STUDIA MATHEMATICA, T. XCVI (1990)

# Gradient homotopies of gradient vector fields

by

A. PARUSIŃSKI (Gdańsk)

**Abstract.** Consider two gradient vector fields on the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ and nowhere vanishing in $\mathbf{S}^{n-1}$. It is shown that if they are homotopic, then they are gradient homotopic. The same result holds for Hamiltonian vector fields.

**1. Introduction and statement of results.** Assume that we have a continuous vector field $v$ defined on the closed unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ and having no zeroes in $\mathbf{S}^{n-1}$. We are interested in finding a topological criterion which ensures that $v$ has zeroes in $\mathbf{B}^n$. By standard facts from homotopy theory (see for example [N]) we know that if $v$ is not homotopic (in the space of all continuous vector fields on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$) to a constant vector field (i.e. $\deg(v|_{\mathbf{S}^{n-1}}) \neq 0$) then $v$ must have zeroes in $\mathbf{B}^n$. The aim of this paper is to study whether we can obtain a better result if we restrict ourselves to some classes of vector fields. For example, if we know that $v$ is gradient or Hamiltonian, could we deduce from some "homotopic" properties of $v$ that it has zeroes in $\mathbf{B}^n$ even if $\deg(v|_{\mathbf{S}^{n-1}}) = 0$? The most natural approach to this problem is to consider gradient (or Hamiltonian) homotopies.

DEFINITION 1. Two gradient vector fields $\operatorname{grad} f$, $\operatorname{grad} g$ on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$ are said to be *gradient homotopic* if there exists a $C^1$-function $F(x, t)$ on $\mathbf{B}^n \times \mathbf{I}$ such that:

(1) $F(*, 0) \equiv f$, $F(*, 1) \equiv g$.

(2) $\operatorname{grad}_x F$ has no zeroes in $\mathbf{S}^{n-1} \times \mathbf{I}$.

PROPOSITION 1. *Let $v = \operatorname{grad} f$, where $f \in C^1(\mathbf{B}^n)$, and let $v$ be nowhere zero in $\mathbf{B}^n$. Then $v$ is gradient homotopic to a constant vector field.*

Proof. We can assume that $f$ is smooth. By the Hadamard Lemma

$$f(x) = f(0) + \sum_{i=1}^{n} x_i g_i(x),$$

where the $g_i$ are smooth and $v(0) = (g_1(0), \ldots, g_n(0))$. Then

$$F(x, t) = f(0) + \sum_{i=1}^{n} x_i \cdot g_i(tx)$$

is the required homotopy between $v$ and $v(0)$. ∎

The problem which we consider is the following: If two gradient vector fields are homotopic, are they necessarily gradient homotopic? If the answer were negative, it would mean that considering gradient homotopies we obtain a better criterion for vanishing of $v = \operatorname{grad} f$ at some point than using the ordinary ones. But we will show that the answer is affirmative.

THEOREM 1. *Assume that we have on $\mathbf{B}^n$ two gradient vector fields nowhere vanishing in $\mathbf{S}^{n-1}$ and homotopic (in the space of all continuous vector fields on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$). Then they are gradient homotopic.*

Let $n = 2k$. We call a vector field $v$ *Hamiltonian* if $v_i = -\partial H/\partial x_{k+i}$, $v_{k+i} = \partial H/\partial x_i$, for $i = 1, \ldots, k$, and some $H \in C^1(\mathbf{B}^n)$. Hamiltonian vector fields occur naturally in mechanics (see e.g. [A]). For Hamiltonian vector fields one can define Hamiltonian homotopies analogously to the gradient case. Note that a vector field $v$ is Hamiltonian iff $v_i = -w_{k+i}$ and $v_{k+i} = w_i$, $i = 1, \ldots, k$, for some gradient vector field $w$, so Theorem 1 implies:

THEOREM 2. *Assume that we have on $\mathbf{B}^n$ two Hamiltonian vector fields nowhere vanishing in $\mathbf{S}^{n-1}$ and homotopic (in the space of all continuous vector fields on $\mathbf{B}^n$ nowhere vanishing in $\mathbf{S}^{n-1}$). Then they are Hamiltonian homotopic.*

**Notation and remarks.** 1. In this paper the character $\mathbf{I}$ is reserved for the interval $[0, 1]$.

2. By a smooth map (function, vector field) on a subset of an affine space we mean the restriction to this set of a $C_\infty$-smooth map (function, vector field) defined in an open neighbourhood of the set.

