STUDIA MATHEMATICA, T. XCVI (1990)

Quantitative unconditionality of Banach spaces E for which $\mathscr{K}(E)$ is an M-ideal in $\mathscr{L}(E)$

by

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Abstract. Let E be a separable Banach space with AP; if $\mathscr{K}(E)$ is an M-ideal in $\mathscr{L}(E)$, then for every $\varepsilon > 0$, E is isometric to a $(1+\varepsilon)$-complemented subspace in a space with a$(1+\varepsilon)$-unconditional shrinking FDD; in particular, E is isometric to a subspace of a space with a $(1+\varepsilon)$-unconditional basis, and $\operatorname{dist}(E, L^p(0, 1)) \geqslant \max(p-1, 1/(p-1))$.

1. Introduction. Since E. Alfsen and E. Effros introduced the notion of an M-ideal [1], many authors have studied for which Banach spaces E, $\mathscr{K}(E)$, the space of compact operators on E, is an M-ideal in $\mathscr{L}(E)$, the space of all continuous operators on E. J. Hennefeld [17] had already proved that if $E = c_0$ or l^p, $1 < p < \infty$, $\mathscr{K}(E)$ is an M-ideal in $\mathscr{L}(E)$; actually for $E = l^2$ this had been proved a long time ago by J. Dixmier [8]. This was improved by Å. Lima: if $(E_\alpha)_{\alpha\in A}$ is a family of finite-dimensional spaces and $E = (\bigoplus_{\alpha\in A} E_\alpha)_{l^p}$, $1 < p < \infty$, or $E = (\bigoplus_{\alpha\in A} E_\alpha)_{c_0}$, then $\mathscr{K}(E)$ is an M-ideal in $\mathscr{L}(E)$ ([21], p. 31); moreover, if F is a subspace of $E = (\bigoplus_{\alpha\in A} E_\alpha)_{l^p}$, $1 < p < \infty$, C. M. Cho and W. B. Johnson showed that $\mathscr{K}(F)$ is an M-ideal in $\mathscr{L}(F)$ if and only if F has the compact approximation property [5]. On the other hand, $\mathscr{K}(E)$ is not an M-ideal in $\mathscr{L}(E)$ when a) $E = l^1$, $E = l^\infty$ [31], b) E is a space with a 1-symmetric basis which is not l^p or c_0 [18], c) $E = \mathscr{C}(K)$, K infinite compact [9], d) $E = L^1(\mu)$, $\dim L^1(\mu) = \infty$ [9], [20], e) E is a Lindenstrauss space which is not $c_0(I)$, f) $E = L^p(\mu)$, $1 < p < \infty$, $p \neq 2$, μ not purely atomic [20]. Some authors also studied when $\mathscr{K}(X, Y)$ is an M-ideal in $\mathscr{L}(X, Y)$ ([2], [9], [12], [20], [26], for instance).

After this concrete examples, structure theorems were sought for Banach spaces E for which $\mathscr{K}(E)$ is an M-ideal in $\mathscr{L}(E)$; Å. Lima proved that E is then an M-ideal in its bidual E^{**} [21], a result which also yields the previous examples a), c), d), e) [16]. Lately G. Godefroy and P. Saab showed that E (supposed separable with AP) has got to be isomorphic to a complemented subspace of a space with a shrinking unconditional finite-dimensional decomposition [15].

1985 *Mathematics Subject Classification* 46B20, 47D15, 47D30, 46B15, 46B10.

The present work is a quantitative version of [15]; we prove that if E is a separable Banach space with AP such that $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then for every $\varepsilon > 0$, E is isometric to a $(1+\varepsilon)$-complemented subspace in a space with a $(1+\varepsilon)$-unconditional shrinking FDD; in particular, E is isometric to a subspace of a space with a $(1+\varepsilon)$-unconditional basis and

$$\operatorname{dist}(E, L^p(0,1)) \geqslant \max\left(p-1, \frac{1}{p-1}\right).$$

This answers a question of G. Godefroy.

Let us note, however, that the main argument of [15] (numerical radius) cannot give this quantitative result and it is replaced by the fundamental lemma 4.1 of E. Alfsen and E. Effros [1].

2. Notations and preliminaries. The Banach spaces considered in this paper are real or complex; all the subspaces are supposed to be closed. If X is a Banach space, its closed unit ball is denoted by X_1, and the weak*-topology on the dual X^* by ω^*; in any case, the notations are standard and are those of [7] or [23].

If E is a Banach space, $\mathcal{K}(E)$ denotes the space of compact operators on E, $\mathcal{L}(E)$ is the space of all continuous linear operators on E, and $\mathcal{F}(E)$ the space of finite rank ones.

A subspace J of a Banach space X is said to be an *M-ideal* if there is an L-projection Q of X^* with range $J^\perp$, the annihilator of J in X^*, that is, a projection Q such that $\|x\| = \|Qx\| + \|x-Qx\|$ for every $x \in X^*$.

