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Added in proof (January 1990). D. Werner (*Remarks on M-ideals of compact operators*, to appear in Quart. J. Math. Oxford) proved that the result of C.-M. Cho and W. B. Johnson also holds for subspaces of c_0-sums of finite-dimensional spaces. A similar result is given by E. Oja (C. R. Acad. Sci. Paris 309 (1989), 983–968).

Very recently, P. G. Casazza and N. J. Kalton (*Notes on approximation properties in separable Banach spaces*, preprint) have introduced the notion of μ-ideal which is more general than that of M-ideal and have given a characterization of separable reflexive Banach spaces X with AP for which $\mathscr{K}(X)$ is a μ-ideal in $\mathscr{L}(X)$.

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Partial differential operators of infinite order with constant coefficients on the space of analytic functions on the polydisc

by

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Abstract. For a componentwise radial plurisubharmonic function $p\colon \mathbf{C}^N \to \mathbf{R}_+$ satisfying some technical conditions we consider the (DFN)-space $A_{p,1}(\mathbf{C}^N) := \{f \in A(\mathbf{C}^N) \mid \exists k \in \mathbf{N}\colon \|f\|_k = \sup_{z\in\mathbf{C}^N} |f(z)| e^{-(1-1/k)p(z)} < \infty\}$ of analytic functions on $\mathbf{C}^N$. If we put $A_p^0(\mathbf{C}^N) := \{F \mid \forall k \in \mathbf{N}\colon F^k \in A_{p,1}(\mathbf{C}^N)\}$ then $A_{p,1}(\mathbf{C}^N)$ is an $A_p^0(\mathbf{C}^N)$-module such that $F \cdot A_{p,1}(\mathbf{C}^N)$ is a closed subspace of $A_{p,1}(\mathbf{C}^N)$ for each $F \in A_p^0(\mathbf{C}^N)$. We prove that $F \cdot A_{p,1}(\mathbf{C}^N)$ is a complemented subspace of $A_{p,1}(\mathbf{C}^N)$ for each $F \in A_p^0(\mathbf{C}^N)$ iff the strong dual $A_{p,1}(\mathbf{C}^N)'_b$ has the linear topological invariant $(\underline{\mathrm{DN}})$ iff $A_{p,1}(\mathbf{C}^N)$ itself is a complemented subspace of a corresponding weighted (LB)-space $L^2_{p,1}(\mathbf{C}^N)$ of locally square integrable functions on $\mathbf{C}^N$. Applying this result to the function $p(z) = \sum_{j=1}^N |z_j|$, $z \in \mathbf{C}^N$, we deduce that each nonzero linear partial differential operator of infinite order with constant coefficients on the Fréchet space $A(\Delta)$ of all analytic functions on the unit polydisc Δ in $\mathbf{C}^N$ admits a continuous linear right inverse. In our approach we use a sequence space representation of $A_{p,1}(\mathbf{C}^N)'_b$ and elementary function theory to give all the projections by explicit formulas.

For a plurisubharmonic function p on $\mathbf{C}^N$ denote by $A_{p,1}(\mathbf{C}^N)$ the space of all entire functions f satisfying $|f(z)| \leqslant A e^{Bp(z)}$ for some $A > 0$, $1 > B > 0$ depending on f. If we allow arbitrarily large $B > 0$, then we get an algebra denoted by $A_p(\mathbf{C}^N)$. Both spaces, endowed with their natural inductive limit topology, are (DFN)-spaces, provided that p satisfies some technical conditions.

Recently Meise and Taylor ([12], [13]) showed for radial weights p that each principal ideal of $A_p(\mathbf{C}^N)$ is complemented if and only if the strong dual $A_p(\mathbf{C}^N)'_b$ has the linear topological invariant (DN).

In the present paper we find for componentwise radial weights p that each subspace $F \cdot A_{p,1}(\mathbf{C}^N)$ of $A_{p,1}(\mathbf{C}^N)$ is complemented if and only if the strong dual $A_{p,1}(\mathbf{C}^N)'_b$ has the linear topological invariant $(\underline{\mathrm{DN}})$ (see Wagner [21] and Vogt [19]). Our proof also gives rise to a new and more elementary proof of the above-mentioned result of Meise and Taylor.

To state an application of our result, we denote by $A(\Delta)$ the Fréchet space of all analytic functions on the unit polydisc Δ in $\mathbf{C}^N$. Then each nonzero linear partial differential operator

$$L\colon A(\Delta)\to A(\Delta),\qquad L[f]=\sum_{\alpha\in\mathbf{N}_0^N} a_\alpha D^\alpha f,$$

of infinite order with constant coefficients admits a continuous linear right inverse, since $A(\Delta)'_b$ is isomorphic to $A_{p,1}(\mathbf{C}^N)$, $p(z)=\sum_{i=1}^N |z_i|$, via Fourier–Borel transform.

To prove the sufficiency of $(\underline{\mathrm{DN}})$, first we note that – since p is componentwise radial – $A_{p,1}(\mathbf{C}^N)'_b$ is in a natural way isomorphic to a Köthe sequence space $\lambda(A)$, $A=(a_{\alpha,k})_{\alpha\in\mathbf{N}_0^N,k\in\mathbf{N}}$. We give a characterization of $A_{p,1}(\mathbf{C}^N)'_b$ having $(\underline{\mathrm{DN}})$ in terms of A and p, which says that we may assume $a_{\alpha,k}=e^{(1-1/k)p(r_\alpha)}/r_\alpha^\alpha$ with appropriate $r_\alpha\in\mathbf{R}_+^N$. Now, for a given nontrivial subspace $F\cdot A_{p,1}(\mathbf{C}^N)$ we apply the classical minimum modulus theorem for analytic functions to construct sets T_α, $\alpha\in\mathbf{N}_0^N$, close to the distinguished boundary of the polydiscs $\{z\in\mathbf{C}^N\,|\,|z_i|<r_{\alpha,i},\ i=1,\ldots,N\}$ such that the modulus of F is sufficiently large on T_α. Hence from our description of $a_{\alpha,k}$ and by the nuclearity of $\lambda(A)$ we conclude that a continuous projection $P\colon A_{p,1}(\mathbf{C}^N)\to F\cdot A_{p,1}(\mathbf{C}^N)$ is given by the formula

$$P[f](z)=F(z)\cdot\sum_{\alpha\in\mathbf{N}_0^N}\left(\frac{1}{2\pi i}\right)^N\int_{T_\alpha}\frac{f(\zeta)}{F(\zeta)\zeta^{\alpha+1}}\,d\zeta\cdot z^\alpha.$$

To prove the necessity of $(\underline{\mathrm{DN}})$ we make a reduction to the case $N=1$. Using our characterization of the property $(\underline{\mathrm{DN}})$, if $A_{p,1}(\mathbf{C})'_b$ does not have $(\underline{\mathrm{DN}})$, we construct an entire function F with $F\cdot A_{p,1}(\mathbf{C})\subset A_{p,1}(\mathbf{C})$ such that $F\cdot A_{p,1}(\mathbf{C})$ is not complemented. To show this we use a sequence space representation of $A_{p,1}(\mathbf{C})/F\cdot A_{p,1}(\mathbf{C})$ and conclude by standard arguments.

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I. Preliminaries. We use standard notation from complex analysis and functional analysis. We write $\mathbf{R}_+:=[0,\infty[$ and $|z|:=\sum_{i=1}^N|z_i|$ for $z\in\mathbf{C}^N$.

1. Definition. A continuous plurisubharmonic function $p\colon\mathbf{C}^N\to\mathbf{R}_+$ is called a *weight function* if

(1) $\log(1+|z|)=o(p(z))$, $|z|\to\infty$.
(2) $p(2z)=O(p(z))$, $\sup_{|w|\le 1}p(z+w)=(1+o(1))p(z)$, $|z|\to\infty$.
(3) $p(z)=p(|z_1|,\ldots,|z_N|)$, $z\in\mathbf{C}^N$.

2. Remark. For each weight function p on $\mathbf{C}^N$ we have

$$\lim_{C\to 1+}\ \limsup_{|r|\to\infty} p(Cr)/p(r)=1.$$

Proof. Apply [14], Lemma 1.10, to $\omega(t)=p(rt)$, $t\in\mathbf{R}_+$, $r\in\mathbf{R}_+^N$.

3. Definition. Let p be a weight function on $\mathbf{C}^N$, let $\eta\in\{1,\infty\}$ and let $(\eta_k)_{k\in\mathbf{N}}\uparrow\eta$ be a strictly increasing sequence of positive numbers. We define the linear spaces

$$A_{p,\eta}(\mathbf{C}^N)=\{f\in A(\mathbf{C}^N)\,|\,\|f\|_k:=\sup_{z\in\mathbf{C}^N}|f(z)|e^{-\eta_k p(z)}<\infty\ \text{for some}\ k\in\mathbf{N}\},$$

$$L^2_{p,\eta}(\mathbf{C}^N)=\Big\{f\in L^2_{\mathrm{loc}}(\mathbf{C}^N)\,\Big|\,\Big(\int_{\mathbf{C}^N}(|f(z)|e^{-\eta_k p(z)})^2\,dm_{2N}\Big)^{1/2}<\infty\ \text{for some}\ k\in\mathbf{N}\Big\},$$

where $A(\mathbf{C}^N)$ denotes the space of all entire functions. $A_{p,\eta}(\mathbf{C}^N)$ is a linear subspace of $L^2_{p,\eta}(\mathbf{C}^N)$. Endowed with their natural inductive limit topology, $A_{p,\eta}(\mathbf{C}^N)$ and $L^2_{p,\eta}(\mathbf{C}^N)$ are (LB)-spaces. $A_{p,\eta}(\mathbf{C}^N)$ is even a (DFN)-space, i.e. the strong dual of a nuclear Fréchet space. Furthermore, we define

$$A_p^0(\mathbf{C}^N):=\{F\in A(\mathbf{C}^N)\,|\,F^k\in A_{p,1}(\mathbf{C}^N)\ \text{for each}\ k\in\mathbf{N}\}.$$

Obviously

$$A_p^0(\mathbf{C}^N)\cdot A_{p,1}(\mathbf{C}^N)\subset A_{p,1}(\mathbf{C}^N).$$

The algebra $A_{p,\infty}(\mathbf{C}^N)$ is usually called $A_p(\mathbf{C}^N)$.

