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Type and cotype numbers of operators on Banach spaces

by

ALBRECHT PIETSCH (Jena)

Abstract. For every operator acting between Banach spaces, we define the sequence of cotype numbers and the sequence of type numbers. The asymptotic behaviour of these sequences is used to introduce the scales of associated quasi-Banach operator ideals. The significance of these new concepts can be illustrated, for example, by the fact that a Banach space has weak cotype q if and only if the sequence of the cotype numbers of its identity map belongs to the Lorentz space $l_{q,\infty}$.

Since the fundamental work of B. Maurey and G. Pisier [8] in 1976, the concepts of type and cotype have proved to be extremely useful within the structure theory of Banach spaces. A new era began when V. D. Milman and G. Pisier [9] introduced a class of Banach spaces which are said to have weak cotype 2. Subsequently, G. Pisier [15, 16] proposed how to define the concepts of weak type p and weak cotype q with $1 < p \leqslant 2$ and $2 \leqslant q < \infty$, respectively. In the present paper we give a different approach to this subject which is based on operator ideal techniques.

In order to describe the main idea, we need to recall a well-known definition.

Let (g_k) be any sequence of independent normalized standard gaussian random variables on a probability space (Ω, μ). An operator T, acting from a Banach space E into a Banach space F, is called *Gauss-summing* if there exists a constant $c \geqslant 0$ such that

$$\Big(\int_\Omega \Big\| \sum_{k=1}^{n} g_k(\omega) Tx_k \Big\|^2 d\mu(\omega)\Big)^{1/2} \leqslant c \sup\Big\{\Big(\sum_{k=1}^{n} |\langle x_k, a\rangle|^2\Big)^{1/2}:\ \|a\| \leqslant 1\Big\}$$

for all finite families of elements $x_1, \ldots, x_n \in E$. Setting $\|T|\mathfrak{P}_G\| := \inf c$, the class of these operators becomes a Banach ideal, denoted by $\mathfrak{P}_G$.

We are now prepared to define the nth *cotype number* of an operator T by

$$x_n(T|\mathfrak{P}_G) := \sup\{a_n(TX):\ X \in \mathfrak{P}_G(l_2, E),\ \|X|\mathfrak{P}_G\| \leqslant 1\},$$

where $a_n(TX)$ is the nth approximation number of TX.

Replacing $\mathfrak{P}_G$ by the adjoint Banach operator ideal $\mathfrak{P}_G^*$ and switching the test operator from the left to the right, we obtain the nth *type number*

$$y_n(T|\mathfrak{P}_G^*) := \sup\{a_n(BT):\ B \in \mathfrak{P}_G^*(F, l_2),\ \|B|\mathfrak{P}_G^*\| \leqslant 1\}.$$

Next, for $0 < r < \infty$ and $0 < w \leqslant \infty$, the quasi-Banach operator ideals

$$(\mathfrak{P}_G)^{(x)}_{r,w} := \{T\colon\ (x_n(T|\mathfrak{P}_G)) \in l_{r,w}\},$$
$$(\mathfrak{P}^*_G)^{(y)}_{r,w} := \{T\colon\ (y_n(T|\mathfrak{P}^*_G)) \in l_{r,w}\}$$

are defined.

Denoting the identity map of a given Banach space E by I_E, we observe that E is of *weak cotype* q with $2 \leqslant q < \infty$ if and only if $I_E \in (\mathfrak{P}_G)^{(x)}_{q,\infty}$. A similar characterization holds for Banach spaces of *weak type* p with $1 < p \leqslant 2$. In this case, however, we must pass to the conjugate exponent p', which means that $I_E \in (\mathfrak{P}^*_G)^{(y)}_{p',\infty}$.

In conclusion, we state that the concepts of type and cotype numbers do not lead to completely new results; see Theorems 3.11 and 3.13 (Remarks). However, we get new insights into some interesting phenomena such as the relationship between eigenvalue distributions and geometry of Banach spaces [14] and the theory of weak Hilbert spaces [15, 16].

For the standard definitions, the reader is referred to my monographs *Operator Ideals* and *Eigenvalues and s-Numbers*, which are quoted as [OPI] and [EIG]. The notation is adopted from the latter.

