \tag{1.3}$$

and $v > 0$ a.e. on $(-\infty, a)$ if $\int_a^\infty w > 0$, then

$$\|Kf\|_{L^{p,q}(w)} \leqslant C\|f\|_{L^{r,s}(v)}, \quad \text{for all } f. \tag{1.4}$$

THEOREM 2. *Let K be the integral operator defined by* (1.1), *where $K(x, y)$ is nondecreasing in y. Suppose $1 \leqslant p, q \leqslant \infty$, $1 \leqslant r < \infty$, $q = 1$ if $p = 1$, and $r \leqslant \min\{p, q\}$. If*

$$\sup_{a\in\mathbf{R}}\|K(x, a)\chi_{(a,\infty)}(x)\|_{L^{p,q}(w)}\Big(\int_{-\infty}^a v^{1-r'}\Big)^{1/r'} = B < \infty, \tag{1.5}$$

where the second factor on the left side of (1.5) *is interpreted as $\|v^{-1}\chi_{(-\infty,a)}\|_{L^\infty(v)}$ if $r = 1$, and we add in this case the hypothesis $v > 0$ a.e. on $(-\infty, a)$ if $\|K(x, a)\chi_{(a,\infty)}(x)\|_{L^{p,q}(w)} > 0$, then*

$$\|Kf\|_{L^{p,q}(w)} \leqslant C\|f\|_{L^r(v)}, \quad \text{for all } f. \tag{1.6}$$

We also obtain results dual to those above.

THEOREM 3. *Let K^* be the integral operator defined by* (1.2), *where $K(x, y)$ is nondecreasing in y. Suppose p, q, r and s are as in Theorem 1. If*

$$\sup_{a\in\mathbf{R}}\Big(\int_{-\infty}^a w\Big)^{1/p}\|K(y, a)v^{-1}(y)\chi_{(a,\infty)}(y)\|_{L^{r',s'}(v)} = B < \infty, \tag{1.7}$$

and $v > 0$ a.e. on (a, ∞) if $\int_{-\infty}^a w > 0$, then

$$\|K^*f\|_{L^{p,q}(w)} \leqslant C\|f\|_{L^{r,s}(v)}, \quad \text{for all } f. \tag{1.8}$$

THEOREM 4. *Let K^* be the integral operator defined by* (1.2), *where $K(x, y)$ is nonincreasing in x. Suppose p, q and r are as in Theorem 2. If*

$$\sup_{a\in\mathbf{R}}\|K(a, y)\chi_{(-\infty,a)}(y)\|_{L^{p,q}(w)}\Big(\int_a^\infty v^{1-r'}\Big)^{1/r'} = B < \infty, \tag{1.9}$$

where the second factor on the left side of (1.9) *is interpreted as $\|v^{-1}\chi_{(a,\infty)}\|_{L^\infty(v)}$ if $r = 1$, and we add in this case the hypothesis $v > 0$ a.e. on (a, ∞) if $\|K(a, y)\times\chi_{(-\infty,a)}(y)\|_{L^{p,q}(w)} > 0$, then*

$$\|K^*f\|_{L^{p,q}(w)} \leqslant C\|f\|_{L^r(v)}, \quad \text{for all } f. \tag{1.10}$$

These theorems are proved in Section 3. Before that we introduce some definitions and recall some known results in Section 2. Finally, Section 4 deals with the converses of our theorems.

2. Definitions and auxiliary results. Let (M, μ) be a measure space. Given a measurable function f on (M, μ), we define the distribution function λ_f and the nonincreasing rearrangement f^* of f with respect to μ by (see e.g. [6], p. 249)

$$\lambda_f(s) = \mu(\{x\in M: |f(x)| > s\}), \quad f^*(t) = \inf\{s > 0: \lambda_f(s) \leqslant t\}.$$

The Lorentz space $L^{p,q}(\mu)$ consists of all measurable functions f satisfying $\|f\|_{L^{p,q}(\mu)} < \infty$, where

$$\|f\|_{L^{p,q}(\mu)} = \begin{cases} \Big[\int_0^\infty (q/p)t^{q/p-1}f^*(t)^q\,dt\Big]^{1/q}, & 0 < p < \infty,\ 0 < q < \infty,\\ \sup_{t>0} t^{1/p}f^*(t), & 0 < p \leqslant \infty,\ q = \infty. \end{cases}$$

Note that if $p = q$ then $L^{p,q}(\mu)$ is the usual $L^p(\mu)$ space.