A Banach space F is said to have a *finite-dimensional Schauder decomposition* (FDD) if every $x \in F$ can be uniquely written as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in F_n$, and each F_n is a finite-dimensional subspace of F; the FDD is said to be *unconditional* if the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent for every $x \in F$, and the number

$$\sup_{\|x\| \leqslant 1} \sup_{N \geqslant 1} \sup_{|\theta_n| \leqslant 1} \Big\| \sum_{n=1}^{N} \theta_n x_n \Big\|$$

is the *unconditional constant* of the FDD; the FDD is said to be *shrinking* if it induces by duality an FDD of F^* (see [23], I.1.g).

3. Acknowledgements. This work was begun while I was visiting the University of Missouri-Columbia in March-April 1988. I thank the Ministère des Affaires Etrangères for partial financial support, and P. and E. Saab for their warm welcome. I am specially glad to thank M. and G. Godefroy for their very enjoyable hospitality, and G. Godefroy for his kind advice and suggestions.

I thank B. Maurey, F. Piquard and J. Saint Raymond for useful conversations, N. Ghoussoub for the reference [3], and D. Werner for useful remarks.

4. The structure theorem

THEOREM. *Let E be a separable Banach space. We suppose that*

(H) *There exist a subalgebra $\mathcal{A}$ of $\mathcal{L}(E)$, containing $I = \mathrm{Id}_E$, and an ideal $\mathcal{B}$ of $\mathcal{A}$ such that:*

1) $\mathcal{B} \subset \overline{\mathcal{F}(E)}$ *(norm closure),*
2) $(\exists B_n \in \mathcal{B})$ $\|B_n\| \leqslant 1$, $\|B_n x - x\| \to 0$, $\forall x \in E$,
3) $\mathcal{B}$ *is an M-ideal in* $\mathcal{A}$.

Then

(C) *For every $\varepsilon > 0$, there exists a space F with a $(1+\varepsilon)$-unconditional FDD such that E is isometric to a $(1+\varepsilon)$-complemented subspace in F. Moreover, if $E \not\supset l^1$ (resp. E is reflexive) we can take F such that its FDD is shrinking (resp. F reflexive).*

We will prove this theorem in the next section. Let us now present a few consequences.

COROLLARY 1. *Let E be a separable Banach space satisfying* (H). *Then*

(C′) *For every $\varepsilon > 0$, E is isometric to a subspace of a space G with a $(1+\varepsilon)$-unconditional basis. Moreover, if $E \not\supset l^1$ (resp. E is reflexive), then the basis may be assumed to be shrinking (resp. G may be assumed reflexive).*

Proof. This follows from [23], I.1.g.5.: instead of $\|x^* - x^*_{i,n}\| \leqslant 4^{-n}$, we take $\|x^* - x^*_{i,n}\| \leqslant 2^{-n}\alpha$; then $\varphi(Tu) \geqslant 1-\alpha$ and $\varphi(M_\theta Tx) \leqslant 1+\alpha$, so that $\|Tu\| \geqslant (1-\alpha)/(1+\alpha)$ and $\|T\|\,\|T^{-1}\| \leqslant (1+\alpha)/(1-\alpha)$.

Hence, if F admits a $(1+\alpha)$-unconditional FDD, F is isometric to a subspace of G with a $(1+\alpha)^2/(1-\alpha)$-unconditional basis.

We conclude the proof by using the interpolation technique of [11], Theorem 3.3. ∎

COROLLARY 2. *Let E be separable Banach space with the Approximation Property. If $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then for every $\varepsilon > 0$, E is isometric to a $(1+\varepsilon)$-complemented subspace in a space F with a $(1+\varepsilon)$-unconditional shrinking FDD, and F may be assumed reflexive if E is.*

Proof. It suffices to remark that condition (H 2) holds ([16], Lemma 5.1), and to use [21], Cor. 2.5, in order to see that $E \not\supset l^1$. ∎

COROLLARY 3. *If E is a separable Banach space with the Approximation Property and $\mathcal{K}(E)$ is an M-ideal in $\mathcal{L}(E)$, then*

$$\operatorname{dist}(E, L^p(0,1)) \geqslant \max\left(p-1, \frac{1}{p-1}\right).$$

Proof. This follows from the reproducibility of the Haar basis ([22], Th. 4.1, [23], II.2.c.7, II.2.c.8): whenever $L^p(0, 1)$, $1 < p < \infty$, is isometric to a subspace of a space with a basis $(u_n)_{n\geq 1}$, then for every $\varepsilon > 0$, there exists a block basis of $(u_n)_{n\geq 1}$ which is $(1+\varepsilon)$-equivalent to the Haar basis. Hence, the distance from E to $L^p(0, 1)$, $1 < p < \infty$, is at least k_p, the unconditional constant of the Haar basis of $L^p(0, 1)$, and $k_p = \max(p-1, 1/(p-1))$ ([3], Cor. 1, [4]).