3. We often replace continuous vector fields or maps defined on submanifolds of affine spaces by their smooth approximations. This can be done thanks to the approximation theorems (see e.g. [H], Theorem 3.5 or [M]). The obtained vector field (map) is homotopic to the original one. This principle will also be used to approximate homotopies, as maps defined on manifold $\times\, \mathbf{I}$, or elements of homotopy groups, as maps defined on spheres. For example, if we have an element of the $k$th homotopy group $\pi_k$ of the space of continuous vector fields defined on $\mathbf{B}^n$, then it can be represented by a smooth map from $\mathbf{S}^k \times \mathbf{B}^n$ to $\mathbf{R}^n$.

**2. Proof of Theorem 1.** Consider the following spaces:

$$\mathscr{V} = \mathscr{V}(n) = \{v = (v_1, \ldots, v_n) \in (C^0(\mathbf{B}^n))^n;\ v(x) \neq 0 \text{ for } x \in \mathbf{S}^{n-1}\},$$

$$\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(n) = \{(v, h) \in C^0(\mathbf{S}^{n-1}; T\mathbf{S}^{n-1}) \times C^0(\mathbf{S}^{n-1});\ (v(x), h(x)) \neq 0 \text{ for } x \in \mathbf{S}^{n-1}\}$$

with $C^0$-topology (where $C^0(\mathbf{S}^{n-1}; T\mathbf{S}^{n-1})$ denotes the space of continuous sections of the tangent bundle $T\mathbf{S}^{n-1}$),

$$\mathscr{A} = \mathscr{A}(n) = \{f \in C^1(\mathbf{B}^n);\ \operatorname{grad} f \in \mathscr{V}\}$$



with $C^1$-topology, and

$$\tilde{\mathscr{A}} = \tilde{\mathscr{A}}(n) = \{(g, h) \in C^1(\mathbf{S}^{n-1}) \times C^0(\mathbf{S}^{n-1});\ (\operatorname{grad} g, h) \in \tilde{\mathscr{V}}\}$$

with mixed $C^1 \times C^0$-topology.

Theorem 1 says that the gradient map **grad**: $\mathscr{A} \to \mathscr{V}$ induces an injection on the connected components. The spaces $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{V}}$ of (gradient) vector fields restricted to $\mathbf{S}^{n-1}$ are used in the inductive step. Let $\mathbf{n}$ denote the outward unit normal vector field on $\mathbf{S}^{n-1}$.

LEMMA 1. *The map $\varphi$: $\mathscr{A} \to \tilde{\mathscr{A}}$ defined by*

$$\varphi(f) = (f, \partial f/\partial \mathbf{n})|_{\mathbf{S}^{n-1}}$$

*is a homotopy equivalence. The same holds for $\tilde{\varphi}$: $\mathscr{V} \to \tilde{\mathscr{V}}$, given by*

$$\tilde{\varphi}(v) = (v', h),$$

*where* $v|_{\mathbf{S}^{n-1}} = v' + h \cdot \mathbf{n}$.

Proof. Let $\varrho$: $\mathbf{I} \to \mathbf{I}$ be a smooth function equal to 0 in a neighbourhood of 0 and to 1 in a neighbourhood of 1. Define $\psi$: $\tilde{\mathscr{A}} \to \mathscr{A}$ by the formula

$$\psi(g, h)(x) = \varrho(\|x\|)\big(g(x/\|x\|) + (\|x\| - 1)h(x/\|x\|)\big).$$

Then it is easy to check that:

(1) $\varphi \circ \psi = \mathrm{id}_{\tilde{\mathscr{A}}}$.