It is not difficult to see that the following equality holds (see, for example, [8], p. 332):

$$\|f\|_{L^{p,q}(\mu)} = \begin{cases} \Big[\int_0^\infty qs^{q-1}\lambda_f(s)^{q/p}\,ds\Big]^{1/q}, & 0 < p < \infty,\ 0 < q < \infty,\\ \sup_{s>0} s\lambda_f(s)^{1/p}, & 0 < p < \infty,\ q = \infty. \end{cases} \tag{2.1}$$

If $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, or $p = q = 1$ or $p = q = \infty$, it can be shown (see, for example, [5], p. 112) that there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{L^{p,q}(\mu)} \leqslant \sup\{|\textstyle\int fg\,d\mu|: \|g\|_{L^{p',q'}(\mu)} \leqslant 1\} \leqslant C\|f\|_{L^{p,q}(\mu)}. \tag{2.2}$$

We will use the following lemma found in [8], p. 333.

LEMMA 1. *Let (M, μ) be a measure space. Suppose $q \geqslant \max\{r, s\}$ and let $\{E_k\}$ be a sequence of disjoint measurable subsets of M. Then*

$$\sum_k \|\chi_{E_k}f\|^q_{L^{r,s}(\mu)} \leqslant \|f\|^q_{L^{r,s}(\mu)}.$$

The measure spaces (M, μ) that appear in our theorems are such that $M = \mathbf{R}$ and μ is a weighted Lebesgue measure.

3. Proofs of the theorems. For simplicity, we make the following definition.

DEFINITION 1. We say that a nonnegative function f is *K-admissible* (respectively, *K^*-admissible*) if $Kf(x)$ (respectively, $K^*f(x)$) is finite for all x.

The proof of Theorem 1 relies essentially on Lemma 1 and the following results.

LEMMA 2. *Let $K(x, y)$ be nonincreasing in x. We put*

$$K_\delta f(x) = \int_{-\infty}^{x-\delta} K(x, y) f(y)\, dy, \quad \text{for } \delta > 0. \tag{3.1}$$

If $f \geq 0$ is K-admissible, then

$$\lim_{x \to x_0^+} K_\delta f(x) \leq K_\delta f(x_0) \leq \lim_{x \to x_0^-} K_\delta f(x), \quad \text{for all } x_0, \tag{3.2}$$

$$\|Kf\|_{L^{p,q}(\mu)} = \lim_{\delta \to 0^+} \|K_\delta f\|_{L^{p,q}(\mu)}, \quad \text{for all } p, q > 0. \tag{3.3}$$

Proof. Fix x_0 and δ. Since

$$K(x, y) f(y) \chi_{(-\infty, x-\delta)}(y) \leq K(x_0, y) f(y) \chi_{(-\infty, x_0)}(y), \quad \text{for } x_0 < x < x_0 + \delta,$$

and since f is K-admissible and $K(x, y)$ is nonincreasing in x, the Dominated Convergence Theorem implies

$$\lim_{x \to x_0^+} K_\delta f(x) = \int_{-\infty}^{x_0-\delta} K(x_0^+, y) f(y)\, dy \leq K_\delta f(x_0).$$

The remaining inequality in (3.2) follows from a similar argument by considering $x_0 - \delta < x < x_0$. The equality (3.3) follows from the fact that if δ_n decreases monotonically to zero, then for all x, $K_{\delta_n} f(x)$ increases monotonically to $Kf(x)$. Consequently, $\|K_{\delta_n} f\|_{L^{p,q}(\mu)}$ tends to $\|Kf\|_{L^{p,q}(\mu)}$.