Corollary 3 is also true for $p = 1$ and $p = \infty$ since $\mathscr{K}(L^1(0, 1))$ and $\mathscr{K}(L^\infty(0, 1))$ are isomorphic to no space which is an M-ideal in its bidual ([21], Cor. 2.5). ∎

COROLLARY 4. *Let E be a Banach space with a monotone basis $\mathscr{U} = (u_n)_{n\geq 1}$. If $\mathscr{K}_d(E, \mathscr{U})$, the space of compact diagonal operators, is an M-ideal in $\mathscr{L}_d(E, \mathscr{U})$, the space of continuous diagonal operators, then* (C) *holds for E.*

Remark 1. If $E = l^1$ and $\mathscr{U}$ is the canonical basis, $\mathscr{K}_d(l^1)$ is isometric to c_0 and is an M-ideal of $\mathscr{L}_d(l^1)$ which is isometric to l^∞; but l^1 cannot be isomorphic to a subspace of a space with a shrinking FDD. So, the condition $E \not\supset l^1$ in the conclusion (C) of the Theorem cannot be removed.

Remark 2. If $E = L^p(\mathbf{T})$, $1 < p < \infty$, and $\mathscr{U} = \mathscr{T}$ is the trigonometrical basis ([23], II.2.c.16), then $\mathscr{L}_d(L^p(\mathbf{T}), \mathscr{T}) = \mathrm{CV}(L^p(\mathbf{T}))$ (resp. $\mathscr{K}_d(L^p(\mathbf{T}), \mathscr{T}) = \mathrm{CV}_c(L^p(\mathbf{T}))$) is the space of convolution operators (resp. compact convolution operators) on $L^p(\mathbf{T})$; as in Cor. 3, $\mathrm{CV}_c(L^p(\mathbf{T}))$ is not an M-ideal in its bidual $\mathrm{CV}(L^p(\mathbf{T}))$ (another proof of this which was indicated to me by F. Lust-Piquard adapts the one of Å. Lima, [20], Th. 11).

5. Proof of the Theorem. First, we show

PROPOSITION. *Let E, $\mathscr{A}$ and $\mathscr{B}$ be as in the Theorem, and assume the condition:*

(H1′) $\mathscr{B} \subset \mathscr{K}(E)$, *and $\mathscr{B}$ is separable,*

instead of (H1). *Let Q be the L-projection of $\mathscr{A}^*$ with kernel $\mathscr{B}^\perp$. Then, for every $\varepsilon > 0$, there exists $S_n \in \mathscr{B}$, $n \geq 1$, such that*

$$Q^* I = \sum_{n=1}^{\infty} S_n \quad (\sigma(\mathscr{A}^{**}, \mathscr{A}^*)),$$

$$\sup_{|\theta_k| \leq 1} \Big\| \sum_{k=1}^{n} \theta_k S_k \Big\| \leq 1 + \varepsilon.$$

We need the following lemma, which is essentially known ([30], Th. 2.5).

LEMMA 1. *With the hypothesis of the Proposition, for every $A \in \mathscr{A}$,*

$$Q^* A = \lim_{n\to\infty} B_n A \quad (\sigma(\mathscr{A}^{**}, \mathscr{A}^*)).$$

Proof. We use the argument of J. Johnson ([19], Lemma 1). For every ultrafilter $\mathscr{V}$, $\lim_{\mathscr{V}} \psi(B_n)$ exists for every $\psi \in \mathscr{B}^*$. $\mathscr{B}$ being an ideal of $\mathscr{A}$, the map ψ: $S \in \mathscr{B} \mapsto \varphi(SA)$ is an element of $\mathscr{B}^*$ for every $A \in \mathscr{A}$ and every $\varphi \in \mathscr{B}^*$; hence, we can define $\hat{\varphi}(A) = \lim_{\mathscr{V}} \varphi(B_n A)$; then $\hat{\varphi} \in \mathscr{A}^*$, $\|\hat{\varphi}\| \leq \|\varphi\|$.

Since $B_n \xrightarrow{\|\cdot\|} I$ as $n \to \infty$ uniformly on compact subsets of E, we have $\|B_n B - B\| \to 0$ as $n \to \infty$ for $B \in \mathscr{B}$, and so $\hat{\varphi}(B) = \varphi(B)$ for $B \in \mathscr{B}$; hence $\|\hat{\varphi}\| = \|\varphi\|$.

If we denote by R: $\mathscr{A}^* \to \mathscr{B}^*$ the restriction map and by $\Lambda_{\mathscr{V}}$: $\mathscr{B}^* \to \mathscr{A}^*$ the map $\varphi \mapsto \hat{\varphi}$, $\Lambda_{\mathscr{V}} R$ is a contractive projection of $\mathscr{A}^*$ with kernel $\mathscr{B}^\perp$; hence $\Lambda_{\mathscr{V}} R = Q$ ([16], Prop. 2.1.b).