4. Definition. Let $A=(a_{\alpha,k})_{\alpha\in\mathbf{N}_0^N,k\in\mathbf{N}}$ be a matrix of positive numbers with

$$a_{\alpha,k}\le a_{\alpha,k+1}\qquad\text{for all}\ k\in\mathbf{N}\ \text{and}\ \alpha\in\mathbf{N}_0^N.$$

Then

$$\lambda(A):=\Big\{(x_\alpha)_{\alpha\in\mathbf{N}_0^N}\in\mathbf{C}^{\mathbf{N}_0^N}\,\Big|\,\|x\|_k:=\sum_\alpha|x_\alpha|a_{\alpha,k}<\infty\ \text{for all}\ k\in\mathbf{N}\Big\}$$

is called the *Köthe sequence space* determined by A.

In particular, let $\eta\in]0,\infty]$ and let $(\eta_k)_{k\in\mathbf{N}}\uparrow\eta$ be a strictly increasing sequence of positive numbers. If $a_{\alpha,k}=\eta_k^{\beta_\alpha}$, $\alpha\in\mathbf{N}_0^N$, $k\in\mathbf{N}$, for a sequence $\beta=(\beta_\alpha)_{\alpha\in\mathbf{N}_0^N}$ in $\mathbf{R}_+$, then we put $\Lambda_\eta(\beta)=\lambda(A)$ and call this space a *power series space of type* η. Note that $\Lambda_\eta(\beta)\cong\Lambda_1(\beta)$ for each $\eta\in]0,\infty[$.

5. Proposition. *For a weight function p on $\mathbf{C}^N$, for $\eta\in\{1,\infty\}$ and $(\eta_k)_{k\in\mathbf{N}}\uparrow\eta$ put*

$$a_{\alpha,k}:=\inf_{r\in\mathbf{R}_+^N}\frac{e^{\eta_k p(r)}}{r^\alpha},\quad \alpha\in\mathbf{R}_+^N,\ k\in\mathbf{N},\qquad A:=(a_{\alpha,k})_{\alpha\in\mathbf{N}_0^N,k\in\mathbf{N}}.$$

Then the map

$$A_{p,\eta}(\mathbf{C}^N)\to\lambda(A)'_b,\qquad f\mapsto(f^{(\alpha)}(0)/\alpha!)_{\alpha\in\mathbf{N}_0^N},$$

is a (linear topological) isomorphism.

Proof. $\eta = 1$: Use Cauchy's integral formula. On the other hand, note that by Definition 1 for each $k \in \mathbf{N}$ there is $C > 0$ such that

$$\sum_{\alpha \in \mathbf{N}_0^N} a_{\alpha,k} r^\alpha \leqslant C e^{\eta_{k+1} p(r)} \quad \text{for all } r \in \mathbf{R}_+^N.$$

For $\eta = \infty$ see [12], 3.2.

For examples of algebras $A_p(\mathbf{C}^N)$ we refer to [8] and [12].

6. Examples. (1) For the case $N = 1$:

(a) If p is a weight function on $\mathbf{C}$ such that there exists $C > 1$ with $2p(r) \leqslant p(Cr) + C$ for all $r \in \mathbf{R}_+$, then $A_{p,1}(\mathbf{C}) \cong \Lambda_1(\beta)'_b$, $\beta = (j)_{j \in \mathbf{N}_0}$ (see e.g. [8], 2.9).

(b) For $p(z) = \log(1+|z|)^s$, $s > 1$: $A_{p,1}(\mathbf{C}) \cong \Lambda_1(\beta)'_b$, $\beta = (j^{s/(s-1)})_{j \in \mathbf{N}_0}$.

(c) For $p(z) = \log(1+|z|) \log\log(e+|z|)$: $A_{p,1}(\mathbf{C}) \cong \lambda(A)'_b$,

$$A = \left(\exp\left(-\exp\left(\frac{j}{1-1/k}\right)\right)\right)_{j \in \mathbf{N}_0, k \in \mathbf{N}}$$

(see [3], Example 5 (5)).

(2) For the case $N > 1$:

(a) If p_j, $j = 1, \ldots, N$, are weight functions on $\mathbf{C}$, then

$$p(z) = \sum_{j=1}^{N} p_j(z_j), \quad z = (z_1, \ldots, z_N) \in \mathbf{C}^N,$$

is a weight function on $\mathbf{C}^N$. Furthermore, for $\eta \in \{1, \infty\}$ we have

$$A_{p,\eta}(\mathbf{C}^N) = \bigotimes_{j=1}^{N} A_{p_j,\eta}(\mathbf{C}).$$

(b) For $p(z) = |z|$ we get by Examples (1)(a) and (2)(a)

$$A_{p,1}(\mathbf{C}^N) \cong \Lambda_1(|\alpha|)'_b.$$

II. Linear topological invariants. We make use of the following linear topological invariants (DN) and $(\underline{\mathrm{DN}})$ (see [17], 1.1, [21], 1.2, [18], 2.1).

7. Definition. A metrizable locally convex space E with a fundamental system of seminorms $(\| \ \|_k)_{k \in \mathbf{N}}$ has *property* $(\underline{\mathrm{DN}})$ if the following holds: There exists $l \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$, $\varepsilon > 0$ and $D > 0$ with $\| \ \|_k^{1+\varepsilon} \leqslant D \| \ \|_l^\varepsilon \| \ \|_n$.

E has *property* (DN) if E has property $(\underline{\mathrm{DN}})$ and we can always choose $\varepsilon = 1$.

$(\underline{\mathrm{DN}})$ and (DN) are linear topological invariants which are inherited by topological subspaces (see [21], 1.2, [17], 1.1, and [18], 2.2).

In the sequel we need the following characterizations of these invariants.

8. Lemma. *Let p be a weight function on $\mathbf{C}^N$ and let $\alpha \in \mathbf{N}_0^N$ be fixed. For $t > 0$ let $r(t) \in \mathbf{R}_+^N$ satisfy*

$$e^{p(r(t))}/r(t)^{t\alpha} = \inf_{r \in \mathbf{R}_+^N} e^{p(r)}/r^{t\alpha}.$$

Then the function $t \mapsto p(r(t))$ is increasing.

Proof. By the hypothesis we have

$$\exp\left(\frac{p(r(t)) - p(r)}{t}\right) \leqslant \left(\frac{r(t)}{r}\right)^\alpha \quad \text{for all } t > 0 \text{ and } r \in \mathbf{R}_+^N.$$

From this we get for $t_2 > t_1$

$$\exp\left(\frac{p(r(t_2)) - p(r(t_1))}{t_2}\right) \leqslant \left(\frac{r(t_2)}{r(t_1)}\right)^\alpha \leqslant \exp\left(\frac{p(r(t_2)) - p(r(t_1))}{t_1}\right),$$

hence $p(r(t_2)) \geqslant p(r(t_1))$.

9. Lemma. *Let p be a weight function on $\mathbf{C}^N$. Then the following assertions are equivalent:*

(i) *$A_{p,1}(\mathbf{C}^N)'_b$ has property $(\underline{\mathrm{DN}})$.*

(ii) *There exists $l \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$, $\varepsilon > 0$ and $D > 0$ with*

$$\frac{\alpha_{\alpha,k}}{a_{\alpha,n}} \leqslant D \left(\frac{a_{\alpha,l}}{a_{\alpha,k}}\right)^\varepsilon \quad \textit{for all } \alpha \in \mathbf{R}_+^N \ (a_{\alpha,k} \textit{ as in Proposition } 5).$$

(iii) *There exists a sequence $(r_\alpha)_{\alpha \in \mathbf{N}_0^N}$ in $\mathbf{R}_+^N$ such that for each $k \in \mathbf{N}$ there exists $n \in \mathbf{N}$ and $D > 0$ with*

$$e^{(1-1/k)p(r_\alpha)}/r_\alpha^\alpha \leqslant D a_{\alpha,n} \quad \textit{for all } \alpha \in \mathbf{N}_0^N.$$

(iv) *$A_{p,1}(\mathbf{C}^N)'_b$ is isomorphic to some $\Lambda_1(\beta)$ (see Definition 4).*

Proof. (i)$\Rightarrow$(ii). By Proposition 5, $A_{p,1}(\mathbf{C}^N)'_b \cong \lambda(A)$ and $A = (a_{\alpha,k})_{\alpha \in \mathbf{N}_0^N, k \in \mathbf{N}}$. By [21], 1.6, or [18], 4.1, $\lambda(A)$ has property $(\underline{\mathrm{DN}})$ iff (ii) holds for $\alpha \in \mathbf{N}_0^N$. Because of Definition 1(1) and since w.l.o.g. $p|\Delta \equiv 0$ on the unit polydisc Δ, (ii) is valid even for $\alpha \in \mathbf{R}_+^N$.