1. Generalized Weyl and Chang numbers

1.1. Let $\mathfrak{A}$ be any quasi-Banach operator ideal. For $T \in \mathfrak{L}(E, F)$ and $n = 1, 2, \ldots$, we define the nth $\mathfrak{A}$-*Weyl number*

$$x_n(T|\mathfrak{A}) := \sup\{a_n(TX)\colon\ X \in \mathfrak{A}(l_2, E),\ \|X|\mathfrak{A}\| \leqslant 1\}$$

and the nth $\mathfrak{A}$-*Chang number*

$$y_n(T|\mathfrak{A}) := \sup\{a_n(BT)\colon\ B \in \mathfrak{A}(F, l_2),\ \|B|\mathfrak{A}\| \leqslant 1\}.$$

Remark. Letting $\mathfrak{A} = \mathfrak{L}$, we obtain the original Weyl numbers $x_n(T)$ and their dual counterparts $y_n(T)$, introduced by A. Pietsch [13].

1.2. First of all, we list some elementary properties.

PROPOSITION.

(1) $\|T\| = x_1(T|\mathfrak{A}) \geqslant x_2(T|\mathfrak{A}) \geqslant \ldots \geqslant 0$ *for* $T \in \mathfrak{L}(E, F)$.

(2) $x_{m+n-1}(S+T|\mathfrak{A}) \leqslant x_m(S|\mathfrak{A}) + x_n(T|\mathfrak{A})$ *for* $S, T \in \mathfrak{L}(E, F)$.

(3) $x_n(BTX|\mathfrak{A}) \leqslant \|B\|\, x_n(T|\mathfrak{A})\, \|X\|$ *for* $X \in \mathfrak{L}(E_0, E)$, $T \in \mathfrak{L}(E, F)$, *and* $B \in \mathfrak{L}(F, F_0)$.

(4) $x_n(T) = 0$ *if and only if* $\operatorname{rank}(T) < n$.

The $\mathfrak{A}$-Chang numbers have the same properties.

1.3. Next we change the definitions of the generalized Weyl and Chang numbers into more handy forms.

LEMMA. *For* $T \in \mathfrak{L}(E, F)$, *we have*

$$x_n(T|\mathfrak{A}) = \sup\{a_n(TX)\colon\ X \in \mathfrak{L}(l_2^m, E),\ \|X|\mathfrak{A}\| \leqslant 1,\ m \geqslant n\},$$
$$y_n(T|\mathfrak{A}) = \sup\{a_n(BT)\colon\ B \in \mathfrak{L}(F, l_2^m),\ \|B|\mathfrak{A}\| \leqslant 1,\ m \geqslant n\}.$$

Proof. Let $J_m \in \mathfrak{L}(l_2^m, l_2)$ be the injection defined by

$$J_m\colon\ (\xi_1, \ldots, \xi_m) \to (\xi_1, \ldots, \xi_m, 0, \ldots).$$

Note that, by [OPI, 11.5.2], $a_n(TX) = c_n(TX)$ for all $X \in \mathfrak{L}(l_2, E)$. Hence the maximality of the Gel'fand numbers [OPI, 11.10.12] implies that $a_n(TX) = \lim_m a_n(TXJ_m)$.

Analogously, defining the surjection $Q_m \in \mathfrak{L}(l_2, l_2^m)$ by

$$Q_m\colon\ (\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots) \to (\xi_1, \ldots, \xi_m)$$

and letting $B \in \mathfrak{L}(F, l_2)$, we deduce from $a_n(BT) = a_n(T'B')$ and $a_n(Q_m BT) = a_n(T'B'J_m)$ that $a_n(BT) = \lim_m a_n(Q_m BT)$.

1.4. PROPOSITION. *The* $\mathfrak{A}$-*Weyl numbers are injective, and the* $\mathfrak{A}$-*Chang numbers are surjective.*

Proof. By [OPI, 11.5.2 and 11.6.2], we have

$$x_n(T|\mathfrak{A}) = \sup\{c_n(TX)\colon\ X \in \mathfrak{A}(l_2, E),\ \|X|\mathfrak{A}\| \leqslant 1\}$$
$$y_n(T|\mathfrak{A}) = \sup\{d_n(BT)\colon\ B \in \mathfrak{A}(F, l_2),\ \|B|\mathfrak{A}\| \leqslant 1\}.$$

The assertions now follow from the injectivity of the Gel'fand numbers and the surjectivity of the Kolmogorov numbers, respectively.