Therefore, for $A \in \mathscr{A}$ and $\varphi \in \mathscr{A}^*$,

$$\langle Q^* A, \varphi \rangle = \langle A, \Lambda_{\mathscr{V}} R\varphi \rangle = \lim_{\mathscr{V}} (R\varphi)(B_n A) = \lim_{\mathscr{V}} \varphi(B_n A),$$

that is, $Q^* A = \lim_{n\to\infty} B_n A$ $(\sigma(\mathscr{A}^{**}, \mathscr{A}^*))$. ∎

Proof of the Proposition. Let us consider $\mathscr{A}$ with its real Banach space structure. By Lemma 1, $Q^* I = \lim_{n\to\infty} B_n$ $(\sigma(\mathscr{A}^{**}, \mathscr{A}^*))$.

Let K be the compact space $(\mathscr{A}_1^*, \omega^*)$, and let us consider elements of $\mathscr{A}^{**}$ as functions on K. Denoting by χ_L the characteristic function of the set $L \subset K$, we set

$$h = (I\chi_{\mathscr{B}^\perp \cap K}) \vee 0 = (I\chi_{\mathscr{B}^\perp \cap K})^+$$

and let $\hat{h}$ be the smallest concave upper-semicontinuous (u.s.c.) function which is greater than h on K; then ([1], Lemma 4.1)

$$(\forall t \in K) \quad (I - Q^* I)(t) = \hat{h}(t) - \hat{h}(-t).$$

Therefore, setting

$$(\forall t \in K) \quad g(t) = \tfrac{1}{2}[I(t) + 1] - \hat{h}(t),$$

we have

$$(\forall t \in K) \quad I(Qt) = (Q^* I)(t) = g(t) - g(-t),$$

and g is a positive, lower-semicontinuous (l.s.c.) function on K such that

$$(\forall t \in K) \quad g(t) + g(-t) \leq 1.$$

I am indebted to J. Saint Raymond for the next lemma.

LEMMA 2. *Let K be a compact space and F: $K \to \mathbf{R}$ such that $F = G_1 - G_2$, with G_1, G_2 positive l.s.c. functions on K. If there exist a metrizable compact space H, a continuous surjection R: $K \to H$ and a function F': $H \to \mathbf{R}$ such that*

$F = F'\circ R$, then there exist continuous functions $f_n(t)$ on K, $n = 1, 2, \ldots$, such that

$$(\forall t\in K)\quad F(t) = \sum_{n=1}^{\infty} f_n(t),$$

$$\sup_{|\theta_n|\leqslant 1} \Big\|\sum_{n=1}^{N} \theta_n f_n\Big\|_\infty \leqslant \sup_{t\in K}[G_1(t)+G_2(t)].$$

Sketch of proof (see [14], Lemma 3). We set

$$(\forall u\in H)\quad \tilde{G}_i(u) = \inf\{G_i(t)\,|\,R(t) = u\};$$

then $\tilde{G}_1$, $\tilde{G}_2$ are positive and l.s.c. on H, and

$$(\forall u\in H)\quad F'(u) = \tilde{G}_1(u)-\tilde{G}_2(u).$$

Since H is a metrizable space we have

$$\tilde{G}_i(u) = \lim_{n\to\infty}\uparrow g_{i,n}(u) = \sum_{n=1}^{\infty}[g_{i,n+1}(u)-g_{i,n}(u)],$$

with $g_{1,1} = g_{2,1} = 0$, $g_{1,n}$, $g_{2,n}$, $n\geqslant 1$, continuous on H; setting, for $n\geqslant 1$,

$$(\forall u\in H)\quad f_n'(u) = [g_{1,n+1}(u)-g_{1,n}(u)]-[g_{2,n+1}(u)-g_{2,n}(u)],$$

we have

$$(\forall u\in H)\quad F_n'(u) = \sum_{n=1}^{\infty} f_n'(u)$$

and, for $|\theta_n|\leqslant 1$, $u\in H$,

$$\Big|\sum_{n=1}^{N}\theta_n f_n'(u)\Big| \leqslant \sum_{n=1}^{\infty}[g_{1,n+1}(u)-g_{1,n}(u)]+\sum_{n=1}^{\infty}[g_{2,n+1}(u)-g_{2,n}(u)]$$
$$= \tilde{G}_1(u)+\tilde{G}_2(u) \leqslant \sup_{t\in K}[G_1(t)+G_2(t)].$$