(ii)$\Rightarrow$(iii). Put $\eta_k := 1 - 1/k$, $k \geqslant 1$. For $\alpha \in \mathbf{R}_+^N$ we choose $r_\alpha \in \mathbf{R}_+^N$ with

$$e^{p(r_\alpha)}/r_\alpha^\alpha = \inf_{r \in \mathbf{R}_+^N} e^{p(r)}/r^\alpha =: a_{\alpha,\infty}. \tag{1}$$

For $l \in \mathbf{N}$ as in (ii) choose $m \in \mathbf{N}$ with $\eta_m^2 \geqslant \eta_l$. Put $C := \eta_m/\eta_l$. Applying (ii) to $k = m$ we get $\varepsilon > 0$ and $D > 0$ with

$$a_{\alpha,m}/a_{\alpha,\infty} \leqslant D(a_{\alpha,l}/a_{\alpha,m})^\varepsilon \quad \text{for all } \alpha \in \mathbf{R}_+^N. \tag{2}$$

Now, for $k \geqslant 1$ given, put $n := (C/\varepsilon)k$ and $\delta := \varepsilon m/(Ck)$.

With the abbreviation $s_\alpha := r_{\alpha/C\eta_m}$ we get for all $\alpha \in \mathbf{R}_+^N$

$$(3) \qquad e^{-(1-\eta_m)p(r_\alpha)} \leqslant \left(\frac{e^{p(s_\alpha)}}{s_\alpha^{\alpha/C}} \Big/ a_{\alpha/C,\infty}\right) e^{-(1-\eta_m)p(s_\alpha)}$$

$$= \frac{(a_{\alpha/C\eta_m,\infty})^{\eta_m}}{a_{\alpha/C,\infty}} = \frac{a_{\alpha/C,m}}{a_{\alpha/C,\infty}},$$

since $C\eta_m \geqslant 1$ and hence $p(s_\alpha) \leqslant p(r_\alpha)$ by Lemma 8.

Because of $C^2\eta_l \geqslant 1$ we get for all $\alpha \in \mathbf{R}_+^N$

$$a_{C\alpha,\infty} \leqslant (a_{C\alpha/C^2\eta_l,\infty})^{C^2\eta_l} = (a_{\alpha/\eta_m,\infty})^{C\eta_m} = (a_{\alpha,m})^C,$$

$$a_{C\alpha,m} = (a_{C\alpha/\eta_m,\infty})^{\eta_m} = (a_{\alpha/\eta_l,\infty})^{C\eta_l} = (a_{\alpha,l})^C,$$

hence

$$(4) \qquad (a_{\alpha,l}/a_{\alpha,m})^C \leqslant a_{C\alpha,m}/a_{C\alpha,\infty} \quad \text{for all } \alpha \in \mathbf{R}_+^N.$$

Since $\eta_n = 1-\delta+\delta\eta_m$, we have for all $r,\ \alpha \in \mathbf{R}_+^N$

$$a_{\alpha,\infty}^{1-\delta} a_{\alpha,m}^{\delta} \leqslant \frac{e^{(1-\delta)p(r)}}{r^{(1-\delta)\alpha}} \frac{e^{\delta\eta_m p(r)}}{r^{\delta\alpha}} \leqslant \frac{e^{\eta_n p(r)}}{r^\alpha},$$

hence

$$(5) \qquad a_{\alpha,\infty}(a_{\alpha,m}/a_{\alpha,\infty})^\delta \leqslant a_{\alpha,n} \quad \text{for all } \alpha \in \mathbf{R}_+^N.$$

Put $B := D^{(C/\varepsilon)\delta}$. Combining (2)–(5) we get for all $\alpha \in \mathbf{R}_+^N$

$$\frac{e^{(1-1/k)p(r_\alpha)}}{r_\alpha^\alpha} = a_{\alpha,\infty} e^{-(1-\eta_m)(C/\varepsilon)\delta p(r_\alpha)} \leqslant a_{\alpha,\infty}\left(\frac{a_{\alpha/C,m}}{a_{\alpha/C,\infty}}\right)^{(C/\varepsilon)\delta}$$

$$\leqslant Ba_{\alpha,\infty}\left(\frac{a_{\alpha/C,l}}{a_{\alpha/C,m}}\right)^{C\delta} \leqslant Ba_{\alpha,\infty}\left(\frac{a_{\alpha,m}}{a_{\alpha,\infty}}\right)^{\delta} \leqslant Ba_{\alpha,n}.$$

(iii)⇒(iv). Because of the trivial estimate

$$a_{\alpha,n} \leqslant \frac{e^{(1-1/n)p(r_\alpha)}}{r_\alpha^\alpha} \quad \text{for all } \alpha \in \mathbf{N}_0^N \text{ and } n \in \mathbf{N},$$

and the hypothesis (iii) we have

$$A_{p,1}(\mathbf{C}^N)'_b \cong \lambda(A) = \lambda\left(\left(\frac{e^{(1-1/k)p(r_\alpha)}}{[illegible]}\right)_{\alpha,\cdot}\right) \sim [illegible]),$$

where $\beta = (p(r_\alpha))_{\alpha \in \mathbf{N}_0^N}$.

(iv)⇒(i). By [21], 1.8, or [18], 2.3, the spaces $\Lambda_1(\beta)$ have property (DN).

Remark. The proof of Lemma 9 makes use of ideas from [21], 1.22, and [3], 7. Furthermore, the proof shows that $\sup_\alpha \beta_{C\alpha}/(1+\beta_\alpha) < \infty$ for some $C > 1$.

10. Lemma. *Let p be a weight function on $\mathbf{C}^N$. Then the following assertions are equivalent:*

(i) *$A_p(\mathbf{C}^N)'_b$ has property (DN).*

(ii) *There exists $l \in \mathbf{N}$ such that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $D > 0$ with*

$$a_{\alpha,k}^2 \leqslant Da_{\alpha,l}a_{\alpha,n} \quad \textit{for all } \alpha \in \mathbf{N}_0^N \ (a_{\alpha,k} \textit{ as in Proposition 5}).$$

(iii) *There exists a sequence $(r_\alpha)_{\alpha\in\mathbf{N}_0^N}$ in $\mathbf{R}_+^N$ such that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $D > 0$ with*

$$e^{kp(r_\alpha)}/r_\alpha^\alpha \leqslant Da_{\alpha,n} \quad \textit{for all } \alpha \in \mathbf{N}_0^N.$$

(iv) *$A_p(\mathbf{C}^N)'_b$ is isomorphic to some $\Lambda_\infty(\beta)$ (see Definition 4).*

Proof. (i)⇒(ii). Proposition 5 and [17], 2.3.

(ii)⇒(iii). Let $l \in \mathbf{N}$ be as in (ii). For $\alpha \in \mathbf{N}_0^N$ we choose $r_\alpha \in \mathbf{R}_+^N$ with

$$e^{(l+1)p(r_\alpha)}/r_\alpha^\alpha = \inf_{r\in\mathbf{R}_+^N} e^{(l+1)p(r)}/r^\alpha = a_{\alpha,l+1}.$$

Following an idea from [20], 2.7, we iterate the hypothesis and get: For each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $D > 0$ with $a_{\alpha,l+1}^k \leqslant Da_{\alpha,l}^{k-1}a_{\alpha,n}$ for all $\alpha \in \mathbf{N}_0^N$. From this we conclude that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $D > 0$ with

$$\frac{e^{(l+k)p(r_\alpha)}}{r_\alpha^\alpha} = \frac{e^{(l+1)p(r_\alpha)}}{r_\alpha^\alpha} e^{(k-1)p(r_\alpha)}$$

$$\leqslant a_{\alpha,l+1}\left(\frac{a_{\alpha,l+1}}{a_{\alpha,l}}\right)^{k-1} \leqslant Da_{\alpha,n} \quad \text{for all } \alpha \in \mathbf{N}_0^N.$$

(iii)⇒(iv). The same argument as in Lemma 9.

(iv)⇒(i). By [17], 2.4, the spaces $\Lambda_\infty(\beta)$ have property (DN).

For other characterizations of the property (DN) for $A_p(\mathbf{C}^N)'_b$ we refer to [12], 2.12, 3.1, and [9], 2.11.

