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A weighted interpolation problem for analytic functions

by

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Abstract. Given a sequence $\{z_n\}$ of complex numbers ($|z_n| < 1$, $n = 1, 2, \ldots$) let $T = T_{\{z_n\}}$ denote the linear map sending an analytic function f to the sequence $\{f(z_n)\}$. Let $1 \leqslant p \leqslant \infty$ and let v^p be in the A_p class of B. Muckenhoupt (or $v^{-p'}$ in $A_{p'}$) and $w = \{w_n\}$ a sequence of nonnegative numbers. We show $T(H^p(v)) = l^p(w)$ if and only if $\{z_n\}$ satisfies L. Carleson's condition

$$\prod_{n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| \geqslant \delta \quad (k \geqslant 1)$$

and there exist positive constants C_1, C_2 such that $C_1 \leqslant w_n / \|\chi_{I_n}\|_{L^p(v)} \leqslant C_2$ ($n \geqslant 1$) where, for $z_n = r_n e^{i\theta_n}$, $I_n = (\theta_n - (1 - r_n), \theta_n + (1 - r_n)) \bmod 2\pi$.

The case $v \equiv 1$ and $w = \{1\}$ ($p = \infty$) is due to L. Carleson and, later, $v \equiv 1$ and $w = \{(1 - |z_n|)^{1/p}\}$ ($1 \leqslant p \leqslant \infty$) to H. S. Shapiro and A. L. Shields.

More generally, we characterize the containments $T(H^p(v)) \subset l^p(w)$, $T(H^p(v)) \supset l^p(w)$ in terms of the action of certain weighted Carleson measures.

1. Introduction. The following interpolation problem was posed by R. C. Buck: Does there exist a sequence of points $\{z_n\}_{n=1}^{\infty}$ in the unit disk $U = \{z \in \mathbf{C}: |z| < 1\}$ having the property that, given an arbitrary bounded sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$, there exists a bounded analytic function f on U such that

$$f(z_n) = a_n \quad (n = 1, 2, \ldots)\ ? \tag{1.1}$$

The existence of such a sequence, called an *interpolating sequence*, was established independently by L. Carleson [1], W. Hayman [5] and D. J. Newman [10]. In fact, Carleson showed that a sequence $\{z_n\}_{n=1}^{\infty}$ in U is an interpolating sequence if and only if there exists $\delta > 0$ such that

$$\prod_{n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| \geqslant \delta \quad (k = 1, 2, \ldots). \tag{1.2}$$

(For a discussion of independent and related results see H. S. Shapiro and A. L. Shields [13].)

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In what follows, given a sequence $\{z_n\}_{n=1}^{\infty}$ in U, we let $T = T_{\{z_n\}_{n=1}^{\infty}}$ denote the linear map sending an analytic function f on U to the sequence of its values on the z_n, namely $\{f(z_n)\}_{n=1}^{\infty}$. In this setting, Carleson's result says that $T(H^\infty) \supset l^\infty$ (equivalently $T(H^\infty) = l^\infty$) if and only if (1.2) holds.

To state the problem under consideration here, we employ the following notation. Let $1 \leqslant p \leqslant \infty$ and suppose v is a nonnegative measurable function on $\mathbf{T}$, the unit circle. Define $L^p(v) = \{f\colon \mathbf{T} \to \mathbf{C}\colon \|f\|_{L^p(v)} < \infty\}$ where

$$\text{(1.3)} \qquad \|f\|_{L^p(v)} = \begin{cases} \left((2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})v(e^{i\theta})|^p d\theta\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{0 \leqslant \theta < 2\pi} |f(e^{i\theta})v(e^{i\theta})|, & p = \infty. \end{cases}$$

We henceforth restrict our attention to those nonnegative functions v such that v^p $(1 \leqslant p < \infty)$ satisfies the A_p condition of B. Muckenhoupt (see [9], p. 214) or $v^{-p'}$ $(1 < p \leqslant \infty,\ 1/p + 1/p' = 1)$ satisfies $A_{p'}$. Observe that these assumptions coincide when $1 < p < \infty$ and may be summarized for any p, $1 \leqslant p \leqslant \infty$, by

$$\text{(1.4)} \qquad |I| \leqslant \|\chi_I\|_{L^p(v)} \|\chi_I\|_{L^{p'}(v^{-1})} \leqslant C|I|, \quad \text{for all arcs } I \subset \mathbf{T},$$

where χ_I denotes the characteristic function of the arc I, $|I|$ is the length of I, and C is a positive constant independent of I. Observe that, for any p $(1 \leqslant p \leqslant \infty)$, condition (1.4) implies the following doubling condition:

$$\text{(D)} \qquad \|\chi_{2I}\|_{L^p(v)} \leqslant D \|\chi_I\|_{L^p(v)}, \quad \text{for all arcs } I \subset \mathbf{T}.$$

(RI denotes the arc concentric with I, having R times the length of I. The constant D may be taken as double the constant appearing in condition (1.4).) This is a well-known property of A_p weights and will be used repeatedly in the sequel. With (1.4) in force, any $f \in L^p(v)$ is absolutely integrable on $\mathbf{T}$ and we may define, for $1 \leqslant p \leqslant \infty$,

$$H^p(v) = \{f \in L^p(v)\colon \hat{f}(n) = 0 \text{ for } n < 0\}$$

equipped with the norm $\|\cdot\|_{H^p(v)} = \|\cdot\|_{L^p(v)}$. ($\hat{f}(n)$ is the nth Fourier coefficient of f.)

