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Approximation properties of the partial sums of Fourier series of some almost periodic functions

by

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Abstract. Two basic classes Ω_α and Λ_β of functions almost periodic in the Stepanov sense are considered. For functions of these classes, four approximation theorems concerning the pointwise and uniform convergence of their Fourier series are proved. Also, in the case of uniformly almost periodic functions, an estimate of the order of strong summability of these series is obtained.

1. Preliminaries. Let S be the class of all complex-valued functions almost periodic in the sense of Stepanov ([9], Chap. V). Suppose that the Fourier series of a function $f \in S$ is of the form

$$f(x) \sim \sum_{k=-\infty}^{\infty} A_k e^{i\lambda_k x} \quad \text{with} \quad A_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t) e^{-i\lambda_k t}\, dt \tag{1}$$

and that $0 = \lambda_0 < \lambda_k < \lambda_{k+1}$ if $k \in \mathbf{N} = \{1, 2, \ldots\}$, $\lim_{k\to\infty} \lambda_k = \infty$, $\lambda_{-k} = -\lambda_k$, $|A_k|+|A_{-k}| > 0$. Let $\alpha > 0$ and $\beta > 1$ be two fixed numbers. Denote by Ω_α the set of functions of class S, bounded on $\mathbf{R} = (-\infty, \infty)$, whose Fourier exponents satisfy the condition

$$\lambda_{k+1} - \lambda_k \geqslant \alpha \qquad (k \in \mathbf{N}),$$

and by Λ_β the set of those $f \in S$ for which

$$\lambda_{k+1} \geqslant \beta\lambda_k \qquad (k \in \mathbf{N}).$$

Given any function $f \in S$, consider the following partial sums of series (1):

$$S_{\lambda_n}[f](x) = \sideset{}{'}\sum_{|\lambda_k| \leqslant \lambda_n} A_k e^{i\lambda_k x} \qquad (n \in \mathbf{N}). \tag{2}$$

Introduce the auxiliary function

$$\Psi_{\lambda,\eta}(t) = \frac{2}{\pi(\eta-\lambda)} t^{-2} \sin\tfrac{1}{2}(\eta-\lambda)t \sin\tfrac{1}{2}(\eta+\lambda)t \qquad (0 < \lambda < \eta,\ |t| > 0). \tag{3}$$

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Then the sums (2) can be written in the integral form

$$S_{\lambda_n}[f](x) = \int_0^{\infty} \{f(x+t)+f(x-t)\} \Psi_{\lambda_n,\eta_n}(t)\,dt, \tag{4}$$

where $\eta_n = \lambda_{n+1}$ if $f \in A_\beta$ and $\eta_n = \lambda_n+\alpha$ if $f \in \Omega_\alpha$ (see [9], p. 83 and [5], Lemma 2).

In this paper we derive some estimates for the rate of pointwise convergence of sums (2) for functions $f \in \Omega_\alpha$ (or $f \in A_\beta$) having some special properties in the neighbourhood of a given point x. We also present the corresponding estimates for the rate of uniform convergence of $S_{\lambda_n}[f]$ for some uniformly almost periodic functions f. These theorems can be treated as generalizations of main results of [6] and [10] concerning purely periodic functions. Moreover, a contribution to the strong summability of series (1) is given (cf. [8]).

The symbols $c_j(p)$ $(j = 1, \ldots)$ occurring in Sections 3–5 will mean some positive constants depending on the indicated parameter p only.

2. Auxiliary results. Suppose that g is a (complex-valued) function bounded on a finite interval $I = [a, b]$.

Given any positive integer n, let us introduce the *modulus of variation* of g on I:

$$v_n(g; I) = v_n(g; a, b) = \sup_{\Pi_n} \sum_{j=0}^{n-1} |g(x_{2j+1})-g(x_{2j})|,$$

where the supremum is extended over all systems Π_n of n nonoverlapping open intervals (x_{2j}, x_{2j+1}) contained in (a, b). Write $v_0(g; I) = 0$. Some basic properties of this modulus can be found in the papers of Chanturiya [6, 7].

For measurable g and $\Psi_{\lambda,\eta}$ defined by (3), we give useful estimates for the Lebesgue integrals of the product $G_{\lambda,\eta} = g\Psi_{\lambda,\eta}$.