III. Complemented subspaces. We need the following application of the classical minimum modulus theorem for entire functions of one variable ([16], Lehrsatz 11):

11. Lemma. *Let $\varphi\colon [1,\infty[\to]0,\infty[$ be continuous, and let $t \mapsto \varphi(t)/t$ be decreasing. We define an unbounded sequence $(R_n)_{n\in\mathbf{N}_0}$ by*

$$R_0 := 1, \quad R_{n+1} := R_n + \varphi(R_n), \quad n \in \mathbf{N}_0.$$

Let X be a locally compact and σ-compact Hausdorff space and let $f\colon X \times \mathbf{C} \to \mathbf{C}$ be a continuous function with $f(x,\cdot) \in A(\mathbf{C})$ for all $x \in X$ and with $f(\cdot,0) \equiv 1$. Then there is a constant $D \geqslant 1$ depending only on $\varphi(1)$, and for each $n \in \mathbf{N}$ there exists a measurable function $S_n\colon X \to]R_{n-1}, R_n[$ such that for all $x \in X$

and all $z \in \mathbf{C}$ *with* $|z| = S_n(x)$

$$\log|f(x, z)| > -D\frac{|z|}{\varphi(|z|)} \sup_{|w|=D|z|} \log|f(x, w)|.$$

Proof. For $n \in \mathbf{N}$ we put

$$C := \varphi(1), \quad \varepsilon := \tfrac{1}{2}\min\{\tfrac{3}{2}e, (8(1+C)\sup_{x>0} e^{-x}x)^{-1}\}, \quad \eta_n := \varepsilon \exp\left(-\frac{R_{n-1}}{\varphi(R_{n-1})}\right).$$

For $x \in X$ and $r \geqslant 0$ we define

$$M(x, r) := \sup_{|w|=r} |f(x, w)|.$$

By [6], Lehrsatz 11, for each $n \in \mathbf{N}$ and all $x \in X$ there are discs with sum of radii less than or equal to $4\eta_n R_n$ such that for all $z \in \mathbf{C}$ with $|z| \leqslant R_n$ and for z outside these discs we have the estimate

$$(1) \qquad \log|f(x, z)| > -H_n \log M(x, 2eR_n), \qquad H_n = 2 + \log\frac{3e}{2\eta_n}.$$

By the assumption we get

$$(2) \qquad \sup_{r \geqslant 1} \varphi(r)/r = \varphi(1) = C,$$

and hence

$$(3) \qquad R_n \leqslant (1+C)R_{n-1} \quad \text{for all } n \in \mathbf{N}.$$

By the choice of η_n, ε and R_n, we see from (3) that for all $n \in \mathbf{N}$

$$\begin{aligned} 2\cdot 4\eta_n R_n &\leqslant 8\eta_n(1+C)R_{n-1} \\ &= \varepsilon 8(1+C)\exp\left(-\frac{R_{n-1}}{\varphi(R_{n-1})}\right)\frac{R_{n-1}}{\varphi(R_{n-1})}\varphi(R_{n-1}) \\ &< \varphi(R_{n-1}) = R_n - R_{n-1}. \end{aligned}$$

Hence, we deduce from (1) that for each $x \in X$ there is a sequence $(S_n(x))_{n\in\mathbf{N}}$ in $]1, \infty[$ such that $R_{n-1} < S_n(x) < R_n$ for all $n \in \mathbf{N}$ and

$$\log|f(x, z)| > -H_n \log M(x, 2eR_n) \quad \text{if } |z| = S_n(x).$$

By (2), for all $n \in \mathbf{N}$, we have

$$H_n = 2 + \log\frac{3e}{2\varepsilon} + \frac{R_{n-1}}{\varphi(R_{n-1})} \leqslant \left(\left(2 + \log\frac{3e}{2\varepsilon}\right)C + 1\right)\frac{R_{n-1}}{\varphi(R_{n-1})}.$$

Since $t \mapsto t/\varphi(t)$ and $t \mapsto M(x, t)$ are increasing, and since we have (3), this implies the desired estimate.

By the assumption, for each $S \in]1, \infty[$ the function

$$H_S\colon X \to \mathbf{R}_+, \quad x \mapsto H_S(x) = \inf_{|z|=S}\Big(|f(x, z)|\big(\sup_{|w|=D|z|}|f(x, w)|\big)^{D|z|/\varphi(|z|)}\Big),$$

is continuous. Hence, for $n \in \mathbf{N}$ and fixed $x_0 \in X$ we may assume that on a neighborhood of x_0 the number $S_n(x)$ does not depend on x. So we can achieve that locally the functions S_n, $n \in \mathbf{N}$, are measurable simple functions.

12. MAIN LEMMA. *Let p be a weight function on $\mathbf{C}^N$ and let $(r_\alpha)_{\alpha\in\mathbf{N}_0^N}$ be a sequence in $\mathbf{R}_+^N$. Then for each $F \in A_p^0(\mathbf{C}^N)$, resp. $F \in A_p(\mathbf{C}^N)$, with $F(0) = 1$, there are open sets U^i in $\mathbf{C}^{i-1}$ ($U^1 := \{0\}$) and measurable functions ϱ_α^i: $U^i \to [1, \infty[$, $\alpha \in \mathbf{N}_0^N$, $1 \leqslant i \leqslant N$, such that the sets*

$$T_\alpha := \{z \in \mathbf{C}^N \,|\, |z_1| = \varrho_\alpha^1, |z_2| = \varrho_\alpha^2(z_1), \ldots, |z_N| = \varrho_\alpha^N(z_1, \ldots, z_{N-1})\}, \quad \alpha \in \mathbf{N}_0^N,$$

have the following properties:

In the case $F \in A_p^0(\mathbf{C}^N)$:

(a) *For each $\varepsilon > 0$ there exists $A > 0$ such that for all $\alpha \in \mathbf{N}_0^N$*

$$|F(z)|^{-1} \leqslant Ae^{\varepsilon p(z)} \quad \textit{for all } z \in T_\alpha.$$

(b) *For each $\varepsilon > 0$ there exists $a > 0$ such that for all $\alpha \in \mathbf{N}_0^N$ and $1 \leqslant i \leqslant N$*

$$r_{\alpha,i} \leqslant |z_i| \leqslant (1+\varepsilon)r_{\alpha,i} + a \quad \textit{for all } z \in T_\alpha,$$

where $r_\alpha = (r_{\alpha,1}, \ldots, r_{\alpha,N})$.

In the case $F \in A_p(\mathbf{C}^N)$:

(a) *There exist $k \in \mathbf{N}$ and $A > 0$ such that for all $\alpha \in \mathbf{N}_0^N$*

$$|F(z)|^{-1} \leqslant Ae^{kp(z)} \quad \textit{for all } z \in T_\alpha.$$

(b) *There exist $B > 0$ and $a > 0$ such that for all $\alpha \in \mathbf{N}_0^N$ and $1 \leqslant i \leqslant N$*

$$r_{\alpha,i} \leqslant |z_i| \leqslant Br_{\alpha,i} + a \quad \textit{for all } z \in T_\alpha.$$

Proof. The case $F \in A_p^0(\mathbf{C}^N)$: For $r \in \mathbf{R}_+^N$ we put

$$M(r) := \sup\{|F(z)| \,|\, |z_i| = r_i, \ 1 \leqslant i \leqslant N\}.$$

Since $F \in A_p^0(\mathbf{C}^N)$, we have $\log M(r) = o(p(r))$, $|r| \to \infty$. Therefore we can choose a continuous decreasing function ψ: $\mathbf{R}_+ \to]0, 1]$ such that

$$(1) \qquad \left(\frac{\log M(r)}{p(r)}\right)^{1/N} = o(\psi(|r|)) \quad \text{and} \quad \psi(t) = o(1).$$

We define

$$\varphi(t) := \psi(t)t, \quad t \in [1, \infty[.$$

Let $(R_n)_{n\in\mathbf{N}_0}$ be as in Lemma 11, and let $1 \leqslant i \leqslant N$ be fixed. We put

$$U^i := \{z \in \mathbf{C}^{i-1} \mid F(z_1, \ldots, z_{i-1}, 0, \ldots, 0) \neq 0\}$$

and apply Lemma 11 to the function $f\colon U^i \times \mathbf{C} \to \mathbf{C}$,

$$f(z_1, \ldots, z_{i-1}; z_i) := F(z_1, \ldots, z_i, 0, \ldots, 0)/F(z_1, \ldots, z_{i-1}, 0, \ldots, 0).$$

We find that there is $D \geqslant 1$ and a sequence of measurable functions

$$(2) \qquad S_n^i\colon U^i \to]R_{n-1}, R_n[, \quad n \in \mathbf{N},$$

such that for all $n \in \mathbf{N}$, all $(z_1, \ldots, z_{i-1}) \in U^i$ and all $z_i \in \mathbf{C}$ with $|z_i| = S_n^i(z_1, \ldots, z_{i-1})$ we have

$$(3) \quad \log|F(z_1, \ldots, z_i, 0, \ldots, 0)| - \log|F(z_1, \ldots, z_{i-1}, 0, \ldots, 0)|$$

$$> -D\frac{|z_i|}{\varphi(|z_i|)} \log\Big(\sup_{|w| = D|z_i|} |f(z_1, \ldots, z_{i-1}; w)|\Big)$$

$$\geqslant -D\frac{|z_i|}{\varphi(|z_i|)} (\log M(|z_1|, \ldots, |z_{i-1}|, D|z_i|, 0, \ldots, 0) - \log F(z_1, \ldots, z_{i-1}, 0, \ldots, 0)|).$$

We put $S_0^i := 0$ and for $\alpha \in \mathbf{N}_0^N$

$$U_n^i(r_{\alpha,i}) := \{z \in U^i \mid S_{n-1}^i(z) \leqslant r_{\alpha,i} < S_n^i(z)\}, \quad n \in \mathbf{N}.$$

Since $\lim_{n\to\infty} R_n = \infty$, $(U_n^i(r_{\alpha,i}))_{n\in\mathbf{N}}$ is a partition of U^i.

Now, for each $\alpha \in \mathbf{N}_0^N$ and $1 \leqslant i \leqslant N$ we can define

$$\varrho_\alpha^i\colon U^i \to]1, \infty[, \quad \varrho_\alpha^i := \sum_{n\in\mathbf{N}} S_n^i 1_{U_n^i(r_{\alpha,i})},$$

i.e. $\varrho_\alpha^i(z) = S_n^i(z)$, where $n \in \mathbf{N}$ is the smallest index with $S_n^i(z) > r_{\alpha,i}$.