(2) $\psi \circ \varphi \sim \mathrm{id}_{\mathscr{A}}$ by means of the homotopy $t \cdot (\psi \circ \varphi)(f) + (1-t) \cdot f$,

which gives the result for $\varphi$. The proof for $\tilde{\varphi}$ is similar. ∎

PROPOSITION 2. *The map $\iota$: $\tilde{\mathscr{A}} \to \tilde{\mathscr{V}}$ which sends $(g, h)$ to $(\operatorname{grad} g, h)$ induces a one-to-one correspondence between the connected components of these spaces.*

Proof. First we show the injectivity of $\pi_0(\iota)$: $\pi_0(\tilde{\mathscr{A}}) \to \pi_0(\tilde{\mathscr{V}})$ (the induced map on the sets of connected components) by induction on $n$.

*Case* $n = 2$. This is a special case (for many reasons, one of them being that $T\mathbf{S}^1$ is a trivial bundle). Let $s$ be a parametrization by arc length on $\mathbf{S}^1$. Then the vector field $\partial/\partial s$ is a nowhere zero section of $T\mathbf{S}^1$.

The space $\mathscr{A}' = \{(v, h) \in \tilde{\mathscr{V}};\ \int_{\mathbf{S}^1} v\, ds = 0\}$ (with $C^0$-topology) is just the image of $\iota$ and the induced map $\tilde{\mathscr{A}} \to \mathscr{A}'$ is a homotopy equivalence. In fact, $\tilde{\mathscr{A}}$ is homeomorphic to $\mathscr{A}' \times \mathbf{R}$ via $(g, h) \to ((\operatorname{grad} g, h), g(0))$ (the inverse is $((v, h), c) \to (c + \int_0^s v, h)$).

Consider $\mathscr{A}' \subset \tilde{\mathscr{V}}$. The connected components $\tilde{\mathscr{V}}_i$ of $\tilde{\mathscr{V}}$ are classified by the topological degree $\deg(v \cdot \partial/\partial s + h \cdot \mathbf{n}) = i \in \mathbf{Z}$.

LEMMA 2. *For $i \neq 1$ the inclusion $\mathscr{A}' \cap \tilde{\mathscr{V}}_i \subset \tilde{\mathscr{V}}_i$ is a homotopy equivalence.*

Proof. Let $(v, h) \in \tilde{\mathscr{V}}_i$. Consider the continuous functions $v_+(s) = \max_{s \in \mathbf{S}^1}\{v(s), 0\}$, $v_-(s) = \min_{s \in \mathbf{S}^1}\{v(s), 0\}$ and their integrals $c_+(v)$, $c_-(v)$ over

$S^1$. Note that $c_+$ and $c_-$ are continuous functions of $v$ and they nowhere vanish (since $i \neq 1$). Therefore, $\mathscr{A}'_i = \mathscr{A}' \cap \tilde{\mathscr{V}}_i$ is a strong deformation retract of $\tilde{\mathscr{V}}_i$, by means of the deformation

$$\big((v, h), t\big) \to t\cdot(v, h) + (1-t)\cdot\big((L(v)\cdot v_+ + (L(v))^{-1}\cdot v_-), h\big),$$

where $L(v) = (-c_-(v)/c_+(v))^{1/2}$. ∎

For $i = 1$ the above proof does not work. Take $(v_0, h_0) \in \mathscr{A}'_1 = \mathscr{A}' \cap \tilde{\mathscr{V}}_1$. Since $\int_{S^1} v = 0$, $v_0$ has zeroes, and rotating the circle we can assume that $v_0(0) = 0$. Hence $h_0(0) \neq 0$. After perturbing $(v_0, h_0)$ in $\mathscr{A}'_1$, we can also assume that it equals (sin, 1) just near 0. We shall show that it is in fact homotopic (in $\mathscr{A}'_1$) to (sin, 1). By the arguments from the proof of Lemma 2, it suffices to find such a homotopy in $\{(v, h) \in \tilde{\mathscr{V}}_1:\ \min(v) < 0 \text{ and } \max(v) > 0\}$.