Note that the Poisson extension $\mathbf{P}f$, of any $f \in H^p(v)$ $(1 \leqslant p \leqslant \infty)$, is analytic in U and is contained in the usual (unweighted) Hardy space $H^1(U)$. If there is no danger of ambiguity we shall denote the Poisson extension of $f \in H^p(v)$ simply by f.

Finally, for a sequence $w = \{w_n\}_{n=1}^{\infty}$ of nonnegative numbers, let $l^p(w)$ denote the space of sequences $a = \{a_n\}_{n=1}^{\infty}$ satisfying

$$\text{(1.5)} \qquad \|a\|_{l^p(w)} = \begin{cases} \left(\sum_{n=1}^{\infty} |a_n w_n|^p\right)^{1/p} < \infty, & 1 \leqslant p < \infty, \\ \sup_{n \geqslant 1} |a_n w_n| < \infty, & p = \infty. \end{cases}$$

In this note we consider the following weighted interpolation problem: Characterize the sequences $\{z_n\}_{n=1}^{\infty}$ in U such that $T(H^p(v)) = l^p(w)$.

A special case of this problem was solved by Shapiro and Shields ([13]): If $v \equiv 1$ and $w_n = (1 - |z_n|)^{1/p}$ $(n \geqslant 1)$ then $T(H^p) \supset l^p(w)$ (in fact, equivalently $T(H^p) = l^p(w)$) if and only if (1.2) holds.

While our results parallel those of Shapiro and Shields, a technical difficulty must be overcome in the case $p = 1$. The details follow the proof of Theorem 2.

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2. Statement of results

THEOREM 1. *Let $1 \leqslant p \leqslant \infty$ and let $\{z_n\}_{n=1}^{\infty}$ be a sequence in U and suppose* (1.4) *holds. Then $T(H^p(v)) = l^p(w)$ if and only if* (1.2) *holds and there exist positive constants C_1, C_2 such that*

$$\text{(2.1)} \qquad C_1 \leqslant w_n / \|\chi_{I_n}\|_{L^p(v)} \leqslant C_2 \quad (n = 1, 2, \ldots)$$

where, for $z_n = r_n e^{i\theta_n}$, we set $I_n = (\theta_n - (1 - r_n),\ \theta_n + (1 - r_n)) \bmod 2\pi$.

More generally, we have

THEOREM 2. *Let $1 \leqslant p \leqslant \infty$ and let $\{z_n\}_{n=1}^{\infty}$ be a sequence of distinct points in U and suppose* (1.4) *holds. Then*

(A) *$T(H^p(v)) \subset l^p(w)$ if and only if*

$$\text{(2.2)} \qquad \left(\sum \{w_n^p\colon z_n \in S(I)\}\right)^{1/p} \leqslant C \|\chi_I\|_{L^p(v)}, \quad \text{for all arcs } I \subset \mathbf{T},$$

while

(B) *$T(H^p(v)) \supset l^p(w)$ if and only if*

$$\text{(2.3)} \qquad \left[\sum \left\{\left[\frac{1 - |z_n|}{w_n \delta_n}\right]^{p'}\colon z_n \in S(I)\right\}\right]^{1/p'} \leqslant C \|\chi_I\|_{L^{p'}(v^{-1})},$$

for all arcs $I \subset \mathbf{T}$, where

$$\delta_n = \prod_{k \neq n} \left|\frac{z_n - z_k}{1 - \bar{z}_k z_n}\right| \quad (n = 1, 2, \ldots),$$

$$S(I) = \{re^{i\theta}\colon \theta \in I,\ 0 < 1 - r < \min\{1, |I|\}\}.$$

Here, and throughout, C shall denote a positive constant, not necessarily the same at each occurrence.

These results are established with the aid of the following weighted Carleson measure theorems.

THEOREM 3. *Let $1 < p < \infty$ and suppose* (1.4) *holds. If μ is a positive Borel measure on U then*

$$(2.4)\qquad [\int_U |\mathbf{P}f(z)|^p d\mu(z)]^{1/p} \leqslant C\|f\|_{L^p(v)}, \quad \text{for every } f\in L^p(v),$$

if and only if

$$(2.5)\qquad [\int_{S(I)} d\mu(z)]^{1/p} \leqslant C\|\chi_I\|_{L^p(v)}, \quad \text{for all arcs } I\subset \mathbf{T}.$$

The substitute result when $p = 1$ replaces $L^p(v)$ with $H^p(v)$. In this case we have

THEOREM 4. *Suppose μ is a positive Borel measure on U and v satisfies A_1. Then*

$$(2.6)\qquad \int_U |\mathbf{P}f(z)|d\mu(z) \leqslant C\|f\|_{H^1(v)}, \quad \text{for every } f\in H^1(v),$$

if and only if

$$(2.7)\qquad \int_{S(I)} d\mu(z) \leqslant C\|\chi_I\|_{L^1(v)}, \quad \text{for all arcs } I\subset \mathbf{T}.$$