Let $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$ and $1 \leqslant i \leqslant N$. If $i = 1$ let $|z_i| = S_n^i$ for some $n \in \mathbf{N}$. If $1 < i \leqslant N$ let $(z_1, \ldots, z_{i-1})$ be in U^i and let $|z_i| = S_n^i(z_1, \ldots, z_{i-1})$ for some $n \in \mathbf{N}$. Since ψ and M are increasing, we get from (3) ($\log^- x := \min\{0, \log x\}$)

$$\log^-|F(z_1, \ldots, z_i, 0, \ldots, 0)|$$

$$> 2D\frac{|z|}{\varphi(|z|)} \log^-|F(z_1, \ldots, z_{i-1}, 0, \ldots, 0)| - D\frac{|z|}{\varphi(|z|)} \log M(D|z_1|, \ldots, D|z_N|),$$

in particular $(z_1, \ldots, z_i) \in U^{i+1}$ if $1 \leqslant i < N$.

By induction on i we get for all $\alpha \in \mathbf{N}_0^N$, all $z \in T_\alpha$ and all $i = 1, \ldots, N$

$$\log^-|F(z_1, \ldots, z_i, 0, \ldots, 0)| > -E_i\left(\frac{|z|}{\varphi(|z|)}\right)^i \log M(D|z_1|, \ldots, D|z_N|),$$

where $E_1 = D_1$ and $E_{i+1} = E_i \cdot 2D + D$ for $i = 1, \ldots, N-1$, hence

$$(4) \qquad \log|F(z)| > -E_N \frac{\log M(D|z_1|, \ldots, D|z_N|)}{\psi(D|z|)^N}.$$

We prove the properties (a) and (b):

(a) By (1) and Definition 1(2) for each $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$(5) \qquad E_N \frac{\log M(Dr)}{\psi(D|r|)^N} \leqslant \varepsilon p(r) + C(\varepsilon) \quad \text{for all } r \in \mathbf{R}_+^N.$$

From (4) and (5) we get $\log|F(z)| \geqslant -\varepsilon p(z) - C(\varepsilon)$ for all $\alpha \in \mathbf{N}_0^N$ and $z \in T_\alpha$.

(b) For $z \in T_\alpha$ and $\alpha \in \mathbf{N}_0^N$ we have $|z_1| = \varrho_\alpha^1 > r_{\alpha,1}$ and $|z_i| = \varrho_\alpha^i(z_1, \ldots, z_{i-1}) > r_{\alpha,i}$ for $2 \leqslant i \leqslant N$, by the definition of ϱ_α^i.

To prove the upper bounds let $\varepsilon > 0$ be fixed and choose $K(\varepsilon) > 0$ according to (1) such that

$$(6) \qquad \varphi(t) \leqslant \frac{\varepsilon}{2(1+\varphi(1))} t + K(\varepsilon) \quad \text{for all } t \in [1, \infty[.$$

Let $\alpha \in \mathbf{N}_0^N$, $1 \leqslant i \leqslant N$ and $z \in T_\alpha$ be given. From (2), (6), the definition of ϱ_α^i and $(R_n)_{n\in\mathbf{N}_0}$ we get with an appropriate $n = n(i; \alpha; z_1, \ldots, z_{i-1}) \in \mathbf{N}$

$$\begin{aligned}|z_i| = \varrho_\alpha^i(z_1, \ldots, z_{i-1}) &= S_n^i(z_1, \ldots, z_{i-1}) - S_{n-1}^i(z_1, \ldots, z_{i-1}) \\ &\quad + S_{n-1}^i(z_1, \ldots, z_{i-1}) \\ &\leqslant R_n - R_{n-2} + r_{\alpha,i} = \varphi(R_{n-1}) + \varphi(R_{n-2}) + r_{\alpha,i} \\ &\leqslant \frac{\varepsilon}{2(1+\varphi(1))} R_{n-1} + \frac{\varepsilon}{2} R_{n-2} + r_{\alpha,i} + 2K(\varepsilon) \\ &\leqslant \varepsilon R_{n-2} + r_{\alpha,i} + 2K(\varepsilon) \\ &\leqslant \varepsilon S_{n-1}^i(z_1, \ldots, z_{i-1}) + r_{\alpha,i} + 2K(\varepsilon) \\ &\leqslant \varepsilon r_{\alpha,i} + r_{\alpha,i} + 2K(\varepsilon) \\ &= (1+\varepsilon) r_{\alpha,i} + 2K(\varepsilon) \quad \text{if } n \geqslant 2,\end{aligned}$$

and

$$|z_i| = S_n^i(z_1, \ldots, z_{i-1}) < R_1 \quad \text{if } n = 1.$$

Hence, we get the assertion with $a = 2K(\varepsilon) + R_1$.

The proof of the case $F \in A_p(\mathbf{C}^N)$ is analogous. However, in this case the choice of the function φ is simpler: Put always $\varphi(t) := 2t$, $t \in [1, \infty[$.

13. Proposition. *Let p be a weight function on $\mathbf{C}^N$, let $\eta = 1$ (resp. $\eta = \infty$), and let $F \in A_p^0(\mathbf{C}^N)$ (resp. $F \in A_p(\mathbf{C}^N)$). Then $F \cdot A_{p,\eta}(\mathbf{C}^N)$ is a closed subspace of $A_{p,\eta}(\mathbf{C}^N)$.*

Proof. The case $\eta = 1$. Fix a sequence $(\eta_k)_{k\in\mathbf{N}}\uparrow 1$. Let $F\in A_p^0(\mathbf{C}^N)\setminus\{0\}$. We may assume that $F(0)=1$, since otherwise we choose $w\in\mathbf{C}^N$, $|w|\leqslant 1$ with $F(w)\neq 0$ and consider the entire function $\tilde{F}(z):=F(z+w)/F(w)$, $z\in\mathbf{C}^N$. For $g\in A_{p,1}(\mathbf{C}^N)$ with $h=g/F\in A(\mathbf{C}^N)$ we have to prove that $h\in A_{p,1}(\mathbf{C}^N)$.

To this end we apply Lemma 12 to $r_\alpha=\alpha$, $\alpha\in\mathbf{N}_0^N$. For fixed $z\in\mathbf{C}^N$ we choose α such that $\alpha_i\leqslant|z_i|<\alpha_i+1$, $1\leqslant i\leqslant N$. By Cauchy's integral formula we have

$$h(z)=\left(\frac{1}{2\pi i}\right)^N\int\limits_{\zeta\in T_{\alpha+2}}\frac{h(\zeta)}{(\zeta-z)^{\mathbf{1}}}d\zeta,$$

where $\mathbf{1}=(1,\ldots,1)$ and $\mathbf{2}=(2,\ldots,2)\in\mathbf{N}_0^N$. Since $g\in A_{p,1}(\mathbf{C}^N)$, we get by Definition 1(1) and by Lemma 12(a) and (b)

$$|h(z)|\leqslant\sup_{\zeta\in T_{\alpha+2}}|h(\zeta)||\zeta^{\mathbf{1}}|=\sup_{\zeta\in T_{\alpha+2}}\left|\frac{g(\zeta)}{F(\zeta)}\right||\zeta^{\mathbf{1}}|\leqslant C\sup_{\zeta\in T_{\alpha+2}}e^{\eta_k p(\zeta)}$$

for some $k\in\mathbf{N}$ and $C\geqslant 1$, not depending on z. By Remark 2 and Lemma 12(b), we obtain from this

$$|h(z)|<\tilde{C}e^{\eta_{k+1}p(z)},$$

for some $\tilde{C}>C$, not depending on z.

The proof of the case $\eta=\infty$ is analogous.

Remark. In the case $\eta=1$, $N=1$, for some particular weights the assertion of Proposition 13 can be found in [5] or [16]. The result concerning the case $\eta=\infty$ is essentially contained in [2], Thm. 7.1.

14. THEOREM. *Let p be a weight function on $\mathbf{C}^N$ such that $A_{p,1}(\mathbf{C}^N)'_b$ has property* (DN). *Then $F\cdot A_{p,1}(\mathbf{C}^N)$ is complemented in $A_{p,1}(\mathbf{C}^N)$ for each $F\in A_p^0(\mathbf{C}^N)$.*

Proof. Fix a sequence $(\eta_k)_{k\in\mathbf{N}}\uparrow 1$. Let $F\in A_p^0(\mathbf{C}^N)\setminus\{0\}$. Applying a translation if necessary, we may assume that $F(0)\neq 0$, w.l.o.g. $F(0)=1$, By the hypothesis and Lemma 9(iii) we get a sequence $(r_\alpha)_{\alpha\in\mathbf{N}_0^N}$ in $\mathbf{R}_+^N$. Applying Lemma 12 to this family we get sets T_α, $\alpha\in\mathbf{N}_0^N$. We claim that the map

$$P\colon A_{p,1}(\mathbf{C}^N)\to F\cdot A_{p,1}(\mathbf{C}^N),$$

$$P[f](z)=F(z)\cdot\sum_{\alpha\in\mathbf{N}_0^N}\left(\frac{1}{2\pi i}\right)^N\int\limits_{\zeta\in T_\alpha}\frac{f(\zeta)}{F(\zeta)\zeta^{\alpha+\mathbf{1}}}d\zeta\cdot z^\alpha,$$

is a continuous linear projection onto $F\cdot A_{p,1}(\mathbf{C}^N)$ ($\mathbf{1}=(1,\ldots,1)\in\mathbf{N}^N$). To show this let $f\in A_{p,1}(\mathbf{C}^N)$ with $\|f\|_k<\infty$ be given. For each $\alpha\in\mathbf{N}_0^N$ put

$$b_\alpha(f):=\left(\frac{1}{2\pi i}\right)^N\int\limits_{\zeta\in T_\alpha}\frac{f(\zeta)}{F(\zeta)\zeta^{\alpha+\mathbf{1}}}d\zeta.$$