The map $\alpha = (v_0, h_0)\colon S^1 \to R^2\backslash 0 \simeq C\backslash 0$ has degree 0 (since $v_0 + h_0\cdot \mathbf{n}$ has degree 1), so it can be lifted to a map $\tilde{\alpha}\colon S^1 \to C$, i.e. $\exp\circ\tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = \pi i/2$. Let $\gamma\colon S^1 \to C$ satisfy $\exp\circ\gamma = (\sin, 1)$ and $\gamma(0) = \pi i/2$. Then

$$\exp(t\cdot\alpha + (1-t)\cdot\gamma)$$

is a homotopy with the required properties. ∎

*Inductive step.* Let $(g, h) \in \tilde{\mathscr{A}} = \tilde{\mathscr{A}}(n)$. We will deform $(g, h)$ in $\tilde{\mathscr{A}}$ in order to obtain an element of $\tilde{\mathscr{A}}$ of the simplest possible form. The first step is to deform $g$ to a Morse function (see e.g. [H]). So, assume that $g$ is a Morse function and divide the set of its critical points into the set $P$ on which $h$ is positive and $Q$ on which $h$ is negative ($h$ nowhere vanishes on the set of critical points of $g$).

LEMMA 3. *Let $(g, h) \in \tilde{\mathscr{A}}$ be as above. Then*

$$\deg(\operatorname{grad} g + h\cdot\mathbf{n}) = (-1)^n + \sum_{p\in P}(-1)^{i(p)},$$

*where $i(p)$ denotes the index of $g$ at $p$. (If $P = \emptyset$, then we understand the right-hand side to be $(-1)^n$.)*

Proof. Assume that $c = \max_{x\in S^{n-1}} h(x) > 0$. Consider on $U = \{x \in R^n;\ 1 \leqslant \|x\| \leqslant 2\}$ the vector field

$$w(x) = v(x/\|x\|) - 2c\cdot(\|x\|-1)\cdot\mathbf{n}(x/\|x\|),$$

where $v = \operatorname{grad} g + h\cdot\mathbf{n}$. Note that $w|_{S^{n-1}} \equiv v$ and $\deg(w|_{S_2^{n-1}}) = (-1)^n$, where $S_2^{n-1}$ denotes the sphere of radius 2. The point $x\in U$ satisfies $w(x) = 0$ if and only if $x/\|x\| \in P$ and $\|x\| = h(x/\|x\|)/(2c)+1$. The local index of $w$ at such an $x$ equals $-(-1)^{i(x/\|x\|)}$. Therefore, by the properties of the index [H] we have

$$\deg(w|_{S_2^{n-1}}) - \deg(w|_{S^{n-1}}) = -\sum_{p\in P}(-1)^{i(p)}$$

and the lemma follows for $c > 0$. If $c \leqslant 0$, then $\deg(v) = (-1)^n$ and $P = \emptyset$. This ends the proof. ∎



We continue the process of perturbing $(g, h)$ using isotopies of $S^{n-1}$. Fix $p \in P$. Since $n \geqslant 3$, we can move $P$ into a small neighbourhood of $p$ without changing $Q$. Afterwards we move $Q$ into $S^{n-1} \cap \{x \in R^n;\ x_n \leqslant -1/2\}$ and $P$ into $S^{n-1} \cap \{x \in R^n;\ x_n \geqslant 1/2\}$. Since $h$ is positive on $P$ and negative on $Q$ we can change it linearly via $t\cdot h + (1-t)\cdot x_n$ to a new $h \equiv x_n$. We call $(g, h)$ of this form *normal.*

Assume that we have two elements $(g_1, h_1)$ and $(g_2, h_2)$ of $\tilde{\mathscr{A}}$ homotopic in $\tilde{\mathscr{V}}$. Then

$$\deg(\operatorname{grad} g_1 + h_1\cdot\mathbf{n}) = \deg(\operatorname{grad} g_2 + h_2\cdot\mathbf{n}).$$