3. Proof of Theorem 1. Assume, for the moment, Theorem 2.

Necessity. If $T(H^p(v)) = l^p(w)$ then, by Theorem 2, both (2.2) and (2.3) hold. Fix an arbitrary $n \geqslant 1$. Since $z_n\in S(I_n)$ and $|I_n| = 2(1-|z_n|)$ we have, by (2.2) and (2.3),

$$(3.1)\qquad w_n \leqslant C\|\chi_{I_n}\|_{L^p(v)}$$

and

$$(3.2)\qquad \frac{|I_n|}{w_n\delta_n} \leqslant C\|\chi_{I_n}\|_{L^{p'}(v^{-1})}$$

respectively. Thus

$$\frac{1}{\delta_n} \leqslant C\frac{w_n}{|I_n|}\|\chi_{I_n}\|_{L^{p'}(v^{-1})} \quad \text{by (3.2)}$$
$$\leqslant C \quad \text{by (3.1) and (1.4).}$$

Hence $\delta_n \geqslant 1/C > 0$ (for any $n \geqslant 1$), which yields (1.2).

Since $\delta_n \leqslant 1$ (for all $n \geqslant 1$), (3.2) also implies

$$1/w_n \leqslant C\|\chi_{I_n}\|_{L^{p'}(v^{-1})}/|I_n| \leqslant C\|\chi_{I_n}\|^{-1}_{L^p(v)} \quad \text{by (1.4).}$$

This inequality combined with (3.1) yields (2.1).

Sufficiency. Assume first that $1 < p < \infty$. From (1.2) it follows that

$$(3.3)\qquad \sum\{|I_n|:\ z_n\in S(I)\} \leqslant C|I|, \quad \text{for all arcs } I\subset\mathbf{T}.$$

(This implication is due to L. Carleson. A simplified and detailed discussion of the argument given in [1], Theorem 3, may be found in [8], pp. 266–272,

relative to the upper half-plane.) It follows that, for any arc I,

$$(3.4)\qquad \sum_{I_k\subset 4I} |I_k| \leqslant \sum\{|I_n|:\ z_n\in S(4I)\}$$
$$\leqslant C|4I| \quad \text{by (3.3)}$$
$$\leqslant C|I|.$$

Then, since every A_p weight also satisfies Muckenhoupt's A_∞ condition (see, for example, Lemma 3 of [2]), the implication (ii)$\Rightarrow$(iii) of Theorem 3 of [12] shows that (3.4) holds with the Lebesgue measure $|J|$ of an interval J replaced by the v^p-measure, $\int_J v(x)^p dx$. Noting that $I_n \subset 3I$ whenever $z_n\in S(I)$ we then have

$$(3.5)\qquad \sum\{\|\chi_{I_n}\|^p_{L^p(v)}:\ z_n\in S(I)\} \leqslant \sum_{I_n\subset 3I} \|\chi_{I_n}\|^p_{L^p(v)}$$
$$\leqslant C\|\chi_I\|^p_{L^p(v)}, \quad \text{for all arcs } I\subset\mathbf{T}.$$

Now (2.2) follows from (2.1) and (3.5) and so, by Theorem 2, $T(H^p(v)) \subset l^p(w)$. (This part of the proof is also valid for $p = 1$.)

For the reverse inclusion, since (1.2) implies that $\delta_n \geqslant \delta > 0$ (for all $n \geqslant 1$) we have, by (2.1) and (1.4),

$$(3.6)\qquad \frac{1-|z_n|}{w_n\delta_n} \leqslant C\|\chi_{I_n}\|_{L^{p'}(v^{-1})}, \quad \text{for every } n\geqslant 1.$$

Since v^p satisfying A_p is equivalent ($1 < p < \infty$) to $v^{-p'}$ satisfying $A_{p'}$ (and hence A_∞), (3.5) holds with $\|\cdot\|^{p'}_{L^{p'}(v^{-1})}$ replacing $\|\cdot\|^p_{L^p(v)}$. This observation, in conjunction with (3.6), yields (2.3) and so, by Theorem 2, $T(H^p(v)) \supset l^p(w)$. (This part of the proof is also valid for $p = \infty$, that is, $p' = 1$.)

It remains to show that $T(H^p(v)) \subset l^p(w)$, in the case $p = \infty$, and that $T(H^p(v)) \supset l^p(w)$, in the case $p = 1$. These inclusions are, by Theorem 2, equivalent to

$$(3.7)\qquad \sup\{w_n:\ z_n\in S(I)\} \leqslant C\|\chi_I\|_{L^\infty(v)}, \quad \text{for all arcs } I\subset\mathbf{T},$$

and

$$(3.8)\qquad \sup\left\{\frac{1-|z_n|}{w_n\delta_n}:\ z_n\in S(I)\right\} \leqslant C\|\chi_I\|_{L^\infty(v^{-1})}, \quad \text{for all arcs } I\subset\mathbf{T},$$

respectively.

Fix an arbitrary arc $I\subset\mathbf{T}$ and recall that $I_n\subset 3I$ for every n with $z_n\in S(I)$. Also fix any n with $z_n\in S(I)$.