LEMMA 1. *Suppose that $0 < a < b < \infty$, $0 < \lambda < \eta < \infty$ and that m denotes the integral part of λ (i.e., $m = [\lambda]$). Then*

$$\Big|\int_a^b G_{\lambda,\eta}(t)\,dt\Big| \leqslant \frac{4}{\pi\lambda(\eta-\lambda)} a^{-2}\Big\{|g(a)|+\Big(1+\frac{b-a}{2}\Big)v_{m+1}(g; a, b)\Big\}.$$

Proof. Introduce the points

$$t_j = a+j(b-a)/(m+1) \qquad (j = 0, 1, \ldots, m+1). \tag{5}$$

Write

$$\int_a^b G_{\lambda,\eta}(t)\,dt = \sum_{j=0}^{m} g(t_j) \int_{t_j}^{t_{j+1}} \Psi_{\lambda,\eta}(t)\,dt + \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} \{g(t)-g(t_j)\}\Psi_{\lambda,\eta}(t)\,dt$$
$$= P_1+P_2, \quad \text{say.}$$

By the Abel transformation,

$$P_1 = g(a)\int_a^b \Psi_{\lambda,\eta}(t)\,dt + \sum_{j=0}^{m-1} \{g(t_{j+1})-g(t_j)\} \int_{t_{j+1}}^{b} \Psi_{\lambda,\eta}(t)\,dt.$$

But, if $0 < u \leqslant b$, the second mean value theorem yields

$$\Big|\int_u^b \Psi_{\lambda,\eta}(t)\,dt\Big| \leqslant \frac{4}{\pi\lambda(\eta-\lambda)} u^{-2}. \tag{6}$$

Consequently,

$$|P_1| \leqslant \frac{4}{\pi\lambda(\eta-\lambda)} a^{-2}\{|g(a)|+v_m(g; a, b)\}.$$

In view of the obvious inequality

$$|\Psi_{\lambda,\eta}(t)| \leqslant 2t^{-2}/(\pi(\eta-\lambda)) \qquad (t > 0), \tag{7}$$

we have

$$|P_2| = \Big|\int_0^{(b-a)/(m+1)} \sum_{j=0}^{m} \{g(t+t_j)-g(t_j)\}\Psi_{\lambda,\eta}(t+t_j)\,dt\Big|$$
$$\leqslant \frac{2(b-a)}{\pi(m+1)(\eta-\lambda)} a^{-2} v_{m+1}(g; a, b),$$

and this completes the proof.

LEMMA 2. *Let $0 = a < b$ and let λ, η, m be as in Lemma 1. Suppose that $g(0) = 0$.*

(i) *If $\eta > \lambda \geqslant 1$ and $b \leqslant \pi$, then*

$$\Big|\int_0^b G_{\lambda,\eta}(t)\,dt\Big| \leqslant \Big(\frac{4}{b}+\frac{43\lambda}{(\eta-\lambda)b^2}\Big) \sum_{j=1}^{m} \frac{v_j(g; 0, jb/(m+1))}{j^3} + \frac{10\lambda}{(\eta-\lambda)b^2} \frac{v_m(g; 0, b)}{m^2}.$$

(ii) *If $\lambda \geqslant 1$, $\eta-\lambda = \alpha > 0$ and $b \leqslant \min\{\pi, \pi/\alpha\}$, then*

$$\Big|\int_0^b G_{\lambda,\eta}(t)\,dt\Big| \leqslant \frac{6}{b}\sum_{j=1}^{m} \frac{v_j(g; 0, jb/(m+1))}{j^2} + \frac{3}{b}\frac{v_m(g; 0, b)}{m}.$$

Proof. Retain the symbols t_j defined by (5), and write

$$\int_0^b G_{\lambda,\eta}(t)\,dt = \int_0^{t_1} G_{\lambda,\eta}(t)\,dt + \sum_{j=1}^{m} g(t_j) \int_{t_j}^{t_{j+1}} \Psi_{\lambda,\eta}(t)\,dt$$
$$+ \sum_{j=1}^{m} \int_{t_j}^{t_{j+1}} \{g(t)-g(t_j)\}\Psi_{\lambda,\eta}(t)\,dt$$
$$= Z_1+Z_2+Z_3, \quad \text{say.}$$

Taking $h = 2b/(\lambda+\eta+2)$, we have

$$|Z_1| \leqslant \frac{\lambda+\eta}{2\pi}\int_0^h |g(t)|\,dt + \frac{2}{\pi(\eta-\lambda)}\int_h^{t_1} t^{-2}|g(t)|\,dt$$
$$\leqslant \left\{\frac{b}{\pi}+\frac{\eta-\lambda+2}{\pi b(\eta-\lambda)}\right\} v_1(g;\,0,\,t_1).$$

An argument similar to that of Lemma 1, with the help of inequality (6), leads to

$$|Z_2| \leqslant |g(t_1)|\Big|\int_{t_1}^b \Psi_{\lambda,\eta}(t)\,dt\Big| + \sum_{j=1}^{m-1}|g(t_{j+1})-g(t_j)|\Big|\int_{t_{j+1}}^b \Psi_{\lambda,\eta}(t)\,dt\Big|$$
$$\leqslant \frac{4(m+1)^2}{\pi\lambda(\eta-\lambda)b^2}\left\{|g(t_1)-g(0)| + \sum_{j=1}^{m-1}|g(t_{j+1})-g(t_j)|\frac{1}{(j+1)^2}\right\}.$$