1.5. The concepts of Weyl numbers and Chang numbers are dual to each other.

PROPOSITION. *For* $T \in \mathfrak{L}(E, F)$, *we have*

$$x_n(T'|\mathfrak{A}) = y_n(T|\mathfrak{A}'), \qquad y_n(T'|\mathfrak{A}) \geqslant x_n(T|\mathfrak{A}').$$

Proof. Let $B \in \mathfrak{A}'(F, l_2)$ and $\|B|\mathfrak{A}'\| \leqslant 1$. Then $B' \in \mathfrak{A}(l_2, F')$ and $\|B'|\mathfrak{A}\| \leqslant 1$. Hence it follows from

$$x_n(T'|\mathfrak{A}) \geqslant a_n(T'B') = a_n(BT)$$

that $x_n(T'|\mathfrak{A}) \geqslant y_n(T|\mathfrak{A}')$.

Let $Y \in \mathfrak{A}(F', l_2)$ and $\|Y|\mathfrak{A}\| \leqslant 1$. Define $B := Y'K_F$. Since $B' = K_F' Y'' = K_F' K_{F'} Y = Y$, we see that $B \in \mathfrak{A}'(F, l_2)$ and $\|B|\mathfrak{A}'\| \leqslant 1$. Hence it follows from

$$a_n(T'Y) = a_n(T'B') = a_n(BT) \leqslant y_n(T|\mathfrak{A}')$$

that $x_n(T'|\mathfrak{A}) \leqslant y_n(T|\mathfrak{A}')$.

Thus we have proved the left-hand equality. The right-hand inequality can be checked by similar arguments.

Remark. In order to obtain even the equality $y_n(T'|\mathfrak{A}) = x_n(T|\mathfrak{A}')$, we would need the following *principle of local reflexivity*.

For every operator $A \in \mathfrak{L}(E', l_2^m)$, every finite-dimensional subspace M of E' and $\varepsilon > 0$, there exists an operator $X \in \mathfrak{L}(l_2^m, E)$ such that

$$X'a = Aa \text{ for all } a \in M \text{ and } \|X|\mathfrak{A}'\| \leqslant (1+\varepsilon)\|A|\mathfrak{A}\|.$$

This property can easily be verified for every quasi-Banach operator ideal whose quasi-norm is left-splitting in the sense of S. Simons and T. J. Leih [18].

1.6. Let $0 < r < \infty$ and $0 < w \leqslant \infty$. An operator $T \in \mathfrak{L}(E, F)$ is said to be of *$\mathfrak{A}$-Weyl type $l_{r,w}$* if $(x_n(T|\mathfrak{A})) \in l_{r,w}$. The set of these operators is denoted by $\mathfrak{A}_{r,w}^{(x)}(E, F)$. We let

$$\|T|\mathfrak{A}_{r,w}^{(x)}\| := \|(x_n(T|\mathfrak{A}))|l_{r,w}\| \quad \text{for } T \in \mathfrak{A}_{r,w}^{(x)}(E, F).$$

Operators of *$\mathfrak{A}$-Chang type $l_{r,w}$* are defined analogously; the components of the resulting class are denoted by $\mathfrak{A}_{r,w}^{(y)}(E, F)$.

1.7. The following result is obvious.

THEOREM. *$\mathfrak{A}_{r,w}^{(x)}$ and $\mathfrak{A}_{r,w}^{(y)}$ are quasi-Banach operator ideals.*

1.8. We now state an immediate consequence of 1.4 and 1.5.

THEOREM. *The quasi-Banach operator ideal $\mathfrak{A}_{r,w}^{(x)}$ is injective, and $\mathfrak{A}_{r,w}^{(y)}$ is surjective. Moreover,*

$$(\mathfrak{A}_{r,w}^{(x)})' = (\mathfrak{A}')_{r,w}^{(y)}, \qquad (\mathfrak{A}_{r,w}^{(y)})' \subseteq (\mathfrak{A}')_{r,w}^{(x)}.$$