Regarding (3.7) we have

$$w_n \leqslant C\|\chi_{I_n}\|_{L^\infty(v)} \quad \text{by (2.1)}$$
$$\leqslant C\|\chi_{3I}\|_{L^\infty(v)}$$
$$\leqslant C\|\chi_I\|_{L^\infty(v)} \quad \text{by (D)}$$

with a constant C independent of n. Thus (3.7) holds.

Regarding (3.8), since $\delta_n \geqslant \delta > 0$ we have

$$\begin{aligned} \frac{1-|z_n|}{w_n\delta_n} &\leqslant C\frac{|I_n|}{\|\chi_{I_n}\|_{L^1(v)}} \quad \text{by (2.1)} \\ &\leqslant C\|\chi_{I_n}\|_{L^\infty(v^{-1})} \quad \text{by (1.4)} \\ &\leqslant C\|\chi_{3I}\|_{L^\infty(v^{-1})} \\ &\leqslant C\|\chi_I\|_{L^\infty(v^{-1})} \quad \text{by (D)} \end{aligned}$$

with a constant C independent of n. This yields (3.8) and completes the proof of Theorem 1.

4. Proof of the weighted Carleson measure theorems. In this section we prove Theorems 3 and 4 which are used in Section 5 to prove Theorem 2.

Proof of Theorem 3. *Necessity.* For an arc $I \subset \mathbf{T}$, an elementary computation yields a constant C, independent of I, for which $\mathbf{P}\chi_I \geqslant C$ on $S(I)$. Setting $f = \chi_I$, (2.5) follows from (2.4).

Sufficiency. Using a simple estimate on the Poisson kernel, Hölder's inequality, and conditions (2.5) and (1.4) the inequality

$$\int_{|z|\leqslant 1/2} |\mathbf{P}f(z)|^p d\mu(z) \leqslant C\|f\|^p_{L^p(v)}, \qquad f \in L^p(v),$$

is readily obtained. Now suppose $|z| > 1/2$. The following argument is, in the case $v \equiv 1$, due to E. M. Stein ([14], p. 236).

Let $\Gamma(e^{it})$ denote the interior of the square whose diagonal has endpoints $z = 0$ and $z = e^{it}$, and consider the nontangential maximal function

$$Nf(e^{it}) = \sup_{z\in\Gamma(e^{it})} |\mathbf{P}f(z)|, \qquad f \in L^p(v).$$

To obtain an inequality between distribution functions, momentarily fix $\lambda > 0$. Since Nf is lower-semicontinuous, the set $\{t\colon Nf(e^{it}) > \lambda\}$ may be decomposed into a disjoint union $\bigcup I_j$ of component arcs in $\mathbf{T}$. Suppose there exists z with $|z| > \frac{1}{2}$ and $|\mathbf{P}f(z)| > \lambda$. If I_z denotes the arc centered at $z/|z|$ of length $2(1-|z|)$ then we have $z \in \Gamma(e^{it})$ for every $t \in I_z$ and so $Nf > \lambda$ on I_z. Thus there exists a component arc I_j containing I_z and also $z \in S(I_z) \subset S(I_j)$. Hence

$$\{z \in U\colon |z| > \tfrac{1}{2},\ |\mathbf{P}f(z| > \lambda\} \subset \bigcup_j S(I_j)$$

and this, in conjunction with (2.5), yields

$$\int_{\{z\in U: |z|>1/2, |\mathbf{P}f(z)|>\lambda\}} d\mu(z) \leqslant C \int_{\{t: Nf(e^{it})>\lambda\}} v(e^{it})^p\,dt.$$

This inequality between distribution functions leads to

$$\int_{\{z\in U: |z|>1/2\}} |\mathbf{P}f(z)|^p d\mu(z) \leqslant C\int_0^{2\pi} |Nf(e^{it})v(e^{it})|^p\,dt$$

and the latter expression is bounded by

$$C\int_0^{2\pi} |Mf(e^{it})v(e^{it})|^p\,dt$$

since Nf is dominated by the Hardy–Littlewood maximal function Mf (see, for example, J. García-Cuerva and J. L. Rubio de Francia [4], p. 109). Since M is bounded on $L^p(v)$ for $1 < p < \infty$ when v^p satisfies A_p (see [9]), the result follows.

Proof of Theorem 4. *Sufficiency.* As noted in the introduction, $f \in H^1(v)$ (v satisfying A_1) implies $\mathbf{P}f \equiv F \in H^1(U)$ where $H^1(U)$ is the usual unweighted Hardy space on the disk U. Thus we have (see, for example, Y. Katznelson [7], p. 85) the canonical factorization $F = BG$, where B is the Blaschke product corresponding to the zeros of F and $G \in H^1(U)$ never vanishes. Then $\mathbf{P}f = F = (BG^{1/2})G^{1/2} \equiv \mathbf{P}g_1\mathbf{P}g_2$ (where g_1 and g_2 are the almost everywhere existing radial limits of $BG^{1/2}$ and $G^{1/2}$ respectively). It is easily checked that $g_i \in H^2(v^{1/2})$ and $\|g_i\|_{H^2(v^{1/2})} = \|f\|^{1/2}_{H^1(v)}$ $(i = 1, 2)$ (cf. [4], p. 108). Thus

$$\begin{aligned} \int_U |\mathbf{P}f(z)|\,d\mu(z) &= \int_U |\mathbf{P}g_1(z)\mathbf{P}g_2(z)|\,d\mu(z) \\ &\leqslant \Big[\int_U |\mathbf{P}g_1(z)|^2\,d\mu(z)\Big]^{1/2} \Big[\int_U |\mathbf{P}g_2(z)|^2\,d\mu(z)\Big]^{1/2} \end{aligned}$$

by Hölder's inequality.