By the above, we can assume that $(g_i, h_i)$, $i = 1, 2$, are normal (in particular $h_1 \equiv h_2 \equiv x_n$). Let $\varphi\colon B^{n-1} \to S_-^{n-1}$ be a diffeomorphism onto the southern hemisphere and let $P(i)$ $(i = 1, 2)$ denote the set of critical points of $g_i$ on which $h_i \equiv x_n$ is positive. By Lemma 3

$$\sum_{p\in P(1)}(-1)^{i(p)} = \sum_{p\in P(2)}(-1)^{i(p)}.$$

Consequently, $\operatorname{grad}(g_1\circ\varphi)$ is homotopic to $\operatorname{grad}(g_2\circ\varphi)$ in $\mathscr{V}(n-1)$ and, by the inductive assumption and Lemma 1, they are also gradient homotopic. Let $G\colon B^{n-1}\times I \to R$ be such a homotopy. Consider on $(S_-^{n-1}\times I) \cup (S_-^{n-1}\times\{0, 1\})$ the function equal to $G\circ\varphi^{-1}$ on $S_-^{n-1}\times I$, to $g_1$ on $S^{n-1}\times\{0\}$ and to $g_2$ on $S^{n-1}\times\{1\}$. By [H], it can be extended to a smooth function $G'$ on $S^{n-1}\times I$ and now $(G', x_n)$ is the required homotopy. This ends the proof of the injectivity of $\pi_0(\iota)$ and hence, by Lemma 1, the proof of Theorem 1.

The surjectivity of $\pi_0(\iota)$ follows from Lemma 1 and the following lemma.

LEMMA 4. *For each $d \in Z$ there is $f \in \mathscr{A}$ such that* $\deg(\operatorname{grad} f|_{S^{n-1}}) = d$.

Proof. Let $\varrho\colon I \to I$ be a smooth functional equal to 0 near 0 and to 1 near 1. For $n = 2$ and $d \neq 1$ we define $f_d(z) := \varrho(\|z\|)\operatorname{Re}(z^{1-d})$, where $z \in B^2 \subset R^2 \simeq C$. By the Cauchy–Riemann formula $d(z^{1-d})/dz = (\partial f/\partial x, -\partial f/\partial y)$ near $S^1$. Therefore, $\deg(\operatorname{grad} f|_{S^{n-1}}) = d$. For $d = 1$ we put $f_1(x, y) = x^2 + y^2$.

Assume $n > 2$. Define $f(x_1, x_2, \ldots, x_n) = f_d(x_1, x_2) + x_3^2 + \ldots + x_n^2$. Then $\operatorname{grad} f = (\partial f_d/\partial x_1, \partial f_d/\partial x_2, 2x_3, \ldots, 2x_n)$ and by the properties of suspension (see e.g. [H], [S])

$$\deg(\operatorname{grad} f|_{S^{n-1}}) = \deg(\operatorname{grad} f_d|_{S^1}) = d. ∎$$

## 3. Questions.

Assume that we have a continuous family $v\colon B^n \times B^m \to R^n$ of continuous vector fields on $B^n$ such that the restriction $\tilde{v}$ of $v$ to $\partial(B^n\times B^m) = S^{n-1}\times B^m \cup B^n \times S^{m-1}$ nowhere vanishes. If $\tilde{v}$ gives a nontrivial element in $\pi_{n+m-1}(R^n\backslash 0) = \pi_{n+m-1}(S^{n-1})$, which is equivalent to saying that $v$ is not homotopic (in the space of all families of vector fields as above) to a constant (family of) vector field(s) $v(x, y) \equiv w \in R^n\backslash 0$, then $v$ must have zeroes (see [N]). As in the case of one vector field one can ask whether in the case of $v$ gradient (i.e. $v(x, y) = \mathrm{grad}_x f(x, y)$ for some $f \in C^1(\mathbf{B}^n \times \mathbf{B}^m)$) there is a better invariant which shows that $v$ has zeroes. First we note that simply by repeating the proof of Proposition 1 we obtain:

PROPOSITION 3. *Let $v(x, y) = \mathrm{grad}_x f$ be as above. If $v$ has no zeroes in $\mathbf{B}^n \times \mathbf{B}^m$, then $v$ is gradient homotopic (i.e. in the space of all families of gradient vector fields nowhere vanishing in $\mathbf{B}^n \times \mathbf{B}^m$) to a constant vector field.*

QUESTION 1. *Are two families of gradient vector fields gradient homotopic if they are homotopic?*

This question seems to be much more difficult than the problem we have considered earlier. If the answer is negative, it means that there exists some new obstruction for families of gradient vector fields which can be used in looking for their zeroes. Question 1 is closely related to the homotopic properties of the map **grad**: $\mathscr{A} \to \mathscr{V}$. We have shown (Proposition 2 and Lemma 1) that $\pi_0(\mathbf{grad})$ is a bijection.