1.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be arbitrary quasi-Banach operator ideals. Combining the definition of $\mathfrak{A}$-Weyl numbers with that of $\mathfrak{B}$-Chang numbers, for $T \in \mathfrak{L}(E, F)$, we put

$$z_n(T|\mathfrak{A}, \mathfrak{B}) := \sup\left\{a_n(BTX): \begin{array}{ll} X \in \mathfrak{A}(l_2, E), & \|X|\mathfrak{A}\| \leqslant 1 \\ B \in \mathfrak{B}(F, l_2), & \|B|\mathfrak{B}\| \leqslant 1 \end{array}\right\}.$$

If $\mathfrak{A} = \mathfrak{L}$ and $\mathfrak{B} = \mathfrak{L}$, then the well-known *Hilbert numbers* result. The case $\mathfrak{A} = \mathfrak{P}_2'$ and $\mathfrak{B} = \mathfrak{P}_2$ is of special interest in the theory of eigenvalue distributions of nuclear operators; see [14]. To simplify notation, we write $z_n(T) := z_n(T|\mathfrak{P}_2', \mathfrak{P}_2)$.

1.10. LEMMA. *For $T \in \mathfrak{L}(E, F)$, we have*

$$n^{1/2} z_{2n-1}(T) \leqslant x_n(T|\mathfrak{P}_2') \leqslant n^{1/2}\Big(\prod_{k=1}^{n} z_k(T)\Big)^{1/n},$$

$$n^{1/2} z_{2n-1}(T) \leqslant y_n(T|\mathfrak{P}_2) \leqslant n^{1/2}\Big(\prod_{k=1}^{n} z_k(T)\Big)^{1/n}.$$

Proof. Let $X \in \mathfrak{P}_2'(l_2, E)$ and $B \in \mathfrak{P}_2(F, l_2)$ such that $\|X|\mathfrak{P}_2'\| \leqslant 1$ and $\|B|\mathfrak{P}_2\| \leqslant 1$. Then, by [EIG, 2.4.17],

$$a_{2n-1}(BTX) = x_{2n-1}(BTX) \leqslant x_n(B)x_n(TX) = x_n(B)a_n(TX).$$

From [EIG, 2.7.3], we know that $n^{1/2}x_n(B) \leqslant \|B|\mathfrak{P}_2\| \leqslant 1$. Hence

$$n^{1/2}a_{2n-1}(BTX) \leqslant x_n(T|\mathfrak{P}_2').$$

This proves that $n^{1/2}z_{2n-1}(T) \leqslant x_n(T|\mathfrak{P}_2')$.

Let $X \in \mathfrak{P}_2'(l_2, E)$ and $\|X|\mathfrak{P}_2'\| \leqslant 1$. Given $\varepsilon > 0$, by [EIG, 2.10.3], we find $u_1, \ldots, u_n \in l_2$ and $b_1, \ldots, b_n \in F'$ such that $\|u_i\| \leqslant 1$, $\|b_j\| \leqslant 1$,

$$a_k(TX) = c_k(TX) \leqslant (1+\varepsilon)|\langle TXu_k, b_k\rangle|,$$

$$\langle TXu_i, b_j\rangle = 0 \quad \text{if } i > j.$$

Define $B \in \mathfrak{L}(F, l_2)$ by $B := \sum_{j=1}^{n} b_j \otimes e_j$. Then $\|B|\mathfrak{P}_2\| \leqslant n^{1/2}$ and $b_j = B'e_j$. We now obtain

$$a_n(TX) \leqslant \Big(\prod_{k=1}^{n} a_k(TX)\Big)^{1/n} \leqslant (1+\varepsilon)|\det(\langle TXu_i, B'e_j\rangle)|^{1/n}$$

$$\leqslant (1+\varepsilon)\Big(\prod_{k=1}^{n} a_k(BTX)\Big)^{1/n} \leqslant (1+\varepsilon)n^{1/2}\Big(\prod_{k=1}^{n} z_k(T)\Big)^{1/n}.$$

Taking the supremum over X and letting $\varepsilon \to 0$ yields

$$x_n(T|\mathfrak{P}_2') \leqslant n^{1/2}\Big(\prod_{k=1}^{n} z_k(T)\Big)^{1/n}.$$

Thus we have proved the first series of inequalities. The second one can be obtained by passing to the dual operator.