By the Abel transformation,

$$|Z_2| \leqslant \frac{8(\lambda+1)}{\pi(\eta-\lambda)b^2}\left\{v_1(g;\,0,\,t_1)+2\sum_{j=1}^{m-2}\frac{v_j(g;\,0,\,t_{j+1})}{(j+1)^2(j+2)}+\frac{v_m(g;\,0,\,b)}{m^2}\right\}$$
$$\leqslant \frac{16\lambda}{\pi(\eta-\lambda)b^2}\left\{2\sum_{j=1}^{m-1}\frac{v_j(g;\,0,\,t_j)}{j^3}+\frac{v_m(g;\,0,\,b)}{m^2}\right\}.$$

In view of (7),

$$|Z_3| = \Big|\sum_{j=1}^{m}\int_0^{b/(m+1)}\{g(t+t_j)-g(t_j)\}\Psi_{\lambda,\eta}(t+t_j)\,dt\Big|$$
$$\leqslant \frac{2(m+1)^2}{\pi(\eta-\lambda)b^2}\int_0^{b/(m+1)}\sum_{j=1}^{m}|g(t+t_j)-g(t_j)|\frac{1}{j^2}\,dt$$
$$\leqslant \frac{2(\lambda+1)}{\pi(\eta-\lambda)b}\left\{\sum_{j=1}^{m-1}v_j(g;\,0,\,t_{j+1})\left(\frac{1}{j^2}-\frac{1}{(j+1)^2}\right)+\frac{v_m(g;\,0,\,b)}{m^2}\right\}$$
$$\leqslant \frac{4\lambda}{\pi(\eta-\lambda)b}\left\{8\sum_{j=2}^{m}\frac{v_j(g;\,0,\,t_j)}{j^3}+\frac{v_m(g;\,0,\,b)}{m^2}\right\}.$$

Collecting the results we get (i).

To show (ii), observe that the function $t^{-2}\cdot\sin\frac{1}{2}\alpha t$ is nonincreasing on $[0,\,\pi/\alpha]$. Hence

$$\Big|\int_u^b \Psi_{\lambda,\eta}(t)\,dt\Big| \leqslant 2/(\pi u\lambda) \qquad (0 < u \leqslant b).$$

In view of (3), $|\Psi_{\lambda,\eta}(t)| \leqslant 1/(\pi t)$ $(t > 0)$. Using these inequalities instead of (6) and (7), and reasoning as previously (see also the proof of Lemma 1 in [11]), we obtain

$$|Z_2| \leqslant \frac{4}{\pi b}\left\{\sum_{j=1}^{m-1}\frac{v_j(g;\,0,\,t_j)}{j^2}+\frac{v_m(g;\,0,\,b)}{m}\right\},$$
$$|Z_3| \leqslant \frac{2}{\pi}\sum_{j=2}^{m}\frac{v_j(g;\,0,\,t_j)}{j^2}+\frac{1}{\pi}\frac{v_m(g;\,0,\,b)}{m}.$$

Since

$$|Z_1| \leqslant \frac{\alpha+2\lambda}{2\pi}\int_0^{t_1}|g(t)-g(0)|\,dt \leqslant \tfrac{3}{2}v_1(g;\,0,\,t_1),$$

the desired estimate is now evident.

3. Pointwise convergence. Consider a (complex-valued) function f defined almost everywhere on the real line. Denote by $f(x\pm 0)$ the one-sided limits of f at a (fixed) point x. Assuming that both these limits are finite, let us introduce the auxiliary functions:

$$g_x^{\pm}(t) = \begin{cases} f(x\pm t)-f(x\pm 0) & \text{if } t \neq 0,\\ 0 & \text{if } t = 0,\end{cases}$$
$$g_x(t) = g_x^{+}(t)+g_x^{-}(t).$$

Write $\|f\|$ instead of $\sup_{-\infty<t<\infty}|f(t)|$ for f bounded on $\mathbf{R}$.