LEMMA 3. *Let $f \geq 0$ be K-admissible with compact support and let $\delta > 0$. Suppose $K(x, y)$ is nonincreasing in x and let $K_\delta f$ be defined as in* (3.1). *For each $k \in \mathbf{Z}$ let $E_k = \{x\colon K_\delta f(x) > 2^k\}$. Then:*

(a) *If E_k is a nonempty set then it is the disjoint union $E_k = \bigcup_i I_i^k$ of intervals with nonempty interiors.*

(b) *Either $I_i^k \cap I_j^{k-1} = \emptyset$ or $I_i^k \subseteq I_j^{k-1}$.*

(c) *If $a_i^k = \inf\{t\colon t \in I_i^k\}$ then $a_i^k > -\infty$ and $K_\delta f(a_i^k) = 2^k$.*

(d) *If $\Gamma(k, j) = \{i\colon I_i^k \subseteq I_j^{k-1}\}$ is a nonempty set then we can choose α_j^k such that $a_j^{k-1} < \alpha_j^k \leq a_i^k$ for all $i \in \Gamma(k, j)$, $\{(a_j^{k-1}, \alpha_j^k)\}_{k,j}$ is a disjoint collection of intervals and $K_\delta f(\alpha_j^k) = 2^k$.*

Proof. (a) The connected components $\{I_i^k\}$ of E_k are intervals with nonempty interiors. In fact, it follows from the second inequality in (3.2) that for each $x_0 \in E_k$ there exists an $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0]$ is contained in E_k.

(b) Since $E_k \subseteq E_{k-1}$ and I_j^{k-1} is a connected component of E_{k-1}, we deduce (b) easily.

(c) Since f is supported on a compact set, we have $a_i^k > -\infty$. Following the argument of the proof of (a) we see that a_i^k does not belong to E_k. On the other hand,

$$2^k \leq \lim_{x \to (a_i^k)^+} K_\delta f(x) \leq K_\delta f(a_i^k)$$

by (3.2). Thus (c) follows.

(d) Let $\alpha_j^k = \inf\{a_i^k\colon i \in \Gamma(k, j)\}$. Then using the first inequality in (3.2), we deduce that $K_\delta f(\alpha_j^k) \geq 2^k$. So $\alpha_j^k > a_j^{k-1}$ holds. Since $(a_j^{k-1}, \alpha_j^k) \subseteq E_{k-1} - E_k$, we have $K_\delta f(\alpha_j^k) \leq 2^k$ by the second inequality in (3.2). It just remains to prove that the collection $\{(a_j^{k-1}, \alpha_j^k)\}_{k,j}$ is disjoint. But $(a_j^{k-1}, \alpha_j^k) \subseteq I_j^{k-1} \cap (E_{k-1} - E_k)$, and the latter sets are clearly pairwise disjoint.

Proof of Theorem 1. It is quite clear that we can restrict ourselves to $f \geq 0$, $f \in L^{r,s}(v)$, f with compact support and $w \not\equiv 0$. Moreover, it is enough to prove the theorem for K-admissible functions. To see this, we take $f \geq 0$, $f \in L^{r,s}(v)$ and define $f_\varepsilon = f\chi_{(-\infty, x_1 - \varepsilon)}$, where $\varepsilon > 0$ and $x_1 = \inf\{x\colon \int_x^\infty w = 0\}$. Note that if $\int_x^\infty w > 0$ for some x, from (1.3) and (2.2) we have

$$Kf(x) = \int_{-\infty}^{x} K(x, y) f(y) v^{-1}(y) v(y)\, dy$$
$$\leq C \|K(x, y) v^{-1}(y) \chi_{(-\infty, x)}(y)\|_{L^{r',s'}(v)} \|f\|_{L^{r,s}(v)} < \infty.$$

Then, it turns out that f_ε is *K-admissible* for all $\varepsilon > 0$, since $Kf_\varepsilon(x) = Kf(x) < \infty$ when $x \leq x_1 - \varepsilon$ and $Kf_\varepsilon(x) \leq Kf(x_1 - \varepsilon) < \infty$ otherwise. Assuming the inequality (1.4) to be true for all f_ε, we may deduce the same inequality for f, since from (2.1) we have

$$\|Kf\|_{L^{p,q}(w)} \leq \|K(f\chi_{(-\infty, x_1)})\|_{L^{p,q}(w)} = \lim_{\varepsilon \to 0^+} \|K(f_\varepsilon)\|_{L^{p,q}(w)}.$$

Let $K_\delta f$ be defined by (3.1). We will prove that

$$\|K_\delta f\|_{L^{p,q}(w)} \leq C \|f\|_{L^{r,s}(v)}, \tag{1.4'}$$

where C does not depend on δ or f. Then, from (3.3) we obtain (1.4).