Applying Theorem 3 with $p = 2$ and $v^{1/2}$ replacing v (and noting that since v satisfies A_1 it also satisfies A_2) this expression is bounded by the product $C\|g_1\|_{H^2(v^{1/2})}\|g_2\|_{H^2(v^{1/2})}$ and this product equals $C\|f\|_{H^1(v)}$ as desired.

Necessity. Let $I \subset \mathbf{T}$ and choose $a = |a|e^{i\varphi} \in U$ with $|a| \geqslant 1/2$ and such that $I = J_a = (\varphi - 2\pi(1-|a|),\ \varphi + 2\pi(1-|a|))$. The following lemma provides a function $f \in H^1(v)$ and a fixed integer $n \geqslant 1$ such that $\|f\|_{H^1(v)} \leqslant C_n\|\chi_{J_a}\|_{L^1(v)}/|J_a|^n$ while $|f(z)| > C_n|J_a|^{-n}$ on $S(J_a)$. Thus, by (2.6)

$$\begin{aligned} C_n|J_a|^{-n}\int_{S(J_a)} d\mu(z) &< \int_{S(J_a)} |\mathbf{P}f(z)|\,d\mu(z) \leqslant C\|f\|_{H^1(v)} \\ &\leqslant C_n\|\chi_{J_a}\|_{L^1(v)}/|J_a|^n, \end{aligned}$$

which yields (2.7).

Lemma 1. *Fix* $a = |a|e^{i\varphi}$ *with* $\frac{1}{2} \leqslant |a| < 1$ *and let* $J_a = (\varphi - 2\pi(1-|a|),\ \varphi + 2\pi(1-|a|))$. *Then the function* $g(z) = (1-\bar{a}z)^{-1}$ *satisfies*:

(a) $|g(z)| > C|J_a|^{-1}$ *on* $S(J_a)$.

(b) *If* v^{-1} *satisfies* A_1 *then* $\|g\|_{H^\infty(v)} \leqslant C/\|\chi_{J_a}\|_{L^1(v^{-1})}$.

(c) *If* v *satisfies* A_1 *and* $n \geqslant 1$ *is sufficiently large then* $f = g^n$ *satisfies*

$$\|f\|_{H^1(v)} \leqslant C_n\|\chi_{J_a}\|_{L^1(v)}/|J_a|^n.$$

Proof. A simple estimate yields (a) and (b) follows from the inequality

$$|g(e^{i\theta})| \leqslant \frac{C}{(|J_a|^2+|\theta-\varphi|^2)^{1/2}}, \qquad |\theta-\varphi| < \pi.$$

To prove (c), choose n such that $2^n > D$ (the constant appearing in condition (D)) and use the previous estimate for g.

5. Proof of Theorem 2. *Part* (A). Assume first that $1 < p < \infty$. Then $T(H^p(v)) \subset l^p(w)$ is equivalent to

$$(5.1) \qquad \Big(\sum_{n=1}^{\infty} |f(z_n)w_n|^p\Big)^{1/p} \leqslant C\|f\|_{H^p(v)}, \quad \text{for every } f\in H^p(v).$$

Since v^p satisfies A_p we know (see Hunt, Muckenhoupt and Wheeden [6], Theorem 1) that the conjugation operator is bounded on $L^p(v)$ and so (5.1) holds for every $f\in L^p(v)$ (with f interpreted as $\mathbf{P}f$), whenever it holds for every $f\in H^p(v)$, and is equivalent to

$$(5.2) \qquad \Big(\sum_{n=1}^{\infty} |\mathbf{P}f(z_n)w_n|^p\Big)^{1/p} = \Big(\int_U |\mathbf{P}f(z)|^p d\Big[\sum_{n=1}^{\infty} w_n^p\delta_{z_n}(z)\Big]\Big)^{1/p}$$
$$\leqslant C\|f\|_{L^p(v)}, \quad \text{for every } f\in L^p(v).$$

With $d\mu(z) = d[\sum_{n=1}^{\infty} w_n^p\delta_{z_n}(z)]$ Theorem 3 shows that (5.2) and (2.2) are equivalent.

With $d\mu(z)$ as above and $p=1$, Theorem 4 shows that (5.1) and (2.2) are equivalent.