By Lemma 12 and Remark 2 there are $A_2\geqslant A_1\geqslant 1$ such that for all $\alpha\in\mathbf{N}_0^N$

$$|b_\alpha(f)|\leqslant\sup_{\zeta\in T_\alpha}\frac{|f(\zeta)|}{|F(\zeta)||\zeta^\alpha|}\leqslant\sup_{\zeta\in T_\alpha}\frac{\|f\|_k e^{\eta_k p(\zeta)}}{|F(\zeta)||\zeta^\alpha|}$$
$$\leqslant A_1\|f\|_k\sup_{\zeta\in T_\alpha}\frac{e^{\eta_{k+1}p(\zeta)}}{r_\alpha^\alpha}\leqslant A_2\|f\|_k\frac{e^{\eta_{k+2}p(r_\alpha)}}{r_\alpha^\alpha}.$$

By Lemma 9(iii) there exist $m\geqslant k+2$ and $A_3\geqslant A_2$ such that we get

$$|b_\alpha(f)|\leqslant A_3\|f\|_k a_{\alpha,m}\quad\text{for all }\alpha\in\mathbf{N}_0^N.$$

As in the proof of Proposition 5 we get for some $A_4\geqslant A_3$, not depending of f,

$$\sum_{\alpha\in\mathbf{N}_0^N}|b_\alpha(f)||z^\alpha|\leqslant A_3\|f\|_k\sum_{\alpha\in\mathbf{N}_0^N}a_{\alpha,m}|z^\alpha|$$
$$\leqslant A_4\|f\|_k e^{\eta_{m+1}p(z)}\quad\text{for all }z\in\mathbf{C}^N.$$

From this we conclude easily that P is continuous.

Remark. The formula for the projection P is motivated by the formula for a continuous linear right inverse of a nonzero convolution operator on $A(\mathbf{C})$, which has been communicated to the author by Prof. B. A. Taylor. This formula is closely related to [4], (43).

15. THEOREM. *Let p be a weight function on $\mathbf{C}^N$ such that $A_p(\mathbf{C}^N)'_b$ has property* (DN). *Then each principal ideal is a complemented subspace of $A_p(\mathbf{C}^N)$.*

Proof. Analogous to the proof of Theorem 14.

Remark. Let p be a *radial* weight function on $\mathbf{C}^N$, i.e. $p(z)=p\big((\sum_{i=1}^N|z_i|^2)^{1/2}\big)$, $z\in\mathbf{C}^N$. Then by Definition 1(2), p is equivalent to the weight function $\tilde{p}$,

$$\tilde{p}(z)=\sum_{l=1}^N p(|z_l|),\qquad z=(z_1,\ldots,z_N)\in\mathbf{C}^N,$$

i.e. $A_p(\mathbf{C}^N)=A_{\tilde{p}}(\mathbf{C}^N)$. Hence, Theorem 15 gives an extension of [13], Cor. 15, where an abstract splitting theorem of Vogt and Wagner (see [20]) has been applied. In the situation of Theorem 14 a corresponding general result is not available. However, in the case $N=1$ one knows more details on the structure of the spaces involved, hence, we can apply a result of Vogt (see [19]) to get a second proof for the one-dimensional part of Theorem 14.

16. PROPOSITION. *Let p be a weight function on $\mathbf{C}$, and let $(\eta_k)_{k\in\mathbf{N}}\uparrow 1$. If $F\in A_p^0(\mathbf{C})$ has precisely the zeros $(a_j)_{j\in\mathbf{N}}$ (counted with respect to multiplicities), then*

$$(A_{p,1}(\mathbf{C})/F\cdot A_{p,1}(\mathbf{C}))'_b\cong\lambda\big((e^{\eta_k p(a_j)})_{j,k\in\mathbf{N}}\big).$$

Proof. Analogous to the proof of [8], 3.7, using the estimates of Lemma 12.

17. Remark. In the case $N = 1$ we give a second proof of Theorem 14: Since $A_{p,1}(\mathbf{C})$ is a (DFN)-space, we have to show for given $F \in A_p^0(\mathbf{C})\setminus\{0\}$ that the following short exact sequence splits:

$$0 \to (A_{p,1}(\mathbf{C})/F \cdot A_{p,1}(\mathbf{C}))'_b \to A_{p,1}(\mathbf{C})'_b \to (F \cdot A_{p,1}(\mathbf{C}))'_b \to 0.$$

W.l.o.g. F has infinitely many zeros. To apply [19], in Definition 3 we fix $\eta_k = 1 - 1/k$, $k \geqslant 1$. Using the notions from [19] it suffices to prove that all the three spaces involved are isomorphic to power series spaces $\lambda((e^{-(1/k)\beta_j})_{j,k\in\mathbf{N}})$ of type 1 in a *linear tame* way. The proof of Proposition 16 shows that this is true for the first one. The proof of Proposition 13 shows that the second and the third one are *tamely* isomorphic. Now, by the proof of Proposition 5, $A_{p,1}(\mathbf{C})'_b$ is *tamely* isomorphic to $\lambda(A)$. By the proof of Lemma 9 (ii)$\Rightarrow$(iii), $\lambda(A)$ is *linear tamely* isomorphic to some space of the prescribed type. Hence by [19] the sequence splits.

By Examples 6(1)(c) and (2)(a) there are weight functions p on $\mathbf{C}^N$ for which $A_{p,1}(\mathbf{C}^N)'_b$ fails to have property ($\underline{\mathrm{DN}}$). We will now prove that for weight functions as in Example 6(2)(a) the hypothesis that $A_{p,1}(\mathbf{C}^N)'_b$ has property ($\underline{\mathrm{DN}}$) is even *necessary* for all subspaces $F \cdot A_{p,1}(\mathbf{C}^N)$, $F \in A_p^0(\mathbf{C}^N)$, to be complemented.

We first consider the case $N = 1$.

18. Lemma. *Let p be a weight function on $\mathbf{C}$ and let $(\eta_k)_{k\in\mathbf{N}} \uparrow 1$. For k, $n \in \mathbf{N}$ and $s > 0$ we put*

$$\varphi_{k,n}(s) := \sup_{0<r<s} \frac{\eta_k p(s) - \eta_n p(r)}{\log(s/r)}, \qquad \Phi_{k,n}(s) := \inf_{r>s} \frac{\eta_k p(s) - \eta_n p(r)}{\log(s/r)}.$$

Let $F \in A_p^0(\mathbf{C})$ and assume that F has only simple zeros $(a_j)_{j\in\mathbf{N}}$ and that (*) *holds:*

(*) *There are $m \in \mathbf{N}$ and $C > 0$ such that*

$$|F'(a_j)|^{-1} < Ce^{\eta_m p(a_j)} \quad \textit{for all } j \in \mathbf{N}.$$

If $F \cdot A_{p,1}(\mathbf{C})$ is complemented in $A_{p,1}(\mathbf{C})$, then for each $k \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that for each $\tilde{k} \in \mathbf{N}$ there exist $\tilde{n} \in \mathbf{N}$ and $j_0 \in \mathbf{N}$ with

$$\varphi_{\tilde{k},\tilde{n}}(a_j) \leqslant \Phi_{k,n}(a_j) \quad \textit{for } j \geqslant j_0.$$

Proof. Put $B := (e^{\eta_k p(a_j)})_{j,k\in\mathbf{N}}$ and define q: $A_{p,1}(\mathbf{C}) \to \lambda(B)'_b$ by $q[f] := (f(a_j))_{j\in\mathbf{N}}$. Then a suitable variation of the arguments in [1], Thm. 4, shows that q is onto and that $\ker q = F \cdot A_{p,1}(\mathbf{C})$. Since $F \cdot A_{p,1}(\mathbf{C})$ is complemented by hypothesis, there is a continuous linear right inverse R for q. Since R is continuous, the subharmonic functions

$$u_j(z) := \sup_{|w|=|z|} \log|R[(\delta_{ij})_{i\in\mathbf{N}}](w)|, \quad z \in \mathbf{C},\ j \in \mathbf{N},$$

satisfy the following estimates (see [12], 2.13): For each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $j_k \in \mathbf{N}$ such that

(1) $$u_j(r) \leqslant \eta_n p(r) - \eta_k p(a_j) \quad \text{for all } r > 0 \text{ and } j \geqslant j_k,$$

(2) $$u_j(a_j) \geqslant 0 \quad \text{for all } j \in \mathbf{N}.$$

By the subharmonicity of u_j, $j \in \mathbf{N}$, and by (2) there is a sequence $(d_j)_{j\in\mathbf{N}}$ in $\mathbf{R}_+$ such that $d_j \log(r/|a_j|) \leqslant u_j(r)$ for all $r > 0$ and $j \in \mathbf{N}$. From this and (1) we deduce that for each $k \in \mathbf{N}$ there exist $n \in \mathbf{N}$ and $j_k \in \mathbf{N}$ such that

$$d_j \log(r/|a_j|) \leqslant \eta_n p(r) - \eta_k p(a_j) \quad \text{for all } r > 0 \text{ and } j \geqslant j_k,$$

hence

$$\sup_{0<r<a_j} \frac{\eta_k p(a_j) - \eta_n p(r)}{\log(a_j/r)} \leqslant d_j \leqslant \inf_{r>a_j} \frac{\eta_k p(a_j) - \eta_n p(r)}{\log(a_j/r)} \quad \text{for all } j \geqslant j_k.$$

19. Lemma. *Let p be a weight function on $\mathbf{C}$ and $(\eta_k)_{k\in\mathbf{N}} \uparrow 1$. Let $\varphi_{k,n}$ and $\Phi_{k,n}$, k, $n \in \mathbf{N}$, be as in Lemma 18. If for each $k \in \mathbf{N}$ there exists $n \in \mathbf{N}$ such that for each $\tilde{k} \in \mathbf{N}$ there exists $\tilde{n} \in \mathbf{N}$ such that*

$$\varphi_{\tilde{k},\tilde{n}}(s) \leqslant \Phi_{k,n}(s) \quad \textit{for large } s \in \mathbf{R}_+;$$

then $A_{p,1}(\mathbf{C})'_b$ has property ($\underline{\mathrm{DN}}$).