In order to show the validity of (1.4)′, we take $\delta > 0$ and a nonnegative K-admissible function f in $L^{r,s}(v)$ supported on a compact set. Suppose $1 \leq q < \infty$. Then (2.1) and Lemma 3 imply

$$\|K_\delta f\|_{L^{p,q}(w)}^q = q \int_0^\infty s^{q-1} \lambda_{K_\delta f}(s)^{q/p}\, ds \leq C \sum_k 2^{kq} \Big(\int_{\{K_\delta f > 2^k\}} w \Big)^{q/p}$$
$$= C \sum_k 2^{kq} \Big(\sum_j \sum_{i \in \Gamma(k,j)} \int_{I_i^k} w \Big)^{q/p} \leq I,$$

where

$$I = C\sum_k \Big(\sum_j 2^{kp} \int_{\alpha_j^k}^{\infty} w\Big)^{q/p}.$$

By (c) and (d) of Lemma 3 and by the fact that $K(x, y)$ is nonincreasing in x, we obtain

$$2^k = C[K_\delta f(\alpha_j^k) - K_\delta f(a_j^{k-1})] \leqslant C \int_{a_j^{k-1}-\delta}^{\alpha_j^k - \delta} K(\alpha_j^k, y) f(y)\,dy.$$

Thus

$$I \leqslant C\sum_k \Big[\sum_j \Big(\int_{a_j^{k-1}-\delta}^{\alpha_j^k-\delta} K(\alpha_j^k, y) f(y)\,dy\Big)^p \Big(\int_{\alpha_j^k}^{\infty} w\Big)\Big]^{q/p}.$$

Then (2.2) and (1.3) give

$$I \leqslant C\sum_k \Big[\sum_j \|f\chi_{(a_j^{k-1}-\delta,\alpha_j^k-\delta)}\|_{L^{r,s}(v)}^p \|K(\alpha_j^k, y)v^{-1}(y)\chi_{(-\infty,\alpha_j^k)}(y)\|_{L^{r',s'}(v)}^p \Big(\int_{\alpha_j^k}^{\infty} w\Big)\Big]^{q/p}$$
$$\leqslant CB^q \sum_k \Big(\sum_j \|f\chi_{(a_j^{k-1}-\delta,\alpha_j^k-\delta)}\|_{L^{r,s}(v)}^p\Big)^{q/p} \leqslant CB^q \|f\|_{L^{r,s}(v)}^q.$$

The last estimate follows from Lemma 1 and (d) of Lemma 3. The case $q = \infty$ with $p < \infty$ is established by a simple modification of the above argument. The case $q = p = \infty$ is deduced easily. Thus Theorem 1 is proved.

The following lemmata are needed in the proof of Theorem 2.

LEMMA 4. *Let $K(x, y)$ be nondecreasing in y. We put*

(3.4) $$K_\delta^* f(x) = \int_{x+\delta}^{\infty} K(y, x) f(y)\,dy, \quad \textit{for } \delta > 0.$$

If $f \geqslant 0$ is K^*-admissible then

(3.5) $$\lim_{x\to x_0^-} K_\delta^* f(x) \leqslant K_\delta^* f(x_0) \leqslant \lim_{x\to x_0^+} K_\delta^* f(x), \quad \textit{for all } x_0,$$

(3.6) $$\|K^* f\|_{L^{p,q}(\mu)} = \lim_{\delta\to 0^+} \|K_\delta^* f\|_{L^{p,q}(\mu)}, \quad \textit{for all } p, q > 0.$$

The proof of this result is almost identical to that of Lemma 2.

LEMMA 5. *Let $f \geqslant 0$ be K^*-admissible with compact support and let $\delta > 0$. Suppose $K(x, y)$ is nondecreasing in y and let $K_\delta^* f$ be defined by (3.4). For each $k \in \mathbf{Z}$ let $E_k^* = \{x: K_\delta^* f(x) > 2^k\}$. Then:*

(a) *If E_k^* is a nonempty set then it is the disjoint union $E_k^* = \bigcup_i J_i^k$ of intervals with nonempty interiors.*

(b) *Either $J_i^k \cap J_j^{k-1} = \emptyset$ or $J_i^k \subseteq J_j^{k-1}$.*

(c) *If $b_i^k = \sup\{t: t \in J_i^k\}$ then $b_i^k < \infty$ and $K_\delta^* f(b_i^k) = 2^k$.*

(d) *If $\Gamma^*(k, j) = \{i: J_i^k \subseteq J_j^{k-1}\}$ is a nonempty set then we can choose β_j^k so that $b_i^k \leqslant \beta_j^k < b_j^{k-1}$ for all $i \in \Gamma^*(k, j)$, $\{(\beta_j^k, b_j^{k-1})\}_{k,j}$ is a disjoint collection of intervals and $K_\delta^* f(\beta_j^k) = 2^k$.*

The proof of this result is similar to that of Lemma 3, only now we apply Lemma 4 instead of Lemma 2. The details are omitted.