THEOREM 1. *Suppose that $f\in\Omega_\alpha$, $0<\delta\leqslant\min\{\pi,\,\pi/\alpha\}$ and that at a fixed point x the limits $f(x\pm 0)$ exist. Then, for every n such that $m = [\lambda_n] \geqslant 1$,*

$$|S_{\lambda_n}[f](x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}|$$
$$\leqslant \frac{6}{\delta}\sum_{j=1}^{m}\frac{v_j(g_x;\,0,\,j\delta/(m+1))}{j^2}+r_m(x,\,\alpha,\,\delta,\,f),$$

where

$$r_m(x,\,\alpha,\,\delta,\,f) = \frac{1}{m\alpha}\left\{\frac{18}{\delta^2}v_m(g_x;\,Y_0)+\sum_{k=1}^{\infty}\frac{v_m(g_x;\,Y_k)}{k^2}+\|f\|\right\}$$

and $Y_k = [k\pi,\,(k+1)\pi]$ $(k = 0,\,1,\,2,\,\ldots)$.

Proof. Identity (4) and the well-known property of the function $\Psi_{\lambda,\eta}$ (see [9], p. 75, (1.10.1)) lead to

$$S_{\lambda_n}[f](x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \sum_{k=0}^{\infty}\int_{Y_k} g_x(t)K_n(t)\,dt$$

with $K_n = \Psi_{\lambda,\eta}$, $\lambda = \lambda_n$, $\eta = \lambda_n+\alpha$. Divide the interval Y_0 into two intervals: $[0, \delta]$ and $[\delta, \pi]$. The term corresponding to the first one can be estimated as in Lemma 2(ii). In view of Lemma 1,

$$\Big|\int_\delta^\pi g_x(t)K_n(t)\,dt\Big| \leqslant \frac{4}{\pi\lambda_n\alpha\delta^2}\Big\{|g_x(\delta)-g_x(0)|+\Big(1+\frac{\pi}{2}\Big)v_{m+1}(g_x;\ \delta,\ \pi)\Big\}$$
$$\leqslant \frac{4}{\pi m\alpha\delta^2}(3+\pi)v_m(g_x;\ 0,\ \pi) \leqslant \frac{8}{m\alpha\delta^2}v_m(g_x;\ 0,\ \pi).$$

Consequently,

$$\Big|\int_{Y_0} g_x(t)K_n(t)\,dt\Big| \leqslant \frac{6}{\delta}\sum_{j=1}^{m}\frac{v_j(g_x;\ 0,\ j\delta/(m+1))}{j^2}+\frac{18}{\alpha\delta^2}\frac{v_m(g_x;\ Y_0)}{m}.$$

Also, by Lemma 1,

$$\sum_{k=1}^{\infty}\Big|\int_{Y_k} g_x(t)K_n(t)\,dt\Big| \leqslant \frac{4}{\pi^3\lambda_n\alpha}\sum_{k=1}^{\infty}\frac{1}{k^2}\Big\{|g_x(k\pi)|+\Big(1+\frac{\pi}{2}\Big)v_{m+1}(g_x;\ Y_k)\Big\}$$
$$\leqslant \frac{2}{3\pi m\alpha}\sup_{k\geqslant 1}|g_x(k\pi)|+\frac{1}{m\alpha}\sum_{k=1}^{\infty}\frac{v_m(g_x;\ Y_k)}{k^2}.$$

Thus, we obtain our assertion.

Let Φ be a continuous, convex and strictly increasing function on $[0, \infty)$, such that $\Phi(0) = 0$. Denote by $V_\Phi(g;\ a,\ b)$ the total Φ-variation of a function g on $[a, b]$ (defined as in [6] or [11]). If g is of bounded Φ-variation on $[a, b]$, then, for every integer n,

$$v_n(g;\ a,\ b) \leqslant n\Phi^{-1}\Big(\frac{1}{n}V_\Phi(g;\ a,\ b)\Big) \tag{8}$$

([6], p. 537), where Φ^{-1} is the function inverse to Φ.

THEOREM 2. *Let a function f of class Ω_α be of bounded Φ-variation on $[x-\delta, x+\delta]$ with some positive $\delta \leqslant \min\{\pi, \pi/\alpha\}$. Then, for every integer n such that $m = [\lambda_n] \geqslant 1$,*

$$|S_{\lambda_n}[f](x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}|$$
$$\leqslant \frac{48}{\delta}\sum_{j=1}^{m}\frac{1}{j}\Big\{\Phi^{-1}\Big(\frac{j}{m}V_\Phi\Big(g_x^+;0,\frac{\delta}{j}\Big)\Big)+\Phi^{-1}\Big(\frac{j}{m}V_\Phi\Big(g_x^-;0,\frac{\delta}{j}\Big)\Big)\Big\}+r_m(x,\alpha,\delta,f),$$

where $r_m(x, \alpha, \delta, f)$ is defined as in Theorem 1.