Fix $* = x_1 \in \mathscr{A}$ and $* = \partial/\partial x_1 \in \mathscr{V}$.

QUESTION 2. *Is $\pi_m(\mathbf{grad})\colon \pi_m(\mathscr{A}, *) \to \pi_m(\mathscr{V}, *)$ injective?*

PROPOSITION 4. *If $\pi_m(\mathbf{grad})$ is injective, then every family of gradient vector fields (on $\mathbf{B}^n$) which is homotopic to a constant vector field is gradient homotopic to a constant vector field. The converse holds for $m < n-1$.*

Proof. Take any family $v(x, y) = \mathrm{grad}_x f(x, y)$ of gradient vector fields. We can assume that $f$ is smooth. By the Hadamard Lemma

$$f(x, y) = f(0, y) + \sum_{i=1}^{n} x_i \cdot g_i(x, y),$$

where the $g_i$'s are smooth and $\mathrm{grad}_x f(0, y) = (g_1(0, y), \ldots, g_n(0, y))$. If $N$ is sufficiently large, then $v(x, y) \neq 0$ for $\|x\| + \|y\|^{2N} \geq 1$, so the homotopy

$$F(x, y, t) = (1-t) f(0, y) + \sum_{i=1}^{n} x_i \cdot g_i((1-t)x + t(1-\|y\|^{2N})x, y)$$

joins $f$ to

$$h(x, y) = \sum_{i=1}^{n} x_i \cdot g_i((1-\|y\|^{2N})x, y)$$

But the gradient (with respect to $x$) of $h$ for $y \in \mathbf{S}^{m-1}$ does not depend on $x$ and equals $w(y) = (g_1(0, y), \ldots, g_n(0, y))$. So, we have just proved that any family of gradient vector fields is gradient homotopic to a family which does not depend on $x$ for $y \in \mathbf{S}^{m-1}$. The same argument works for homotopies of gradient vector fields and so it is easy to see that the homotopy class of $w\colon \mathbf{S}^{m-1} \to \mathbf{R}^n \setminus 0$ only depends on the gradient homotopy class of $v(x, y) = \mathrm{grad}_x f(x, y)$.



Assume that $\pi_m(\mathbf{grad})$ is injective and take a $v(x, y) = \mathrm{grad}_x f(x, y)$ homotopic to a constant vector field. We can assume that $f$ is smooth and that $v$ restricted to $\mathbf{B}^n \times \mathbf{S}^{m-1}$ does not depend on $y$. We denote this map as above by $w$. We know that $v$ is homotopic to a constant vector field, say $\partial/\partial x_1$. Let

$$V\colon (\mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I},\ \partial(\mathbf{B}^n \times \mathbf{B}^m) \times \mathbf{I}) \to (\mathbf{R}^n, \mathbf{R}^n \setminus 0)$$

be such a homotopy. By the same arguments as at the beginning of the proof, we can change $V$ in such a way that $V|_{\mathbf{B}^n \times \mathbf{S}^{m-1} \times \mathbf{I}}$ does not depend on $y$ (in particular, it is gradient with respect to $x$).

Let $\tilde{f}\colon \mathbf{B}^n \times \mathbf{B}^m \to \mathbf{R}$ be defined as follows:

(1) $\tilde{f}(x, y) = f(x, 2y)$ if $\|y\| \leq 1/2$.
(2) $\mathrm{grad}_x \tilde{f}(x, y) = V(x, y/\|y\|, 2\|y\| - 1)$ if $\|y\| \geq 1/2$.
(3) $\tilde{f}|_{\mathbf{B}^n \times \mathbf{S}^{m-1}} \equiv x_1$.