Proof of Theorem 2. It is not difficult to see that under our hypotheses the inequality (1.6) is equivalent to the dual inequality

(1.6)′ $$\|v^{-1}K^*(gw)\|_{L^{r'}(v)} \leqslant C\|g\|_{L^{p',q'}(w)}, \quad \text{for all } g,$$

where K^* is the integral operator defined by (1.2).

Suppose that $1 < r < \infty$, $v \not\equiv \infty$ and take $g \geqslant 0$, $g \in L^{p',q'}(w)$ with compact support so that gw is K^*-admissible. It is sufficient to prove (1.6)′ for such g. This reduction is similar to that in the proof of Theorem 1. It is accomplished by showing that $g_\varepsilon w$ is K^*-admissible, where g is nonnegative, $g \in L^{p',q'}(w)$ and $g_\varepsilon = g\chi_{(x_1+\varepsilon,\infty)}$ with $\varepsilon > 0$ and $x_1 = \sup\{x: \int_{-\infty}^{x} v^{1-r'} = 0\}$.

Let $\delta > 0$ and define K_δ^* by (3.4). We will show that

(1.6)″ $$\|v^{-1}K_\delta^*(gw)\|_{L^{r'}(v)} \leqslant C\|g\|_{L^{p',q'}(w)},$$

where C is independent of δ and g. Then letting $\delta \to 0^+$, we get (1.6)′.

Applying Lemma 5 to the function $f = gw$, we get

$$\|v^{-1}K_\delta^*(gw)\|_{L^{r'}(v)}^{r'} \leqslant C\sum_k 2^{kr'}\Big(\int_{\{K_\delta^*(gw) > 2^k\}} v^{1-r'}\Big)$$
$$= C\sum_k 2^{kr'} \sum_j \sum_{i\in\Gamma^*(k,j)} \Big(\int_{J_i^k} v^{1-r'}\Big) \leqslant I,$$

where

$$I = C\sum_k \sum_j 2^{kr'}\Big(\int_{-\infty}^{\beta_j^k} v^{1-r'}\Big).$$

Using parts (c) and (d) of Lemma 5, and recalling that $K(x, y)$ is nondecreasing in y we have

$$2^k = C[K_\delta^*(gw)(\beta_j^k) - K_\delta^*(gw)(b_j^{k-1})] \leqslant C \int_{\beta_j^k+\delta}^{b_j^{k-1}+\delta} K(y, \beta_j^k) g(y) w(y)\,dy.$$

Thus

$$I \leqslant C\sum_k \sum_j \Big[\int_{\beta_j^k+\delta}^{b_j^{k-1}+\delta} K(y, \beta_j^k) g(y) w(y)\,dy\Big]^{r'} \Big[\int_{-\infty}^{\beta_j^k} v^{1-r'}\Big]$$
$$\leqslant C\sum_k \sum_j \|g\chi_{(\beta_j^k+\delta, b_j^{k-1}+\delta)}\|_{L^{p',q'}(w)}^{r'} \|K(y, \beta_j^k)\chi_{(\beta_j^k,\infty)}(y)\|_{L^{p,q}(w)}^{r'} \Big(\int_{-\infty}^{\beta_j^k} v^{1-r'}\Big)$$
$$\leqslant CB^{r'} \sum_k \sum_j \|g\chi_{(\beta_j^k+\delta, b_j^{k-1}+\delta)}\|_{L^{p',q'}(w)}^{r'} \leqslant CB^{r'} \|g\|_{L^{p',q'}(w)}^{r'}$$

by (2.2), (1.5), part (d) of Lemma 5 and Lemma 1.