Proof. Clearly, both g_x^+ and g_x^- are of bounded Φ-variation on $[0, \delta]$. Using inequality (8), we obtain

$$v_j\Big(g_x;\ 0,\ \frac{j\delta}{m+1}\Big) \leqslant j\Big\{\Phi^{-1}\Big(\frac{1}{j}V_\Phi\Big(g_x^+;\ 0,\ \frac{j\delta}{m+1}\Big)\Big)+\Phi^{-1}\Big(\frac{1}{j}V_\Phi\Big(g_x^-;\ 0,\ \frac{j\delta}{m+1}\Big)\Big)\Big\}.$$

Now, our assertion follows easily from Theorem 1 (see also Lemma 2 in [11]).

Remark 1. If the function f possesses finite one-sided limits at each point of the real line, then $\lim_{m\to\infty} m^{-1}v_m(g_x;\ Y_k) = 0$ for every k (see [7]). Consequently, $\lim_{m\to\infty} r_m(x, \alpha, \delta, f) = 0$. In particular, this holds if f is of bounded Φ-variation on each finite interval. If moreover

$$\sup_k V_\Phi(f;\ k\pi,\ (k+2)\pi) \leqslant B < \infty \quad (k = 0,\ \pm 1,\ \pm 2,\ \ldots), \tag{9}$$

then

$$r_m(x,\alpha,\delta,f) \leqslant \frac{2}{\alpha}\Big(\frac{18}{\delta^2}+\frac{\pi^2}{6}\Big)\Phi^{-1}\Big(\frac{B}{m}\Big)+\frac{1}{m\alpha}\|f\|.$$

Remark 2. Since the functions $g_x^\pm$ are right-continuous at 0, we have $\lim_{t\to 0+} V_\Phi(g_x^\pm;\ 0,\ t) = 0$. Hence, under the additional assumption

$$\sum_{k=1}^{\infty}\frac{1}{k}\Phi^{-1}\Big(\frac{1}{k}\Big) < \infty, \tag{10}$$

we have

$$\lim_{m\to\infty}\sum_{j=1}^{m}\frac{1}{j}\Phi^{-1}\Big(\frac{j}{m}V_\Phi\Big(g_x^\pm;\ 0,\ \frac{\delta}{j}\Big)\Big) = 0.$$

Now, we will present an estimate of the rate of pointwise convergence of sums (2) for functions f of class Λ_β. The symbol $\|f\|_S$ will denote the norm in the space S, defined by

$$\|f\|_S = \sup_{-\infty<u<\infty}\frac{1}{\pi}\int_u^{u+\pi}|f(t)|\,dt.$$

THEOREM 3. *Suppose that $f \in \Lambda_\beta$ and that, at a (fixed) point x, the limits $f(x\pm 0)$ are finite. If, for a positive number $\delta \leqslant \pi$, the function f is bounded on $[x-\delta, x+\delta]$, then*

$$|S_{\lambda_n}[f](x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}| \leqslant \frac{13\beta+31}{\delta^2(\beta-1)}\sum_{j=1}^{m}\frac{v_j(g_x;\ 0,\ j\delta/(m+1))}{j^3}$$
$$+\frac{10}{\delta^2(\beta-1)}\frac{v_m(g_x;\ 0,\ \delta)}{m^2}+\frac{6}{\delta^2(\beta-1)m}\|g_x\|_S,$$

whenever $m = [\lambda_n] \geqslant 1$.

Proof. Clearly,

$$S_{\lambda_n}[f](x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\} = \sum_{k=0}^{\infty}\int_{Y_k} g_x(t)Q_n(t)\,dt,$$

where $Y_k = [k\pi, (k+1)\pi]$, $Q_n(t) = \Psi_{\lambda,\eta}(t)$, $\lambda = \lambda_n$, $\eta = \lambda_{n+1}$.

Observing that $\lambda_{n+1}-\lambda_n \geqslant \lambda_n(\beta-1)$ and applying Lemma 2(i), we obtain

$$\Big|\int_0^\delta g_x(t)Q_n(t)\,dt\Big| \leqslant \frac{13\beta+31}{\delta^2(\beta-1)}\sum_{j=1}^{m}\frac{v_j(g_x;\,0,\,j\delta/(m+1))}{j^3}+\frac{10}{\delta^2(\beta-1)}\,\frac{v_m(g_x;\,0,\,\delta)}{m^2}.$$

Further, by inequality (7),

$$\Big|\int_\delta^\pi g_x(t)Q_n(t)\,dt\Big| \leqslant \frac{2}{\delta^2(\beta-1)m}\,\|g_x\|_S,$$

and

$$\Big|\sum_{k=1}^{\infty}\int_{Y_k} g_x(t)Q_n(t)\,dt\Big| \leqslant \frac{2}{\pi^2(\beta-1)\lambda_n}\sum_{k=1}^{\infty}\frac{1}{k^2}\|g_x\|_S \leqslant \frac{\|g_x\|_S}{3(\beta-1)m}.$$

Collecting the results we get the desired estimate.