This gives the lemma with $f_n = f_n'\circ R$. ■

We use Lemma 2 with $K = (\mathscr{A}_1^*, \omega^*)$, $H = (\mathscr{B}_1^*, \omega^*)$, $F = Q^*I$, and R the restriction map. Since $Q^*I = F'\circ R$, where

$$F'(u) = \hat{u}(I) = \lim_{\mathscr{V}} u(B_n) = \lim_{k\to\infty} u(B_{n_k}),$$

we get by Lemma 2 functions $f_n\in\mathscr{C}(K)$ such that

$$(\forall t\in K)\quad (Q^*I)(t) = \sum_{n=1}^{\infty} f_n(t),$$

$$\sup_{|\theta_n|\leqslant 1}\Big\|\sum_{n=1}^{N}\theta_n f_n\Big\|_\infty \leqslant \sup_{t\in K}[g(t)+g(-t)] \leqslant 1.$$

In order to conclude the proof of the Proposition, it suffices to apply the following quantitative version of a classical result of A. Pełczyński (a similar result can be found in [10], Lemma 1), with $Y = \mathscr{B}$, $X = \mathscr{C}(K)$.

LEMMA 3. *Let X be a Banach space and let Y be a subspace of X. If $f\in Y^{\perp\perp}$ is such that*

$$f = \sum_{n=1}^{\infty}{}^{\omega^*} x_n,\quad x_n\in X,$$

$$\sup_{|\theta_n|\leqslant 1}\Big\|\sum_{n=1}^{N}\theta_n x_n\Big\| \leqslant C\|f\|,$$

then, for every $\varepsilon > 0$, there are $y_n\in Y$, $n\geqslant 1$, such that

$$f = \sum_{n=1}^{\infty}{}^{\omega^*} y_n,\quad \sup_{|\theta_n|\leqslant 1}\Big\|\sum_{n=1}^{N}\theta_n y_n\Big\| \leqslant (C+\varepsilon)\|f\|.$$

Sketch of proof (see [29], Lemma 15.7, p. 446). There exist $z_n\in Y$, $n\geqslant 1$, such that $f = \omega^*\text{-}\lim_{n\to\infty} z_n$. Setting $v_n = z_n-\sum_{i=1}^{n} x_i$, $n\geqslant 1$, we can define a strictly increasing sequence of integers $0 = p_0 < p_1 < \ldots$, a sequence of real numbers $\lambda_n\geqslant 0$, such that $\sum_{i=p_{n-1}+1}^{p_n}\lambda_i = 1$ and

$$u_n = \sum_{i=p_{n-1}+1}^{p_n}\lambda_i v_i,\quad \|u_n\|\leqslant \varepsilon\|f\|/2^{n+1}.$$

Setting now

$$w_n = \sum_{i=p_{n-1}+1}^{p_n}\lambda_i z_i,\qquad n\geqslant 1,$$
$$y_1 = w_1,\quad y_{n+1} = w_{n+1}-w_n,$$

we have

$$y_1 = \sum_{j=1}^{p_1}\mu_j^0 x_j+u_1,$$

$$y_{n+1} = \sum_{j=p_{n-1}+1}^{p_{n+1}}\mu_j^n x_j+u_{n+1}-u_n,\quad n\geqslant 1,$$

with

$$\mu_j^0 = \sum_{i=j}^{p_1}\lambda_i.$$

$$\mu_j^n = \begin{cases} 1-\sum_{i=j}^{p_n}\lambda_i, & p_{n-1}+1\leqslant j\leqslant p_n,\\ \sum_{i=j}^{p_{n+1}}\lambda_i, & p_n+1\leqslant j\leqslant p_{n+1},\end{cases}\qquad n\geqslant 1.$$

Since $0 \leqslant \mu_j^n \leqslant 1$, $\forall j$, $\forall n \geqslant 0$, we obtain

$$\Big\|\sum_{n=1}^{N} \theta_n y_n\Big\| \leqslant \Big\|\sum_{n=1}^{N-1} \sum_{j=p_{n-1}+1}^{p_n} (\theta_n \mu_j^{n-1} + \theta_{n+1}\mu_j^n)x_j + \sum_{j=p_{N-1}+1}^{p_N} \theta_N \mu_j^{N-1} x_j\Big\| + \varepsilon\|f\| \leqslant C\|f\| + \varepsilon\|f\|$$

because $\mu_j^{n-1} + \mu_j^n = 1$ for $p_{n-1}+1 \leqslant j \leqslant p_n$, and hence $|\theta_n\mu_j^{n-1} + \theta_{n+1}\mu_j^n| \leqslant 1$. ∎

The conclusion of the proof of the Theorem is along the lines of [15].

We begin by noting that, for $x \in E$, $x^* \in E^*$, $x \otimes x^*$ induces an element of $\mathscr{A}^*$:

$$(\forall A \in \mathscr{A}) \qquad \langle x \otimes x^*, A\rangle = \langle x^*, Ax\rangle;$$

we have $\|x\otimes x^*\|_{\mathscr{A}^*} \geqslant \|R(x\otimes x^*)\|_{\mathscr{B}^*}$; on the other hand, for every $A \in \mathscr{A}$, $B_n A \in \mathscr{B}$ and condition (H2) gives

$$\langle x^*, Ax\rangle = \lim_{n\to\infty} \langle x^*, B_n Ax\rangle;$$

hence $\|x\otimes x^*\|_{\mathscr{A}^*} = \|R(x\otimes x^*)\|_{\mathscr{B}^*}$.