Proof. For $k \in \mathbf{N}$ choose by induction numbers $n(k) \in \mathbf{N}$, $n(k+1) \geqslant n(k)$, $\eta_{n(k)} \geqslant \eta_k/\eta_{k+1}$, such that for each $k \in \mathbf{N}$ there exists $s_k > 0$ with

(1) $$\varphi_{k,n(k)}(s) \leqslant \min_{l\leqslant k} \Phi_{l,n(l)}(s) =: \psi_k(s) \quad \text{for all } s \geqslant s_k.$$

We may assume $\psi_k(s_k) \uparrow \infty$. We put $\psi_0(s_0) := 0$. For $j \in \mathbf{N}$ with $\psi_k(s_k) \leqslant j < \psi_{k+1}(s_{k+1})$ we can choose $r_j \geqslant s_k$ (since ψ_k is unbounded) with $j = \psi_k(r_j)$. From (1) we get for $\psi_k(s_k) \leqslant j < \psi_{k+1}(s_{k+1})$

$$\varphi_{k,n(k)}(r_j) \leqslant j \leqslant \Phi_{k,n(k)}(r_j),$$

and for $\psi_{k+1}(s_{k+1}) \leqslant j < \psi_{k+2}(s_{k+2})$

$$\varphi_{k,n(k)}(r_j) \leqslant \varphi_{k+1,n(k+1)}(r_j) \leqslant j = \psi_{k+1}(r_j) \leqslant \Phi_{k,n(k)}(r_j),$$

since

$$\varphi_{k,n(k)}(s) = \eta_{n(k)} \sup_{r<s} \frac{(\eta_k/\eta_{n(k)}) p(s) - p(r)}{\log(s/r)}$$

and $\eta_k/\eta_{n(k)} \leqslant \eta_{k+1}/\eta_{n(k+1)}$. By induction we get for all $k \in \mathbf{N}$

$$\varphi_{k,n(k)}(r_j) \leqslant j \leqslant \Phi_{k,n(k)}(r_j) \quad \text{for all } j \geqslant \psi_k(s_k),$$

hence $e^{\eta_k p(r_j)}/r_j^j \leqslant C a_{j,n(k)}$ for all $j \geqslant \psi_k(s_k)$. By Lemma 9, $A_{p,1}(\mathbf{C})'_b$ has property ($\underline{\mathrm{DN}}$).

20. Lemma. *Let p be a weight function on $\mathbf{C}$ such that $F\cdot A_{p,1}(\mathbf{C})$ is complemented in $A_{p,1}(\mathbf{C})$ for each $F\in A_p^0(\mathbf{C})$ that satisfies the condition $(*)$ of Lemma 18. Then $A_{p,1}(\mathbf{C})'_b$ has property $(\underline{DN})$.*

Proof. We show that the hypothesis of Lemma 19 is fulfilled. Assume not; then there exists $k\in\mathbf{N}$ such that for each $n\in\mathbf{N}$ there exists $\tilde{k}\in\mathbf{N}$ such that for each $\tilde{n}\in\mathbf{N}$ there is an unbounded sequence $(x_j)_{j\in\mathbf{N}}=(x_j(n,\tilde{n}))_{j\in\mathbf{N}}$ in $\mathbf{R}_+$ with

$$(1)\qquad \varphi_{\tilde{k},\tilde{n}}(x_j)>\Phi_{k,n}(x_j)\quad\text{for all } j\in\mathbf{N}.$$

Now choose a strictly increasing sequence $(a_j)_{j\in\mathbf{N}}$ in $]0,\infty[$ having infinitely many members in common with $(x_j(n,\tilde{n}))_{j\in\mathbf{N}}$ for all $n,\tilde{n}\in\mathbf{N}$ and such that for each $\varepsilon>0$ there is $C>0$ with

$$\left|\frac{\prod_{i\neq j}(1-z/a_i)}{\prod_{i\neq j}(1-a_j/a_i)}\right|\leqslant Ce^{\varepsilon p(z)}\quad\text{for all } z\in\mathbf{C}\text{ and } j\in\mathbf{N}.$$

This is possible, since $\log(1+r)=o(p(r))$ by Definition 1(1) (see e.g. [6], Kap. I, Hilfssatz 3). Put

$$F(z):=\prod_{i=1}^{\infty}(1-z/a_i),\qquad z\in\mathbf{C}.$$

Then F fulfills $(*)$ of Lemma 18, and hence, $F\cdot A_{p,1}(\mathbf{C})$ is complemented by hypothesis. By the choice of $(a_j)_{j\in\mathbf{N}}$, we get a contradiction to (1) from Lemma 18.

21. Theorem. *Let p_l, $l=1,\ldots,N$, be weight functions on $\mathbf{C}$. For the weight function $p(z)=\sum_{l=1}^N p_l(z_l)$, $z=(z_1,\ldots,z_N)\in\mathbf{C}^N$, on $\mathbf{C}^N$ the following assertions are equivalent:*

(i) *$A_{p,1}(\mathbf{C}^N)'_b$ has property $(\underline{DN})$.*
(ii) *$F\cdot A_{p,1}(\mathbf{C}^N)$ is complemented in $A_{p,1}(\mathbf{C}^N)$ for each $F\in A_p^0(\mathbf{C}^N)$.*
(iii) *$A_{p,1}(\mathbf{C}^N)$ is a complemented subspace of $L^2_{p,1}(\mathbf{C}^N)$.*

Proof. (i)$\Rightarrow$(ii): Theorem 14.

(ii)$\Rightarrow$(i). Since p has this particular form, it is easy to see that for each $1\leqslant l\leqslant N$ and each $F\in A_{p_l}^0(\mathbf{C})$ the subspace $F\cdot A_{p_l,1}(\mathbf{C})$ is complemented in $A_{p_l,1}(\mathbf{C})$. Hence, for $1\leqslant l\leqslant N$ the space $A_{p_l,1}(\mathbf{C})'_b$ has $(\underline{DN})$ by Lemma 20. Since

$$A_{p,1}(\mathbf{C}^N)=\hat{\bigotimes}_{l=1}^{N}A_{p_l,1}(\mathbf{C}),$$

$A_{p,1}(\mathbf{C}^N)'_b$ has $(\underline{DN})$, too.

(i)$\Rightarrow$(iii). Choose $(r_\alpha)_{\alpha\in\mathbf{N}_0^N}$ according to Lemma 9(iii). Then we get the desired projection $P\colon L^2_{p,1}(\mathbf{C}^N)\to A_{p,1}(\mathbf{C}^N)$ by the following formula:

$$P[f](z)=\sum_{\alpha\in\mathbf{N}_0^N}a_\alpha^{-1}\int_{A_\alpha}\frac{f(\zeta)}{\zeta^\alpha}\,dm_{2N}\cdot z^\alpha,$$

where

$$A_\alpha:=\{\zeta\in\mathbf{C}^N\,|\,r_{\alpha,1}<\zeta_1<r_{\alpha,1}+1,\ldots,r_{\alpha,N}<\zeta_N<r_{\alpha,N}+1\},$$

$$a_\alpha:=\prod_{l=1}^{N}\{\pi((r_{\alpha,l}+1)^2-r_{\alpha,l}^2)\}\geqslant\pi^N,\qquad\alpha\in\mathbf{N}_0^N.$$

Similarly to the proof of Theorem 14 we conclude that P is well defined and continuous.

(iii)$\Rightarrow$(i). Let $1\leqslant l\leqslant N$ be fixed. From (iii) we see that $A_{p_l,1}(\mathbf{C})$ is a complemented subspace of $L^2_{p_l,1}(\mathbf{C})$. Let P_l be a projection onto $A_{p_l,1}(\mathbf{C})$. From [1], Thm. 1, we conclude that

$$R_l[f]=g-P_l[g],\qquad\text{where } g\in L^2_{p_l,1}(\mathbf{C})\text{ with }\bar{\partial}g=f,$$

is a continuous linear right inverse for the $\bar{\partial}$-operator

$$\bar{\partial}\colon\{f\in L^2_{p_l,1}(\mathbf{C})\,|\,\bar{\partial}f\in L^2_{p_l,1}(\mathbf{C})\}\to L^2_{p_l,1}(\mathbf{C}).$$

As indicated in [15], § 1, from this we find that for each $F\in A^0_{p_l,1}(\mathbf{C})$ satisfying the hypothesis $(*)$ of Lemma 18, $F\cdot A_{p_l,1}(\mathbf{C})$ is complemented. Hence, as in the previous part of the proof we get (i).

22. Theorem. *Let p_l, $l=1,\ldots,N$, be weight functions on $\mathbf{C}$. For the weight function $p(z)=\sum_{l=1}^N p_l(z_l)$, $z=(z_1,\ldots,z_N)\in\mathbf{C}^N$, on $\mathbf{C}^N$ the following assertions are equivalent:*

(i) *$A_p(\mathbf{C}^N)'_b$ has property (DN).*
(ii) *Each principal ideal of $A_p(\mathbf{C}^N)$ is complemented.*
(iii) *$A_p(\mathbf{C}^N)$ is a complemented subspace of $L^2_p(\mathbf{C}^N)$.*

Proof. Analogous to the proof of Theorem 21. Instead of Lemma 20 use Lemma 10 and [12], Prop. 2.15.