When $p=\infty$ we wish to show that $T(H^\infty(v)) \subset l^\infty(w)$ is equivalent to

$$(5.3) \qquad \sup\{w_n\colon z_n\in S(I)\} \leqslant C\|\chi_I\|_{L^\infty(v)}, \quad \text{for all arcs } I\subset \mathbf{T}.$$

Necessity. Proceeding as in the proof of Theorem 4, let $I\subset\mathbf{T}$ and choose $a\in U$ $(1/2\leqslant|a|<1)$ such that $I=J_a$. By Lemma 1 there exists $f\in H^\infty(v)$ with $\|f\|_{H^\infty(v)} \leqslant C\|\chi_{J_a}\|^{-1}_{L^1(v^{-1})}$ and such that $|f(z)| > C|J_a|^{-1}$ on $S(J_a)$. Then, for any $z_n\in S(J_a)$, we have

$$w_n < Cw_n|f(z_n)||J_a| \leqslant C\|f\|_{H^\infty(v)}|J_a| \leqslant C\|\chi_{J_a}\|^{-1}_{L^1(v^{-1})}|J_a|$$
$$\leqslant C\|\chi_{J_a}\|_{L^\infty(v)} \quad \text{by (1.4)}$$

with a constant C independent of n.

Sufficiency. We show $|f(z_n)|w_n \leqslant C\|f\|_{H^\infty(v)}$, for any $n\geqslant 1$. Fix any $n\geqslant 1$ and let $z_n = r_n e^{i\theta_n}$. If $r_n\leqslant\frac{1}{2}$ the result follows from a simple estimate on the Poisson kernel, Hölder's inequality and conditions (1.4) and (5.3). If $r_n > \frac{1}{2}$ we use the estimate

$$P_{r_n}(\theta_n - t) \leqslant \frac{C(1-r_n^2)}{(1-r_n)^2+(\theta_n-t)^2} \qquad (t\in[0,2\pi])$$

to obtain, for $k\geqslant 1$,

$$\Big|\int_{2^kI_n\setminus 2^{k-1}I_n} P_{r_n}(\theta_n-t)f(e^{it})\,dt\Big| \leqslant \frac{C}{2^{2k}|I_n|}\int_{2^kI_n}|f(e^{it})|\,dt$$
$$\leqslant C\|f\|_{L^\infty(v)}\frac{\|\chi_{2^kI_n}\|_{L^1(v^{-1})}}{|2^kI_n|}2^{-k}$$
$$\leqslant C\|f\|_{L^\infty(v)}\|\chi_{2^kI_n}\|^{-1}_{L^\infty(v)}2^{-k} \quad \text{by (1.4)}$$
$$\leqslant C\|f\|_{L^\infty(v)}\|\chi_{I_n}\|^{-1}_{L^\infty(v)}2^{-k}.$$

Thus

$$|f(z_n)|w_n \leqslant C\|f\|_{H^\infty(v)}\|\chi_{I_n}\|^{-1}_{L^\infty(v)}w_n\Big[1+\sum_{k=1}^{\infty}2^{-k}\Big]$$
$$\leqslant C\|f\|_{H^\infty(v)} \quad \text{by (5.3) since } z_n\in S(I_n).$$

This completes the proof of part (A).

Part (B). The following argument is, in the case $v\equiv 1$, due to Shapiro and Shields ([13], p. 517). Let $1\leqslant p\leqslant\infty$ and suppose v^p $(1\leqslant p<\infty)$ satisfies A_p or $v^{-p'}$ $(1<p\leqslant\infty)$ satisfies $A_{p'}$. $T(H^p(v)) \supset l^p(w)$ is equivalent to

$$(5.4) \qquad \sup_{n\geqslant 1} m_n(a) \leqslant C\|a\|_{l^p(w)}, \quad \text{for every } a\in l^p(w),$$

where $m_n(a) = \inf\{\|f\|_{H^p(v)}\colon f(z_k)=a_k,\ 1\leqslant k\leqslant n\}$.

Following Shapiro and Shields, set

$$B_n(z) = \prod_{j=1}^{n}\frac{z-z_j}{1-\bar z_j z}, \qquad B_{nk}(z) = B_n(z)\frac{1-\bar z_k z}{z-z_k}$$

and $b_{nk} = B_{nk}(z_k)$ $(k\leqslant n)$. Let $a=\{a_k\}_{k=1}^{\infty}\in l^p(w)$ and define

$$\Phi_n(z) = \sum_{k=1}^{n}\frac{a_k}{b_{nk}}\Big(\frac{1-\bar z_k z}{z-z_k}\Big).$$

Then $f_n(z) = B_n(z)\Phi_n(z)$ satisfies $f_n(z_k)=a_k$, $1\leqslant k\leqslant n$, and so

$$m_n(a) = \inf_{g\in H^p(v)}\|f_n - B_n g\|_{H^p(v)} = \inf_{g\in H^p(v)}\|\Phi_n - g\|_{L^p(v)},$$

since $|B_n|\equiv 1$ on $\mathbf{T}$. Since $H^p(v) = H^{p'}(v^{-1})^\perp$ $(1<p\leqslant\infty)$ a duality relation (see (2), p. 516, of [13]) gives, for $1<p\leqslant\infty$,