Therefore, since $\mathscr{A}^* = \mathscr{B}^\perp \oplus_1 N$ with N isometrically isomorphic to $\mathscr{B}^*$, we have $Q(x\otimes x^*) = x\otimes x^*$, and the Proposition allows us to write

$$\langle x^*, x\rangle = \sum_{n=1}^{\infty} \langle x^*, S_n x\rangle.$$

By condition (H1), we can find a sequence $(R_n)_{n\geqslant 1}$ of finite rank operators such that $\|S_n - R_n\| \leqslant \varepsilon/2^n$, $n \geqslant 1$; setting (as in [27], proof of Prop. 3, or [15], proof of Cor. 6)

$$(\forall x \in E) \qquad S(x) = \sum_{n=1}^{\infty}{}^{\omega^*} R_n(x),$$

we obtain an operator from E into E^{**}; S is actually an operator from E into E because for every $x \in E$ and $n \geqslant 1$ we have

$$\operatorname{dist}(S(x), E) \leqslant \Big\|S(x) - \sum_{k=1}^{n} R_k(x) - x + \sum_{k=1}^{n} S_k(x)\Big\| \leqslant \sum_{k=n+1}^{\infty} \|R_k - S_k\|\,\|x\| \leqslant \varepsilon\|x\|/2^n;$$

moreover, since $\|I - S\| \leqslant \varepsilon$, S is an invertible operator and $\|S^{-1}\| \leqslant 1/(1-\varepsilon)$.

Now, the finite rank operators $U_n = S^{-1}R_n$ satisfy the conditions

$$x = \sum_{n=1}^{\infty} U_n(x), \qquad \sup_{|\theta_n|\leqslant 1} \Big\|\sum_{n=1}^{N} \theta_n U_n(x)\Big\| \leqslant \frac{1+2\varepsilon}{1-\varepsilon}\|x\|, \qquad \forall x \in E.$$

It follows that $(U_n)_{n\geqslant 1}$ is a $\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)$-unconditional finite-dimensional expansion of identity of E.

Now, by [24], Lemma 1.2, there is a Banach space F with a 1-unconditional FDD such that E is $\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)$-isomorphic to a $\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)$-complemented subspace in F. By renorming F, we deduce that F has a $\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)$-unconditional FDD and that E is isometric to a $\left(\frac{1+2\varepsilon}{1-\varepsilon}\right)^2$-complemented subspace of F.

In order to complete the proof let F be a Banach space with a $(1+\varepsilon)$-unconditional FDD, $(P_n)_{n\geqslant 1}$ the associated projections, J an isometry from the Banach space E into F, and P a projection of F with range $J(E)$ and $\|P\| \leqslant 1+\varepsilon$.

As in [15], adapting the proof of [11], Th. 3.3, we set

$$W = \overline{\operatorname{conv}}\Big(\bigcup_{N\geqslant 1} \{(\sum_{n=1}^{N} \theta_n P_n)(JE_1);\ |\theta_n| \leqslant 1\}\Big).$$

The set W is weakly conditionally compact (w.c.c.) if $E \not\supset l^1$ (resp. W is weakly compact (w.c.) if E is reflexive) ([11], Lemma 3.2); in particular, every sequence in W has a weakly Cauchy subsequence. Define ([6], Lemma 1)

$$W_n = \tfrac{1}{2}(2^n W + 2^{-n} F_1), \qquad n \geqslant 0,$$

and $|||x||| = (\sum_{n=0}^{\infty} \|x\|_n^2)^{1/2}$, where $\|\cdot\|_n$ is the gauge of W_n; then $F_0 = \{x \in F \mid |||x||| < \infty\}$ is a Banach space and the canonical injection $j\colon F_0 \to F$ is continuous ([6], Lemma 1 (ii)). Since $W \subset (1+\varepsilon)F_1$, it follows that $\|j\| \leqslant 1+\varepsilon$ because $|||x||| \leqslant 1 \Rightarrow \|x\|_0 \leqslant 1 \Rightarrow x \in \frac{1}{2}(W + F_1) \subset (1+\varepsilon)F_1$.

Since $JE_1 \subset W$, we have $J = j \circ A$, where $A\colon E \to F_0$ is an isomorphic embedding, with $\|A\| \leqslant 1$, $\|A^{-1}\| \leqslant 1+\varepsilon$.