Remark 3. Observe that

$$\sum_{j=1}^{m}\frac{v_j(g_x;\,0,\,j\delta/(m+1))}{j^3} \leqslant \sum_{j=1}^{m}\frac{v_1(g_x;\,0,\,j\delta/(m+1))}{j^2} \leqslant \frac{4}{m+1}\sum_{j=1}^{m} v_1\Big(g_x;\,0,\,\frac{\delta}{j}\Big),$$

$$\frac{v_m(g_x;\,0,\,\delta)}{m^2} \leqslant \frac{4}{m}\cdot\sup_{x-\delta\leqslant t\leqslant x+\delta}|f(t)|.$$

Consequently, in view of the continuity of g_x at $t=0$, the right-hand side of the inequality in Theorem 3 converges to zero as $n\to\infty$.

Corollary 1. *If $f\in\Lambda_\beta$ is of bounded Φ-variation on $[x-\delta,\,x+\delta]$, then*

$$|S_{\lambda_n}[f](x)-\tfrac{1}{2}\{f(x+0)+f(x-0)\}|$$
$$\leqslant \frac{c_1(\beta)}{m\delta^2}\Bigg(\sum_{j=1}^{m}\Big\{\Phi^{-1}\Big(\frac{j}{m}V_\Phi\Big(g_x^+;\,0,\,\frac{\delta}{j}\Big)\Big)+\Phi^{-1}\Big(\frac{j}{m}V_\Phi\Big(g_x^-;\,0,\,\frac{\delta}{j}\Big)\Big)\Big\}+\|g_x\|_S\Bigg).$$

4. Uniform convergence. Suppose now that a function f of class S is continuous on $\mathbf{R}$, and denote by $\omega(\delta, f; I)$ its ordinary modulus of continuity on a finite interval $I=[a, b]$. Write

$$\omega(\delta, f) = \sup_k \omega(\delta, f;\, I_k),$$

where $I_k=[k\pi,\,(k+2)\pi]$, $k=0,\,\pm1,\,\pm2,\,\ldots$ Retaining the symbol of the modulus of variation of f on I, used in Sects. 2–3, let us introduce the quantity

$$v_j(f) = \sup_k v_j(f;\, I_k).$$

As is known ([6] or [10]),

$$v_j(f;\,I) \leqslant 2j\omega((b-)a/j,\,f;\,I) \qquad (j\in\mathbf{N}). \tag{11}$$

Therefore, if f is uniformly continuous on $\mathbf{R}$,

$$v_j(f) \leqslant 4j\omega(\pi/j,\,f) \qquad (j\in\mathbf{N}).$$

Applying the above inequalities one can estimate the expressions on the right-hand side of the estimate in Theorem 1. Namely, we have

$$v_j\Big(g_x;\,0,\,\frac{j\delta}{m+1}\Big) \leqslant 2j\Big\{\omega\Big(\frac{\pi}{m},\,g_x^+;\,Y_0\Big)+\omega\Big(\frac{\pi}{m},\,g_x^-;\,Y_0\Big)\Big\} \leqslant 4j\omega\Big(\frac{\pi}{m},\,f\Big)$$

and

$$v_j\Big(g_x;\,0,\,\frac{j\delta}{m+1}\Big) \leqslant v_j(g_x^+;\,0,\,\pi)+v_j(g_x^-;\,0,\,\pi) \leqslant 2v_j(f)$$

for $j=1,\,\ldots,\,m$. Moreover,

$$v_m(g_x;\,Y_k) \leqslant 4m\omega(\pi/m,\,f) \quad \text{for } k=0,\,1,\ldots$$

Putting $\delta=\min\{\pi,\,\pi/\alpha\}$ we get the following result analogous to Theorem 1 of [6]:

Theorem 4. *If f is a uniformly continuous function of class Ω_α and if $m=[\lambda_n]\geqslant 1$, then*

$$\sup_{-\infty<x<\infty}|S_{\lambda_n}[f](x)-f(x)|$$
$$\leqslant c_2(\alpha)\Bigg(\inf_{1\leqslant\mu\leqslant m}\Big\{\omega\Big(\frac{\pi}{m},\,f\Big)\sum_{j=1}^{\mu}\frac{1}{j}+\sum_{j=\mu+1}^{m}\frac{v_j(f)}{j^2}\Big\}+\frac{\|f\|}{m}\Bigg).$$

From this theorem some sufficient conditions for the uniform convergence of sums (2) can be deduced (see [6]).