$$(5.5) \qquad m_n(a) = \sup\Big\{\Big|\int_{\mathbf{T}}\Phi_n(z)f(z)\frac{dz}{2\pi i}\Big|\colon f\in H^{p'}(v^{-1}),\ \|f\|_{H^{p'}(v^{-1})}=1\Big\}$$
$$= \sup\Big\{\Big|\sum_{k=1}^{n}\frac{a_k}{b_{nk}}(1-|z_k|^2)f(z_k)\Big|\colon \|f\|_{H^{p'}(v^{-1})}=1\Big\}$$

by Cauchy's integral formula. That (5.5) is also valid in the case $p = 1$ is the content of Corollary 2 of Lemma 2. Thus, since $|b_{nk}| \geqslant \delta_k$, we have, for $1 \leqslant p \leqslant \infty$,

$$\sup_{n\geqslant 1} m_n(a) \leqslant C \sup\Big\{\sum_{k=1}^{\infty} |a_k w_k| \frac{(1-|z_k|)|f(z_k)|}{\delta_k w_k}:\ \|f\|_{H^{p'}(v^{-1})} = 1\Big\}. \tag{5.6}$$

Set $W_k = (1-|z_k|)/(\delta_k w_k)$ and $V = v^{-1}$ (and compare with Lemma 4 of [12]). By Hölder's inequality and (5.6) we obtain

$$\sup_{n\geqslant 1} m_n(a) \leqslant C\|a\|_{l^p(w)} \sup\{(\sum_{k=1}^{\infty} |f(z_k)W_k|^{p'})^{1/p'}:\ \|f\|_{H^{p'}(V)} = 1\}$$
$$\leqslant C\|a\|_{l^p(w)}$$

if and only if $\sup\{(\sum_{k=1}^{\infty} |f(z_k)W_k|^{p'})^{1/p'}:\ \|f\|_{H^{p'}(V)} = 1\} \leqslant C$.

This latter inequality holds if and only if $T(H^{p'}(V)) \subset l^{p'}(W)$ and, by Theorem 2 (A), this is equivalent to (2.3) as desired.

Regarding inequality (5.5) in the case $p = 1$: Denote by $C(\mathbf{T})$ the collection of all continuous fnctions $f: \mathbf{T} \to \mathbf{C}$ and by $\mathcal{M}$ its conjugate space. Assume for the moment that both v, v^{-1} are continuous (in addition to v satisfying A_1) and define

$$C(v^{-1}) = \{f \in C(\mathbf{T}):\ \|f\|_{L^\infty(v^{-1})} < \infty\}, \qquad \mathcal{M}(v) = \{\mu \in \mathcal{M}:\ \|\mu\| < \infty\},$$

where $\|\mu\|$ denotes the total variation of the measure $v\,d\mu$. Let $P(v^{-1})$ denote the subspace of $C(v^{-1})$ generated by $1, e^{i\theta}, e^{2i\theta}, \ldots$ Upon incorporating the above definitions, the duality argument in [3] (pp. 130–131) establishes, for any $k \in L^1(v)$, the equality

$$\inf_{g\in H^1(v)} \|k-g\|_{L^1(v)} = \sup\Big\{\Big|\int_{\mathbf{T}} k(z) f(z) \frac{dz}{2\pi i}\Big|:\ f \in H^\infty(v^{-1}),\ \|f\|_{H^\infty(v^{-1})} = 1\Big\} \tag{5.7}$$

with the infimum on the left being attained.

The few remaining details are devoted to establishing (5.7) when v, v^{-1} are not assumed continuous. In what follows, let $\varphi_n = \frac{1}{2}n\chi_{(-1/n,1/n)}$ and set $v_n = \varphi_n * v$ $(n = 1, 2, \ldots)$. Then each v_n is continuous and satisfies A_1.

Lemma 2. *For any* $k \in L^1(v)$, $\lim_{n\to\infty} \|\varphi_n * k - k\|_{L^1(v)} = 0$.

Proof. Let $\varepsilon > 0$. Since v satisfies A_1 we have $\varphi_n * v \leqslant Cv$ a.e. $(n = 1, 2, \ldots)$ for a fixed constant $C > 1$. Since $C(\mathbf{T})$ is dense in $L^1(v)$ (see, for example, W. Rudin [11], p. 68] we may choose $g \in C(\mathbf{T})$ such that $\|k-g\|_{L^1(v)} < \varepsilon/C$. That $\lim_{n\to\infty} \|\varphi_n * g - g\|_{L^1(v)} = 0$ is immediate, and so, for sufficiently large n,

$$\|\varphi_n * k - k\|_{L^1(v)} \leqslant \|\varphi_n * k - \varphi_n * g\|_{L^1(v)} + \|\varphi_n * g - g\|_{L^1(v)} + \|g-k\|_{L^1(v)}$$
$$\leqslant \frac{1}{2\pi}\int (\varphi_n * |k-g|) v\, d\theta + \varepsilon + \varepsilon$$
$$= \frac{1}{2\pi}\int |k-g| (\varphi_n * v)\, d\theta + 2\varepsilon < 3\varepsilon.$$

Corollary 1. *For any* $k \in L^1(v)$,

$$\inf_{g\in H^1(v)} \|k-g\|_{L^1(v)} \leqslant \liminf_{n\to\infty} \inf_{g\in H^1(v_n)} \|k-g\|_{L^1(v_n)}.$$