Then $\tilde{f}$ is gradient homotopic to $f$ and $V$ can be considered as an (ordinary) homotopy between $\mathrm{grad}_x \tilde{f}$ and $\partial/\partial x_1$.

The function $\tilde{f}$ induces, via the identification $(\mathbf{S}^m, *) \simeq (\mathbf{B}^m/\mathbf{S}^{m-1}, \mathbf{S}^{m-1})$, a function $h\colon \mathbf{B}^n \times \mathbf{S}^m \to \mathbf{R}$ satisfying $h(x, *) \equiv x_1$. So, $h$ defines $\sigma_h \in \pi_m(\mathscr{A}, *)$ which gives (by means of $V$, see above) a trivial element of $\pi_m(\mathscr{V}, *)$. By assumption, $\sigma_h$ is trivial in $\pi_m(\mathscr{A}, *)$. Let $H\colon \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{R}$ be a homotopy between $h$ and the constant map. Let

$$\alpha\colon \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{B}^n \times (\mathbf{B}^m/\mathbf{S}^{m-1}) \times \mathbf{I} = \mathbf{B}^n \times \mathbf{S}^m \times \mathbf{I}$$

be the canonical projection. Then $H \circ \alpha$ is a gradient homotopy between $\tilde{f}$ and a constant vector field.

Now we assume that $m < n$ and every family of gradient vector fields which is homotopic to a constant vector field is gradient homotopic to a constant vector field. Let $h\colon (\mathbf{S}^m, *) \to (\mathscr{A}, *)$ be such that $\mathrm{grad} \circ h$ is homotopic to a constant map. We can treat $h$ as a function on $\mathbf{B}^n \times \mathbf{S}^m$. By the identification $(\mathbf{S}^m, *) \simeq (\mathbf{B}^m/\mathbf{S}^{m-1}, \mathbf{S}^{m-1})$, $h$ gives a function $f$ on $\mathbf{B}^n \times \mathbf{B}^m$ such that $v(x, y) = \mathrm{grad}_x f(x, y)$ is homotopic, by means of a homotopy $V\colon \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{R}^n$, to a constant vector field. By assumption, we can find a gradient homotopy $F\colon \mathbf{B}^n \times \mathbf{B}^m \times \mathbf{I} \to \mathbf{R}$ such that

(1) $F(x, y, 0) \equiv f(x, y)$ (in particular $F(x, y, 0) \equiv x_1$ for $\|y\| = 1$),
(2) $F(x, y, 1) \equiv x_1$,
(3) $\mathrm{grad}_x F(x, y, t) \neq 0$ if $(x, y) \in \partial(\mathbf{B}^n \times \mathbf{B}^m)$.

In order to prove the proposition it suffices to change $F$ in such a way that it equals identically $x_1$ for $\|y\| = 1$.

Using the Hadamard Lemma, as at the beginning of the proof, we change $F$ in such a way that $\mathrm{grad}_x F(x, y, t)$ does not depend on $x$ for $\|y\| = 1$ and still equals $x_1$ on $\mathbf{B}^n \times \mathbf{S}^{m-1} \times \{0, 1\}$. Put $w(y, t) = \mathrm{grad}_x F(x, y, t)$ for $\|y\| = 1$. Adding to $F$ a function not depending on $x$ we change it so that now $F(0, y, t) \equiv 0$. This implies $F(x, y, t) = \langle x, w(y, t) \rangle$ if $\|y\| = 1$. By the assumption on $m$ and $n$, there exists a homotopy $W\colon \mathbf{S}^{m-1}\times\mathbf{I}\times\mathbf{I}\to\mathbf{R}^n\setminus 0$ between $w$ and $\partial/\partial x_1$ which equals identically $\partial/\partial x_1$ on $\mathbf{S}^{m-1}\times\{0, 1\}$. Now, in order to obtain a homotopy with the required properties, it suffices to glue $F$ to $\langle x, w(y, t)\rangle$. One can do it, for example, as follows. Choose a parametrization $\alpha\colon \mathbf{I}\to\mathbf{I}\times\mathbf{I}$ of the union of three sides of the square, for example

$$\alpha(t)=\begin{cases}(0, 3t) & \text{if } t\leqslant 1/3,\\ (3t-1, 1) & \text{if } 1/3\leqslant t\leqslant 2/3,\\ (1, 3-3t) & \text{if } 2/3\leqslant t.\end{cases}$$