1.11. The following result is now obvious; see [EIG, 2.1.8 and 2.1.9].

THEOREM. *Let $0 < r < 2$ and $0 < s < \infty$ such that $1/r = 1/s + 1/2$. Then*

$$\mathfrak{L}_{r,w}^{(z)} = (\mathfrak{P}_2')_{s,w}^{(x)} = (\mathfrak{P}_2)_{s,w}^{(y)}.$$

1.12. We conclude this chapter with two multiplication formulas which will not be used in the present paper. However, they may become of interest for future research. We first generalize [EIG, 2.4.17].

LEMMA. *Let $T \in \mathfrak{L}(E, F)$ and $S \in \mathfrak{L}(F, G)$. Then*

$$x_{m+n-1}(ST|\mathfrak{A}) \leqslant x_m(S)x_n(T|\mathfrak{A}).$$

Proof. Assume that $X \in \mathfrak{A}(l_2, E)$ and $\|X|\mathfrak{A}\| \leqslant 1$. Given $\varepsilon > 0$, we choose $U \in \mathfrak{L}(l_2, F)$ such that

$$\|TX - U\| \leqslant (1+\varepsilon)a_n(TX) \quad \text{and} \quad \operatorname{rank}(U) < n.$$

Next we choose $V \in \mathfrak{L}(l_2, G)$ such that

$$\|S(TX-U)-V\| \leqslant (1+\varepsilon)a_m(S(TX-U)) \quad \text{and} \quad \operatorname{rank}(V) < m.$$

Then

$$a_{m+n-1}(STX) \leqslant \|STX-SU-V\| \leqslant (1+\varepsilon)a_m(S(TX-U))$$
$$\leqslant (1+\varepsilon)x_m(S)\|TX-U\| \leqslant (1+\varepsilon)^2 x_m(S)a_n(TX),$$

which in turn implies that

$$x_{m+n-1}(ST|\mathfrak{A}) \leqslant (1+\varepsilon)^2 x_m(S)x_n(T|\mathfrak{A}).$$

Letting $\varepsilon \to 0$ yields the required inequality.

1.13. In order to state the second multiplication formula, we need to recall a well-known concept.

Given any quasi-Banach operator ideal $\mathfrak{A}$, the nth *$\mathfrak{A}$-approximation number* of $T \in \mathfrak{A}(E, F)$ is defined by

$$a_n(T|\mathfrak{A}) := \inf\{\|T-L|\mathfrak{A}\|: \ L \in \mathfrak{L}(E, F), \ \operatorname{rank}(L) < n\};$$

see [EIG, 2.8.1].

LEMMA. *Let* $T \in \mathfrak{A}(E, F)$ *and* $S \in \mathfrak{L}(F, G)$. *Then*

$$x_{m+n-1}(ST) \leqslant x_m(S|\mathfrak{A})a_n(T|\mathfrak{A}).$$

Proof. Assume that $X \in \mathfrak{L}(l_2, E)$ and $\|X\| \leqslant 1$. Then it follows from $a_m(STX) \leqslant x_m(S|\mathfrak{A})\|TX|\mathfrak{A}\|$ that $x_m(ST) \leqslant x_m(S|\mathfrak{A})\|T|\mathfrak{A}\|$. Let $L \in \mathfrak{L}(E, F)$ such that $\operatorname{rank}(L) < n$. Then

$$x_{m+n-1}(ST) \leqslant x_m(S(T-L)) + x_n(STL) \leqslant x_m(S|\mathfrak{A})\|T-L|\mathfrak{A}\|.$$

Passing to the infimum over L on the right-hand side completes the proof.

2. Gauss-summing operators

2.1. The *gaussian measure* γ is defined by

$$\gamma(B) := \frac{1}{\sqrt{2\pi}} \int_B e^{-\zeta^2/2}\, d\zeta,$$

where B denotes any Borel subset of the real line. In the complex case, we let

$$\gamma(B) := \frac{1}{\pi} \int_B e^{-(\xi^2+\eta^2)}\, d\xi\, d\eta,$$

where the variable ζ is split into its real and imaginary parts, $\zeta = \xi + i\eta$. In both cases the normalization is chosen such that the second absolute moment takes the value 1.

2.2. Let (g_k) be any *sequence of independent standard gaussian random variables* defined on a probability space (Ω, μ). This means, among other things, that $\mu\{\omega \in \Omega: \ g_k(\omega) \in B\} = \gamma(B)$ for every Borel subset B.