IV. Applications

23. Definition and Remark. Let Δ be the polydisc $\{z\in\mathbf{C}^N\,|\,|z_j|<1,\ j=1,\ldots,N\}$ and let $(a_\alpha)_{\alpha\in\mathbf{N}_0^N}$ be a sequence of complex numbers satisfying the estimates

$$(1)\qquad \sup_{\alpha\in\mathbf{N}_0^N}|a_\alpha|\alpha!e^{k|\alpha|}<\infty\quad\text{for all } k\in\mathbf{N},\text{ i.e. }(a_\alpha\alpha!)_{\alpha\in\mathbf{N}_0^N}\in\Lambda_\infty(|\alpha|).$$

Then some calculation shows that we get a continuous linear operator on the space of all analytic functions on Δ by setting

$$L: A(\Delta)\to A(\Delta), \quad L[f] = \sum_{\alpha\in\mathbf{N}_0^N} a_\alpha D^\alpha f,$$

where $D^\alpha := \partial^{|\alpha|}/\partial z_1^{\alpha_1}\ldots\partial z_N^{\alpha_N}$, $\alpha\in\mathbf{N}_0^N$. L is called a *linear partial differential operator of infinite order with constant coefficients*. By the continuity of L, we get for the associated function

$$(2)\qquad \hat{L}\in A^0_{|z|}(\mathbf{C}^N), \quad \hat{L}(z) = \sum_{\alpha\in\mathbf{N}_0^N} a_\alpha z^\alpha, \quad z\in\mathbf{C}^N.$$

Conversely, (2) implies (1). Now, by the Fourier–Borel transform

$$\mathfrak{F}: A(\Delta)'_b\to A_{|z|,1}(\mathbf{C}^N), \quad \mathfrak{F}[\varphi](z) = \varphi(e^{\langle z,\cdot\rangle}) = \sum_{\alpha\in\mathbf{N}_0^N}\frac{\varphi(w^\alpha)}{\alpha!}z^\alpha,$$

we identify the strong dual of $A(\Delta)$ with $A_{|z|,1}(\mathbf{C}^N)$. By this identification the adjoint operator is the operator $M_{\hat{L}}$ of multiplication with the associated function, i.e.

$$L^t = M_{\hat{L}}: A_{|z|,1}(\mathbf{C}^N)\to A_{|z|,1}(\mathbf{C}^N), \quad M_{\hat{L}}[f] = \hat{L}\cdot f.$$

24. Theorem. *Each nonzero linear partial differential operator L of infinite order with constant coefficients on $A(\Delta)$ admits a continuous linear right inverse.*

Proof. Since $\hat{L}\in A^0_{|z|}(\mathbf{C}^N)\setminus\{0\}$ and $A_{|z|,1}(\mathbf{C}^N)'_b\cong\Lambda_1(|\alpha|)$, by Example 6(2)(b), we find from Theorem 14 that $\operatorname{im} L^t = \hat{L}\cdot A_{|z|,1}(\mathbf{C}^N)$ is complemented, i.e. L^t admits a continuous linear left inverse. Since $A(\Delta)$ is a nuclear Fréchet space, L admits a continuous linear right inverse.

Remark. In the case $N=1$ we get a description of $\ker L$ as in [7], Thm. 3.4, from Proposition 16.

25. Definition and Remark. Let p be a weight function on $\mathbf{C}$ and let $A = (a_{j,k})_{j\in\mathbf{N}_0,k\in\mathbf{N}}$ and $B = (b_{j,k})_{j\in\mathbf{N}_0,k\in\mathbf{N}}$ where

$$a_{j,k} := \inf_{r>0}\frac{e^{(1-1/k)p(r)}}{r^j}, \quad b_{j,k} := \left(\inf_{r>0}\frac{e^{(1/k)p(r)}}{r^j}\right)^{-1}, \quad j\in\mathbf{N}_0,\ k\in\mathbf{N}.$$

Let $\mu = (\mu_n)_{n\in\mathbf{N}_0}\in\lambda(B)$. For the associated function we get (see [11], 1.12)

$$\hat{\mu}\in A^0_p(\mathbf{C}), \quad \hat{\mu}(z) := \sum_{n=0}^\infty \mu_n z^n, \quad z\in\mathbf{C},$$

and vice versa. According to Proposition 5 we identify $A_{p,1}(\mathbf{C})'_b$ with $\lambda(A)$. Then for the adjoint $T_\mu := M^t_{\hat{\mu}}$ of the multiplication operator $M_{\hat{\mu}}$ on $A_{p,1}(\mathbf{C})$ we get (see [10], 3.4(1))

$$T_\mu: \lambda(A)\to\lambda(A), \quad T_\mu[(x_j)_{j\in\mathbf{N}_0}] = \Big(\sum_{n=0}^\infty x_{n+j}\mu_n\Big)_{j\in\mathbf{N}_0}.$$

The continuous linear operator T_μ is called a *Toeplitz operator*. Since $M_{\hat{\mu}}$ has closed range (Proposition 13), T_μ is onto for $\mu\neq 0$. If $\hat{\mu}$ has the zeros $(a_j)_{j\in\mathbf{N}}$ (counted with respect to multiplicities), we get from Proposition 16 (see [10], 3.5)

$$\ker T_\mu \cong \lambda\big((e^{(1-1/k)p(a_j)})_{j,k\in\mathbf{N}}\big).$$

Since $\lambda(A)$ is nuclear, dualizing Theorem 21 we get:

26. Theorem. *Let p be a weight function on $\mathbf{C}$. Let $\lambda(A)$ and $\lambda(B)$ be as in Definition 25. Then $\lambda(A)$ has property $(\underline{DN})$ (i.e. is isomorphic to some $\Lambda_1(\beta)$ by Lemma 9) if and only if each Toeplitz operator T_μ on $\lambda(A)$, $\mu\in\lambda(B)$, $\mu\neq 0$, admits a continuous linear right inverse.*

27. Corollary. *Let $1<s<\infty$ and let $\mu\in\Lambda_\infty(j^s)$, $\mu\neq 0$. Then for each $0<\varrho<1$ the Toeplitz operator*

$$T_\mu: \Lambda_\varrho(j^s)\to\Lambda_\varrho(j^s), \quad T_\mu[(x_j)_{j\in\mathbf{N}_0}] = \Big(\sum_{n=0}^\infty x_{n+j}\mu_n\Big)_{j\in\mathbf{N}_0},$$

admits a continuous linear right inverse.

Proof. Let $1<s<\infty$ be fixed. For $0<\varrho<1$ we consider the weight function

$$p(z) = a\big(\log(1+|z|)\big)^t, \quad z\in\mathbf{C},$$

where

$$t := \frac{s}{s-1} > 1, \quad a := \log R(\varrho) := \left(\frac{t^{1-s}-t^{-s}}{-\log\varrho}\right)^{1/(s-1)} > 0.$$

Since

$$\inf_{r\geq 1} e^{b(\log r)^t}/r^j = \exp\big(-(t^{1-s}-t^{-s})(1/b)^{s-1}j^s\big)$$

for $j\in\mathbf{N}$ and $b>0$, in Theorem 26 we get $\lambda(A) = \Lambda_\varrho(j^s)$ and $\lambda(B) = \Lambda_\infty(j^s)$. Hence the assertion follows from Theorem 26.

Let us compare this result with earlier results (see [8], [10] and [11]):

28. Remark. We use the notation of the proof of Corollary 27. Let $1<s<\infty$ and let $\mu\in\Lambda_\infty(j^s)$, $\mu\neq 0$. Let us assume that the associated function $\hat{\mu}$ (in $A^0_p(\mathbf{C})$) has the zeros $(a_j)_{j\in\mathbf{N}}$ (counted with respect to multiplicities). Then by [10], 3.9 and 3.6, and by Corollary 27 the Toeplitz operator

$$T_\mu[(x_j)_{j\in\mathbf{N}_0}] = \Big(\sum_{n=0}^\infty x_{n+j}\mu_n\Big)_{j\in\mathbf{N}_0}$$

acts continuously and surjectively on the spaces $\Lambda_\infty(j^s)'_b = \bigcup_{0<\varrho<1} \Lambda_\varrho(j^s)$ and $\Lambda_1(j^s) = \bigcap_{0<\varrho<1} \Lambda_\varrho(j^s)$ as well as on the steps $\Lambda_\varrho(j^s)$, $0 < \varrho < 1$. By [10], 3.9 and 3.6, by the proof of Corollary 27 and by Definition 25, we get

$$\ker T_\mu \cong \Lambda_1(\beta)'_b = \bigcup_{0<\varrho<1} \Lambda_{R(\varrho)}(\beta), \quad \ker T_\mu \cong \Lambda_\infty(\beta) = \bigcap_{0<\varrho<1} \Lambda_{R(\varrho)}(\beta),$$

$$\ker T_\mu \cong \Lambda_{R(\varrho)}(\beta), \quad 0 < \varrho < 1,$$

respectively, where $\beta = \big((\log(1+|a_j|))^r\big)_{j\in\mathbb{N}}$. By [11], 3.7(2), and [8], 4.12(1), T_μ does not admit a continuous linear right inverse on the limit spaces $\Lambda_\infty(j^s)'_b$ and $\Lambda_1(j^s)$ respectively, even though it does on each step $\Lambda_\varrho(j^s)$, $0 < \varrho < 1$, as we proved in Corollary 27.

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