Proof. Since each v_n is continuous (and satisfies A_1), there exists a function g_n for which $\inf_{g\in H^1(v_n)} \|k-g\|_{L^1(v_n)}$ is attained $(n = 1, 2, \ldots)$. Set $\tilde{g}_n = \varphi_n * g_n$. Then $\tilde{g}_n \in H^1(v)$ and

$$\|k-\tilde{g}_n\|_{L^1(v)} \leqslant \|k-\varphi_n * k\|_{L^1(v)} + \|\varphi_n * k - \varphi_n * g_n\|_{L^1(v)}$$
$$\leqslant \|k-\varphi_n * k\|_{L^1(v)} + \frac{1}{2\pi}\int (\varphi_n * |k-g_n|) v\, d\theta$$
$$= \|k-\varphi_n * k\|_{L^1(v)} + \frac{1}{2\pi}\int |k-g_n| v_n\, d\theta.$$

Thus

$$\inf_{g\in H^1(v)} \|k-g\|_{L^1(v)} \leqslant \liminf_{n\to\infty} \|k-\tilde{g}_n\|_{L^1(v)}$$
$$\leqslant \liminf_{n\to\infty} \|k-\varphi_n * k\|_{L^1(v)} + \liminf_{n\to\infty} \frac{1}{2\pi}\int |k-g_n| v_n\, d\theta$$
$$= \liminf_{n\to\infty} \frac{1}{2\pi}\int |k-g_n| v_n\, d\theta \qquad \text{by Lemma 2.}$$

Corollary 2. *For any* $k \in L^1(v)$,

$$\inf_{g\in H^1(v)} \|k-g\|_{L^1(v)} = \sup\Big\{\Big|\int_{\mathbf{T}} k(z) f(z) \frac{dz}{2\pi i}\Big|:\ f \in H^\infty(v^{-1}),\ \|f\|_{H^\infty(v^{-1})} = 1\Big\}.$$

Proof. It is clear that, for every $f \in H^\infty(v^{-1})$ and for every $g \in H^1(v)$, we have

$$\int_{\mathbf{T}} k(z) f(z)\, dz = \int_{\mathbf{T}} (k(z)-g(z)) f(z)\, dz.$$

If in addition $\|f\|_{H^\infty(v^{-1})} = 1$, then

$$\Big|\int_{\mathbf{T}} k(z) f(z) \frac{dz}{2\pi i}\Big| \leqslant \|k-g\|_{L^1(v)}$$

follows from Hölder's inequality.

On the other hand, from Corollary 1 and (5.7) we have

$$\inf_{g\in H^1(v)} \|k-g\|_{L^1(v)} \leqslant \liminf_{n\to\infty} \sup\left\{\left|\int_{\mathbf{T}} k(z) f(z)\frac{dz}{2\pi i}\right| : f\in H^\infty(v_n^{-1}),\ \|f\|_{H^\infty(v_n^{-1})}=1\right\}$$

$$\leqslant \sup\left\{\left|\int_{\mathbf{T}} k(z) f(z)\frac{dz}{2\pi i}\right| : f\in H^\infty(v^{-1}),\ \|f\|_{H^\infty(v^{-1})}=1\right\}$$

since $v_n \leqslant Cv$ a.e. implies $H^\infty(v_n^{-1}) \subset H^\infty(v^{-1})$ $(n=1,2,\ldots)$.

Now (5.5) in the case $p=1$ follows from Corollary 2 with Φ_n replacing k.

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A multiplier characterization of analytic UMD spaces

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Abstract. We prove that the Banach spaces X for which analytic martingales converge unconditionally are precisely those for which certain multipliers are bounded on the Hardy space $H_X^1(T)$.

1. Introduction. The purpose of this paper is to characterize the complex Banach spaces X for which analytic martingales converge unconditionally in terms of boundedness of certain translation-invariant operators on the vector-valued Hardy spaces $H_X^1(T)$.

Bourgain [2] and Burkholder [3] have shown that the so-called UMD Banach spaces X, defined to be those in which Walsh–Paley martingales converge unconditionally, are precisely those for which the conjugate function operator is bounded from $L_X^2(T)$ to itself. Their methods are based on transference and we use a refinement of such arguments here.

We remark that the class of Banach spaces for which analytic martingales converge unconditionally is strictly larger than the class UMD and includes such spaces as $L^1(T)$, which do not even enjoy the Radon–Nikodym property [6].

The rest of this paper is arranged as follows. In the second section we introduce some basic definitions and provide a formal statement of the result given in the abstract. We also sketch the proof of the easy half of the theorem.

In the next section we reformulate the problem in probabilistic terms, following where possible an argument of McConnell [8]. In the penultimate section we establish the multiplier theorem. Our argument uses a result of Edgar [5] which allows us to approximate certain Brownian martingales by discrete-parameter analytic martingales. In the final section of this paper we mention some other properties of analytic UMD spaces.

Garling has introduced a more general class of martingales, termed Hardy martingales, which may be used to prove renorming theorems [6]. It is known that the Banach spaces for which analytic martingales converge unconditionally are those for which Hardy martingales converge unconditionally. Indeed, this follows from the techniques of this paper.

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