Corollary 2. *If $f\in\Omega_\alpha$ is uniformly continuous on $\mathbf{R}$, then*

$$\|S_{\lambda_n}[f]-f\| \leqslant c_3(\alpha)\{\omega(\pi/m,\,f)\log(m+1)+m^{-1}\|f\|\},$$

whenever $m=[\lambda_n]\geqslant 1$ (cf. [4], Th. 8).

Corollary 3. *If $f\in\Omega_\alpha$ is uniformly continuous on $\mathbf{R}$ and if*

$$\sum_{j=1}^{\infty} v_j(f)/j^2 < \infty,$$

then the sequence $(S_{\lambda_n}[f])$ converges uniformly on $\mathbf{R}$. In particular, this is true if f is of bounded Φ-variation on each finite interval and if conditions (9) *and* (10) *are fulfilled.*

Remark 4. From Theorem 3 it follows that, for any uniformly continuous function f of class Λ_β,

$$\|S_{\lambda_n}[f]-f\| \leqslant 9\frac{\beta+3}{\beta-1}\omega(\pi/m, f) \quad (m = [\lambda_n] \geqslant 1)$$

(cf. [4], Th. 4).

5. Strong summability. Given any function f of class Ω_α, with Fourier series (1), consider the sums

$$s_\mu[f](x) = \sum_{|\lambda_k| \leqslant \alpha\mu/2} A_k e^{i\lambda_k x} \quad (\mu \in \mathbf{N}). \tag{12}$$

Obviously, the sequences of sums (2) and (12) are equiconvergent. Moreover, the rates of pointwise and uniform convergence of the sums $s_\mu[f](x)$ can be estimated analogously to Theorems 1, 2, 4.

Here, an estimate of the Hardy type expression

$$H_n^q[f](x) = \left\{\frac{1}{n}\sum_{\mu=1}^{n} |s_\mu[f](x)-f(x)|^q\right\}^{1/q} \quad (q \geqslant 2)$$

will be presented.

THEOREM 5. *If a function f of class Ω_α is uniformly continuous on $\mathbf{R}$, then, for every positive integer n,*

$$\sup_{-\infty<x<\infty} H_n^q[f](x) \leqslant c_4(\alpha)q\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left(\omega\left(\frac{\pi}{\mu}, f\right)\right)^q\right\}^{1/q} + \frac{4}{3}\frac{\|f\|}{n^{1/q}},$$

where $\|f\|$ and $\omega(\delta, f)$ have the same meaning as before.

Proof. Denote by $s_\mu^*[f](x)$ the sums of the form (12) such that the interval $(\frac{1}{2}\alpha\mu, \frac{1}{2}\alpha(\mu+1))$ does not contain any λ_ν. Applying Lemma 1.10.2 of [9] we easily verify that

$$s_\mu^*[f](x)-f(x) = \int_0^\infty g_x(t)\Psi_\mu(t)\,dt,$$

where $g_x(t) = f(x+t)+f(x-t)-2f(x)$ and $\Psi_\mu(t) = \Psi_{\lambda,\eta}(t)$ with $\lambda = \frac{1}{2}\alpha\mu$, $\eta = \frac{1}{2}\alpha(\mu+1)$, i.e.,

$$\Psi_\mu(t) = \frac{4}{\pi\alpha}t^{-2}\sin\tfrac{1}{4}\alpha t\sin\tfrac{1}{4}\alpha(2\mu+1)t$$

(see also [2], p. 41). Evidently, if the interval $(\frac{1}{2}\alpha\mu, \frac{1}{2}\alpha(\mu+1))$ contains a Fourier exponent λ_ν, then

$$s_\mu[f](x) = s_{\mu+1}^*[f](x)-(A_\nu e^{i\lambda_\nu x}+A_{-\nu}e^{-i\lambda_\nu x}).$$

As is known, $|A_{\pm\nu}| \leqslant E_{\alpha\mu/2}(f)$, where $E_\sigma(f)$ denotes the constant of the best uniform approximation of f by entire functions of exponential type σ ([3], p. 19). Hence, by the Jackson type theorem,

$$|A_\nu e^{i\lambda_\nu x}+A_{-\nu}e^{-i\lambda_\nu x}| \leqslant c_5(\alpha)\omega(\pi/\mu, f)$$

(see e.g. [1], Sect. 105). Consequently, $H_n^q[f](x)$ can be estimated from above by

$$\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left|\int_0^\infty g_x(t)\Psi_{\mu+\gamma}(t)\,dt\right|^q\right\}^{1/q} + c_5(\alpha)\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left(\omega\left(\frac{\pi}{\mu}, f\right)\right)^q\right\}^{1/q},$$

where $\gamma = \gamma(m)$ equals 0 or 1.