Now the formula for the homotopy can be written as follows:

$$\tilde{F}(x, y, t)=$$
$$\begin{cases}F\big(x, (1+\alpha_2(t))\cdot y, \alpha_1(t)\big) & \text{if } \|y\|\leqslant(1+\alpha_2(t))^{-1},\\ \big\langle x, W\big(y/\|y\|, (1+\alpha_2(t))\cdot\|y\|-1, \alpha_1(t)\big)\big\rangle & \text{otherwise. } \blacksquare\end{cases}$$

**Acknowledgements.** I would like to thank Professor Kazimierz Gęba for suggesting the problem and his help during the preparation of this paper. The revised version of the paper was written during my stay as a fellow of the Alexander von Humboldt Foundation at the Max-Planck-Institut für Mathematik in Bonn. I would like to thank them both for their kind hospitality.

**References**

[A] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York 1978.
[H] M. W. Hirsch, *Differential Topology*, Springer, New York 1976.
[M] J. W. Milnor, *Topology from the Differentiable Viewpoint*, Univ. Press of Virginia, Charlottesville 1965.
[N] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, New York Univ. lecture notes, 1974.
[S] R. M. Switzer, *Algebraic Topology—Homotopy and Homology*, Springer, New York 1975.

INSTYTUT MATEMATYKI UNIWERSYTETU GDAŃSKIEGO
INSTITUTE OF MATHEMATICS, UNIVERSITY OF GDAŃSK
Wita Stwosza 57, 80-952 Gdańsk, Poland

*Received November 7, 1988*
*Revised version March 7, 1989*
(2507)

STUDIA MATHEMATICA, T. XCVI (1990)

# An operator on a separable Hilbert space with all polynomials hypercyclic

by

BERNARD BEAUZAMY (Paris)

**Abstract.** We construct an operator $T$ on a separable Hilbert space with one hypercyclic point $x_0$ and such that for any polynomial $p$ with complex coefficients the point $p(T)x_0$ is also hypercyclic.

Let $x_0$ be a point in a Banach space $E$, and let $T$ be a linear operator on $E$. The *orbit* of $x_0$ under $T$ is just the set of iterates

$$F_{x_0}=\{x_0, Tx_0, T^2x_0, \ldots\}.$$

The point $x_0$ is said to be *cyclic* for $T$ if the vector space generated by $F_{x_0}$ is dense in $E$, and *hypercyclic* if $F_{x_0}$ itself is dense in $E$.

The invariant subspace problem, solved negatively by P. Enflo in Banach spaces (see P. Enflo [5]) and still unsolved in Hilbert spaces, can of course be rephrased as: Let $T$ be an operator; does there exist (besides 0) a point $x_0$ which is not cyclic? In Enflo's example, all nonzero points are cyclic.

So one is naturally led to an investigation of the regularity of the orbits of a linear operator. Trying to find points for which the orbit is regular (meaning, for instance, that $\|T^n x\|\to\infty$ as $n\to\infty$) was done in our book [4], Chap. 3. Here, conversely, we concentrate on irregular orbits: those of hypercyclic points, and try to construct operators with as many hypercyclic points as possible.

The first result in this direction was obtained by S. Rolewicz [8], who constructed on $l_p$ $(1\leqslant p<\infty)$ or $c_0$ an operator with one hypercyclic point. Of course, its iterates are also hypercyclic, but if one considers for instance $(x_0+Tx_0)/2$, nothing says that this vector is still hypercyclic. Indeed, the construction can be modified in order to provide also a finite number of such vectors, but only a finite number.

The following question was raised by P. Halmos [6]: can one produce an operator, in a separable Hilbert space, for which the set of hypercyclic points would contain a vector space? We solve this question here. We do not prove that all points are cyclic, so our example might still have invariant subspaces. We do not know if this is the case or not.