Although the proof for $r = 1$ is similar to that found in Sawyer [8] for the modified Hardy operators, it is worthwhile to include it here for the sake of clarity. We have

$$\|v^{-1}K^*(gw)\|_{L^\infty(v)} \leqslant \sup_{a\in\mathbf{R}} \|v^{-1}\chi_{(-\infty,a)}\|_{L^\infty(v)} K^*(gw)(a)$$
$$= \sup_{a\in\mathbf{R}} \|v^{-1}\chi_{(-\infty,a)}\|_{L^\infty(v)} \int_a^\infty K(y, a)g(y)w(y)\,dy$$
$$\leqslant C\sup_{a\in\mathbf{R}} \|v^{-1}\chi_{(-\infty,a)}\|_{L^\infty(v)} \|K(y, a)\chi_{(a,\infty)}(y)\|_{L^{p,q}(w)} \|g\|_{L^{p',q'}(w)}$$
$$\leqslant CB\|g\|_{L^{p',q'}(w)},$$

where the last two inequalities follow from (2.2) and (1.5) respectively. This concludes the proof of Theorem 2.

The proofs of Theorems 3 and 4 mimic those of Theorems 1 and 2. We discuss this briefly.

Proof of Theorem 3. The proof of (1.8) when $q < \infty$ is similar to that of (1.6)′ in Theorem 2. And a simple modification of this proof deals with the case of $q = \infty$ and $p < \infty$. Finally, the inequality (1.8) in the case of $q = p = \infty$ is easy to establish.

Proof of Theorem 4. Under our hypothesis, the inequality (1.10) is equivalent to

(1.10)′ $$\|v^{-1}K(gw)\|_{L^{r'}(v)} \leqslant C\|g\|_{L^{p',q'}(w)}, \quad \text{for all } g.$$

This inequality in the case of $r > 1$ is proved similarly to Theorem 1. The case of $r = 1$ for (1.10)′ is established with the method for proving Theorem 2 in case $r = 1$.

4. A counterexample. In this section we prove that the converses of Theorems 1 and 2 do not hold. Moreover, our counterexample shows that the converse of Theorem 2.1 of Andersen–Heinig [1] does not hold either. We start with the following.

PROPOSITION. *Let K be the integral operator defined by* (1.1), *where* $K(x, y) = \varphi(x)\psi(y)$ *with* $\varphi \geqslant 0$ *and* $\psi \geqslant 0$. *Suppose* $1 \leqslant p, q \leqslant \infty$, $q = 1$ *if* $p = 1$, $1 \leqslant r < \infty$ *and* $r \leqslant \min\{p, q\}$. *Then*

(4.1) $$\|Kf\|_{L^{p,q}(w)} \leqslant C\|f\|_{L^r(v)}, \quad \textit{for all } f,$$

if and only if

(4.2) $$\sup_{a\in\mathbf{R}} \|\varphi\chi_{(a,\infty)}\|_{L^{p,q}(w)} \Big(\int_{-\infty}^a \psi^{r'}v^{1-r'}\Big)^{1/r'} = B < \infty,$$

where the second factor on the left side of (4.2) *is thought of as* $\|\psi v^{-1}\chi_{(-\infty,a)}\|_{L^\infty(v)}$ *if* $r = 1$, *and, in this case, the hypothesis* $v > 0$ *a.e. in* $(-\infty, a)$ *is added if* $\|\varphi\chi_{(a,\infty)}\|_{L^{p,q}(w)} > 0$.

Proof. (4.2)⇒(4.1). The case of $r > 1$ follows by applying Theorem 2 with $K(x, y) = \varphi(x)$ and weights w and $\psi^{-r}v$. The case of $r = 1$ follows the proof of Theorem 2 in the case of $r = 1$.

The proof of the converse is a simple extension of that given by Sawyer for the modified Hardy operators (see [8], Theorem 4).