Since $P_n(W) \subset W$, $n \geqslant 1$, F_0 has a $(1+\varepsilon)$-unconditional FDD ([6], Lemma 1 (viii), (ix)) which is shrinking (resp. F_0 is reflexive) since $(F_0)_1 = \bigcap_{n\geqslant 0} \frac{1}{2}(2^n W + 2^{-n} F_1)$ is w.c.c. (resp. w.c.) by a classical lemma of Grothendieck ([7], pp. 227, 237).

Renorming F_0 concludes the proof. ∎

6. Concluding remarks and questions

6.1. The Lorentz sequence spaces $d(w, p)$, $1 < p < \infty$, are reflexive Banach spaces with 1-symmetric basis such that $\mathscr{K}(d(w, p))$ is not an M-ideal in $\mathscr{L}(d(w, p))$ [18]; therefore the Theorem of this paper gives a necessary condition on E for $\mathcal{K}(E)$ to be an M-ideal in $\mathcal{L}(E)$ but it is not a sufficient condition. This leads to

QUESTION 1. Is it possible to find a necessary and sufficient condition on E for $\mathcal{K}(E)$ to be an M-ideal in $\mathcal{L}(E)$?

6.2. QUESTION 2. Can $\mathcal{K}(L^p(0,1))$, $1<p<\infty$, $p\neq 2$, be isomorphic to a space which is an M-ideal in its bidual? (see [14], Question III.4).

6.3. Let E be a Banach space with a monotone basis $\mathcal{U}=(u_n)_{n\geq 1}$, and let $\bar{u}_n\in\mathcal{K}_{\rm d}(E,\mathcal{U})$ be defined by $\bar{u}_n(u_k)=u_n$ if $k=n$, and 0 if $k\neq n$; $(\bar{u}_n)_{n\geq 1}$ is a basis of $\mathcal{K}_{\rm d}(E,\mathcal{U})$ ([28], Lemma 5), and it is easy to see that it is equivalent to the canonical basis of c_0 if and only if $\mathcal{U}$ is unconditional. A. Sersouri ([28], Problème 15) asked if $\mathcal{U}$ is unconditional whenever $\mathcal{K}_{\rm d}(E,\mathcal{U})$ is isomorphic to c_0. We may ask: is $\mathcal{U}$ unconditional whenever $\mathcal{K}_{\rm d}(E,\mathcal{U})$ is an M-ideal in $\mathcal{L}_{\rm d}(E,\mathcal{U})$ (resp. $\mathcal{K}_{\rm d}(E,\mathcal{U})$ is isomorphic to a space which is an M-ideal in its bidual)? (Let us remark that $\mathcal{K}_{\rm d}(E,\mathcal{U})^{**}=\mathcal{L}_{\rm d}(E,\mathcal{U})$ if $(\bar{u}_n)_{n\geq 1}$ is shrinking, and, in particular, if $\mathcal{U}$ is shrinking or boundedly complete ([28], Th. 8).) The answer is negative: if $\mathcal{U}$ is a conditional basis of l^p, $1<p<\infty$ ([23], I.2.b.11, II, p. 162), then $\mathcal{K}_{\rm d}(l^p,\mathcal{U})$ is an M-ideal in $\mathcal{L}_{\rm d}(l^p,\mathcal{U})=\mathcal{K}_{\rm d}(l^p,\mathcal{U})^{**}$ since $\mathcal{K}(l^p)$ is an M-ideal in its bidual. A weaker question can be asked (a partial answer is given in [28], Prop. 16):

QUESTION 3. If $\mathcal{K}_{\rm d}(E,\mathcal{U})$ is an M-ideal in $\mathcal{L}_{\rm d}(E,\mathcal{U})$ (resp. $\mathcal{K}_{\rm d}(E,\mathcal{U})$ is isomorphic to a space which is an M-ideal in its bidual), does $\mathcal{U}$ have an unconditional basic subsequence?

For $E=L^p(\mathbf{T})$, $\mathcal{U}=\mathcal{T}$ the trigonometric basis, the existence of $\Lambda(r)$-sets ($r\geq\max(p,p^*)$) implies the existence of an unconditional basic subsequence without the M-ideal hypothesis ([25], Th. 2.11).

6.4. Let us mention that every separable $\mathcal{L}^\infty$-space which is isomorphic to a space which is an M-ideal in its bidual is isomorphic to c_0 [13].

6.5. If we replace condition (H1) in the Theorem by the stronger condition:

(H1′) $\mathcal{F}(E)\cap\mathcal{B}$ is norm dense in $\mathcal{B}$

(which holds if $\mathcal{B}=\mathcal{K}_{\rm d}(E,\mathcal{U})$), we can take in the proof $R_n\in\mathcal{F}(E)\cap\mathcal{B}$, and if $\mathcal{A}$ is closed in the strong operator topology, then $S\in\mathcal{A}$; hence if $\mathcal{A}$ is also a full subalgebra $\mathcal{L}(E)$, then $S^{-1}\in\mathcal{A}$ (these two conditions hold when $\mathcal{A}=\mathcal{L}_{\rm d}(E,\mathcal{U})$); therefore $U_n\in\mathcal{F}(E)\cap\mathcal{B}$: E has a $(1+\varepsilon)$-unconditional finite-dimensional expansion of its identity, $(U_n)_{n\geq 1}$, *with the* U_n *in* $\mathcal{B}$.