Put $h = 2\pi/(\alpha n)$, $J_k = [2k\pi/\alpha, 2(k+1)\pi/\alpha]$; write

$$\begin{aligned}\int_0^\infty g_x(t)\Psi_{\mu+\gamma}(t)\,dt &= \Big(\int_0^h + \int_h^{2\pi/\alpha} + \int_{2\pi/\alpha}^\infty\Big) g_x(t)\Psi_{\mu+\gamma}(t)\,dt\\ &= W_1(\mu)+W_2(\mu)+W_3(\mu), \quad \text{say.}\end{aligned}$$

Clearly,

$$|W_1(\mu)| \leqslant \frac{(2\mu+3)\alpha h}{4\pi}\omega(h, g_x; J_0) \leqslant 5\left(1+\frac{2}{\alpha}\right)\omega\left(\frac{\pi}{n}, f\right).$$

From Lemma 1, with $\lambda = \frac{1}{2}\alpha(\mu+\gamma)$, $\eta = \frac{1}{2}\alpha(\mu+\gamma+1)$, it follows that

$$\begin{aligned}|W_3(\mu)| &\leqslant \sum_{k=1}^\infty \left|\int_{J_k} g_x(t)\Psi_{\mu+\gamma}(t)\,dt\right|\\ &\leqslant \frac{4}{\pi^3(\mu+\gamma)}\sum_{k=1}^\infty \frac{1}{k^2}\left\{\left|g_x\left(\frac{2k\pi}{\alpha}\right)\right| + \left(1+\frac{\pi}{\alpha}\right)v_{m+1}(g_x; J_k)\right\},\end{aligned}$$

where $m = [\alpha(\mu+\gamma)/2]$. Using (11), we get

$$|W_3(\mu)| \leqslant \frac{8\|f\|}{3\pi\mu} + c_6(\alpha)\omega\left(\frac{\pi}{\mu}, f\right)$$

with $c_6(\alpha) \leqslant 16(1+\pi/\alpha)(1+\alpha/2)(1+4/\alpha^2)/\pi^3$. Hence

$$H_n^q[f](x) \leqslant \left\{\frac{1}{n}\sum_{\mu=1}^{n}|W_2(\mu)|^q\right\}^{1/q} + c_7(\alpha)\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left(\omega\left(\frac{\pi}{\mu}, f\right)\right)^q\right\}^{1/q} + \frac{8\|f\|}{3\pi}\left(\frac{2}{n}\right)^{1/q}.$$

The inequality of Hardy and Littlewood ([12], Chap. XII, Th. 5.15.II) yields

$$\left\{\frac{1}{n}\sum_{\mu=1}^{n}|W_2(\mu)|^q\right\}^{1/q}$$

$$\leqslant \frac{8}{\alpha^2}\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left|\frac{\alpha}{2\pi}\int_h^{2\pi/\alpha} g_x(t)t^{-2}\sin\frac{\alpha}{4}t\cos\frac{\alpha}{4}(2\gamma+1)t\sin\frac{\alpha}{2}\mu t\,dt\right|^q\right\}^{1/q}$$

$$+\frac{8}{\alpha^2}\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left|\frac{\alpha}{2\pi}\int_h^{2\pi/\alpha} g_x(t)t^{-2}\sin\frac{\alpha}{4}t\sin\frac{\alpha}{4}(2\gamma+1)t\cos\frac{\alpha}{2}\mu t\,dt\right|^q\right\}^{1/q}$$

$$\leqslant c_8(\alpha)q\left\{\frac{1}{n}\int_h^{2\pi/\alpha}|g_x(t)|^q t^{-2}\,dt\right\}^{1/q},$$

where $c_8(\alpha) = c\cdot\max\{1,\ 1/\sqrt{\alpha}\}$, $c = \text{const}$. Consequently,

$$\left\{\frac{1}{n}\sum_{\mu=1}^{n}|W_2(\mu)|^q\right\}^{1/q} \leqslant c_8(\alpha)q\left\{\frac{1}{n}\int_h^{2\pi/\alpha}(\omega(t,\ g_x;\ J_0))^q t^{-2}\,dt\right\}^{1/q}$$

$$\leqslant c_8(\alpha)q\left\{\frac{\alpha}{2\pi n}\int_1^n\left(\omega\left(\frac{2\pi}{\alpha u},\ g_x;\ J_0\right)\right)^q du\right\}^{1/q}$$

$$\leqslant 2c_8(\alpha)\left(1+\frac{2}{\alpha}\right)\left(\frac{\alpha}{2\pi}\right)^{1/q} q\left\{\frac{1}{n}\sum_{\mu=1}^{n}\left(\omega\left(\frac{\pi}{\mu},f\right)\right)^q\right\}^{1/q}.$$

Thus, the desired result follows.

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