Our counterexample is as follows. For each $1 < r \leqslant p = q < \infty$ we will choose weights w and v and positive functions φ and ψ such that φ is nonincreasing, ψ is nondecreasing, and (4.2) is satisfied while (4.3) is not, where

(4.3) $$\inf_{0\leqslant\beta\leqslant 1} \sup_{a\in\mathbf{R}} \varphi(a)^{1-\beta}\psi(a)^\beta \|\varphi^\beta\chi_{(a,\infty)}\|_{L^q(w)} \Big(\int_{-\infty}^a \psi^{(1-\beta)r'}v^{1-r'}\Big)^{1/r'} = B < \infty.$$

Condition (4.3) is the sufficient condition found by Andersen–Heinig for the case of $K(x, y) = \varphi(x)\psi(y)$. Let

$$w \equiv 1, \quad v(y) = \begin{cases} 1, & y < 1, \\ y^\eta, & y \geqslant 1, \end{cases}$$

where $\eta > 1/(r'-1)$. Let

$$\varphi(x) = \begin{cases} 1, & x < 1, \\ \Big(\sum_{n=1}^\infty 2^{-n}(\alpha_n - 1)/x^{\alpha_n}\Big)^{1/q}, & x \geqslant 1, \end{cases}$$

where $1 < \alpha_n < 2$ and $\alpha_n \downarrow 1$;

$$\psi(y) = \begin{cases} 1, & y \geqslant -1, \\ |y|^{-\gamma}, & y < -1, \end{cases}$$

where $\gamma > 1/q + 1/r'$.

It is routine to show that $\|\varphi\chi_{(a,\infty)}\|_{L^q(w)}$ is bounded by 1 if $a \geqslant 1$, and equal to $(2-a)^{1/q}$ if $a < 1$. The expression $(\int_{-\infty}^a \psi^{r'}v^{1-r'})^{1/r'}$ is bounded independently of a if $a \geqslant -1$, and proportional to $|a|^{1/r'-\gamma}$ if $a < -1$.

Then (4.2) follows since if $a \leqslant -1$ then

$$\|\varphi\chi_{(a,\infty)}\|_{L^q(w)} \Big(\int_{-\infty}^a \psi^{r'}v^{1-r'}\Big)^{1/r'} < C|a|^{1/q+1/r'-\gamma} < \infty,$$

and if $a > -1$ the expression on the LHS is also bounded independently of a.

Finally, it is easy to check that (4.3) does not hold since for all real a the norm $\|\varphi^\beta\chi_{(a,\infty)}\|_{L^q(w)} = \infty$ when $0 \leqslant \beta < 1$ and $\int_{-\infty}^a \psi^{(1-\beta)r'}v^{1-r'} = \int_{-\infty}^a v^{1-r'} = \infty$ when $\beta = 1$.

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An axiomatic definition of the entropy of a $\mathbf{Z}^d$-action on a Lebesgue space

by

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Abstract. We introduce the concept of a principal factor for a $\mathbf{Z}^d$-action and we use a characterization of these factors to obtain an axiomatic definition of the entropy of a $\mathbf{Z}^d$-action, $d \geqslant 2$.

Introduction. The notion of the entropy of a $\mathbf{Z}^d$-action on a Lebesgue probability space has been introduced by A. N. Kolmogorov in [9] for $d = 1$ and then generalized by several authors ([1], [7], [11], [12]) to arbitrary $d \geqslant 1$.

In this paper we give an axiomatic definition of the entropy of a $\mathbf{Z}^d$-action for every $d \geqslant 2$. Our result is an analogue of the result of V. A. Rokhlin ([14]).

To obtain our result we first prove the existence of relatively perfect partitions for a given $\mathbf{Z}^d$-action. Next we introduce a concept of a principal factor and, using the above result, we give a characterization of principal factors by means of entropy. This characterization and the generalized Sinai theorem ([8]) allow us to obtain, by the use of the Rokhlin idea ([14]), our axiomatic definition of entropy.

Our result is an example of a result of ergodic theory obtained by a relativization method also used by other authors (see [2], [3], [10], [16], [17]).

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§ 1. Preliminaries. Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space, let $\mathscr{M}$ be the set of all measurable partitions of X and let $\mathscr{Z}$ be the subset of $\mathscr{M}$ consisting of partitions with finite entropy.

We denote by ε the measurable partition of X into single points and by ν the trivial measurable partition whose only element is X.

Let $\prec$ denote the lexicographical ordering of the group $\mathbf{Z}^d$, $d \geqslant 2$. Let $e^i \in \mathbf{Z}^d$ be the ith standard unit vector. We put

$$\mathbf{Z}^d_n = \{g = (m_1, \ldots, m_d) \in \mathbf{Z}^d;\ m_1 = \ldots = m_n = 0\}, \quad 1 \leqslant n \leqslant d,$$

$$\mathbf{Z}^d_- = \{g \in \mathbf{Z}^d;\ g \prec 0\}.$$

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