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Received October 10, 1988 (2486)
Revised version February 22, 1989

Added in proof (January 1990). D. Werner (*Remarks on M-ideals of compact operators*, to appear in Quart. J. Math. Oxford) proved that the result of C.-M. Cho and W. B. Johnson also holds for subspaces of c_0-sums of finite-dimensional spaces. A similar result is given by E. Oja (C. R. Acad. Sci. Paris 309 (1989), 983–968).

Very recently, P. G. Casazza and N. J. Kalton (*Notes on approximation properties in separable Banach spaces*, preprint) have introduced the notion of μ-ideal which is more general than that of M-ideal and have given a characterization of separable reflexive Banach spaces X with AP for which $\mathscr{K}(X)$ is a μ-ideal in $\mathscr{L}(X)$.

STUDIA MATHEMATICA, T. XCVI (1990)

Partial differential operators of infinite order with constant coefficients on the space of analytic functions on the polydisc

by

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Abstract. For a componentwise radial plurisubharmonic function $p\colon \mathbf{C}^N \to \mathbf{R}_+$ satisfying some technical conditions we consider the (DFN)-space $A_{p,1}(\mathbf{C}^N) := \{f \in A(\mathbf{C}^N) \mid \exists k \in \mathbf{N}\colon \|f\|_k = \sup_{z\in\mathbf{C}^N} |f(z)| e^{-(1-1/k)p(z)} < \infty\}$ of analytic functions on $\mathbf{C}^N$. If we put $A_p^0(\mathbf{C}^N) := \{F \mid \forall k \in \mathbf{N}\colon F^k \in A_{p,1}(\mathbf{C}^N)\}$ then $A_{p,1}(\mathbf{C}^N)$ is an $A_p^0(\mathbf{C}^N)$-module such that $F \cdot A_{p,1}(\mathbf{C}^N)$ is a closed subspace of $A_{p,1}(\mathbf{C}^N)$ for each $F \in A_p^0(\mathbf{C}^N)$. We prove that $F \cdot A_{p,1}(\mathbf{C}^N)$ is a complemented subspace of $A_{p,1}(\mathbf{C}^N)$ for each $F \in A_p^0(\mathbf{C}^N)$ iff the strong dual $A_{p,1}(\mathbf{C}^N)'_b$ has the linear topological invariant $(\underline{\mathrm{DN}})$ iff $A_{p,1}(\mathbf{C}^N)$ itself is a complemented subspace of a corresponding weighted (LB)-space $L^2_{p,1}(\mathbf{C}^N)$ of locally square integrable functions on $\mathbf{C}^N$. Applying this result to the function $p(z) = \sum_{j=1}^N |z_j|$, $z \in \mathbf{C}^N$, we deduce that each nonzero linear partial differential operator of infinite order with constant coefficients on the Fréchet space $A(\Delta)$ of all analytic functions on the unit polydisc Δ in $\mathbf{C}^N$ admits a continuous linear right inverse. In our approach we use a sequence space representation of $A_{p,1}(\mathbf{C}^N)'_b$ and elementary function theory to give all the projections by explicit formulas.

For a plurisubharmonic function p on $\mathbf{C}^N$ denote by $A_{p,1}(\mathbf{C}^N)$ the space of all entire functions f satisfying $|f(z)| \leqslant Ae^{Bp(z)}$ for some $A > 0$, $1 > B > 0$ depending on f. If we allow arbitrarily large $B > 0$, then we get an algebra denoted by $A_p(\mathbf{C}^N)$. Both spaces, endowed with their natural inductive limit topology, are (DFN)-spaces, provided that p satisfies some technical conditions.

Recently Meise and Taylor ([12], [13]) showed for radial weights p that each principal ideal of $A_p(\mathbf{C}^N)$ is complemented if and only if the strong dual $A_p(\mathbf{C}^N)'_b$ has the linear topological invariant (DN).

In the present paper we find for componentwise radial weights p that each subspace $F \cdot A_{p,1}(\mathbf{C}^N)$ of $A_{p,1}(\mathbf{C}^N)$ is complemented if and only if the strong dual $A_{p,1}(\mathbf{C}^N)'_b$ has the linear topological invariant $(\underline{\mathrm{DN}})$ (see Wagner [21] and Vogt [19]). Our proof also gives rise to a new and more elementary proof of the above-mentioned result of Meise and Taylor.

To state an application of our result, we denote by $A(\Delta)$ the Fréchet space of all analytic functions on the unit polydisc Δ in $\mathbf{C}^N$. Then each nonzero