2.3. An operator $T \in \mathfrak{L}(E, F)$ is said to be *Gauss-summing* if there exists a constant $c \geqslant 0$ such that

$$\Big(\int_\Omega \Big\|\sum_{k=1}^n g_k(\omega) Tx_k\Big\|^2 d\mu(\omega)\Big)^{1/2} \leqslant c \sup\Big\{\Big(\sum_{k=1}^n |\langle x_k, a\rangle|^2\Big)^{1/2}: \ \|a\| \leqslant 1\Big\}$$

for all finite families of elements $x_1, \ldots, x_n \in E$. Setting $\|T|\mathfrak{P}_G\| := \inf c$, the class of these operators becomes an injective Banach ideal, denoted by $\mathfrak{P}_G$; see [6, Theorem 1].

Remark. For $X \in \mathfrak{L}(l_2^m, E)$ the norm $\|X|\mathfrak{P}_G\|$ is usually denoted by $l(X)$, and we have

$$l(X) = \Big(\int \ldots \int \Big\|\sum_{k=1}^m \zeta_k X e_k\Big\|^2 d\gamma(\zeta_1) \ldots d\gamma(\zeta_m)\Big)^{1/2};$$

see [2] and [6, Lemma 3].

2.4. In the following we collect some elementary but important inequalities.

LEMMA. $\|B|\mathfrak{P}_2\| \leqslant \|B|\mathfrak{P}_G^*\|$ *for all* $B \in \mathfrak{L}(F, l_2^m)$.

Proof. As is well known from [6, Theorem 6], $T \in \mathfrak{P}_2(E, F)$ implies $T \in \mathfrak{P}_G(E, F)$ and $\|T|\mathfrak{P}_G\| \leqslant \|T|\mathfrak{P}_2\|$. Passing to the adjoint Banach operator ideals and taking into account that $\mathfrak{P}_2$ is self-adjoint, we conclude that $T \in \mathfrak{P}_G^*(F, E)$ implies $T \in \mathfrak{P}_2(F, E)$ and $\|T|\mathfrak{P}_2\| \leqslant \|T|\mathfrak{P}_G^*\|$. This is much more than the desired result.

2.5. The next inequality is due to T. Figiel and N. Tomczak-Jaegermann [2]; see also [19, p. 92].

LEMMA. $\|B|\mathfrak{P}_G^*\| \leqslant \|B|\mathfrak{P}_G'\|$ *for all* $B \in \mathfrak{L}(F, l_2^m)$.

2.6. We now combine the preceding statements.

LEMMA. $\|X|\mathfrak{P}_2'\| \leqslant \|X|\mathfrak{P}_G\|$ *for all* $X \in \mathfrak{L}(l_2^m, E)$.

Proof. Applying 2.4 and 2.5 to the operator $X' \in \mathfrak{L}(E', l_2^m)$ and using the injectivity of $\mathfrak{P}_G$, we obtain

$$\|X|\mathfrak{P}_2'\| = \|X'|\mathfrak{P}_2\| \leqslant \|X'|\mathfrak{P}_G^*\| \leqslant \|X'|\mathfrak{P}_G'\| = \|X''|\mathfrak{P}_G\| = \|X|\mathfrak{P}_G\|.$$

A more direct proof can be obtained as follows. Given $a_1, \ldots, a_n \in E'$, we define $A \in \mathfrak{L}(E, l_2^n)$ by $A := \sum_{i=1}^n a_i \otimes e_i$. Then

$$\|A\| = \sup\Big\{\Big(\sum_{i=1}^n |\langle x, a_i\rangle|^2\Big)^{1/2}: \ \|x\| \leqslant 1\Big\}.$$

Since AX acts between Hilbert spaces, we know that $\|AX|\mathfrak{S}_2\| = \|AX|\mathfrak{P}_G\|$. Hence

$$\Big(\sum_{i=1}^{n} \|X'a_i\|^2\Big)^{1/2} = \Big(\sum_{i=1}^{n} \|X'A'e_i\|^2\Big)^{1/2}$$
$$\leqslant \|X'A'|\mathfrak{P}_2\| = \|X'A'|\mathfrak{S}_2\| = \|AX|\mathfrak{S}_2\| = \|AX|\mathfrak{P}_G\|$$
$$\leqslant \|X|\mathfrak{P}_G\| \sup\Big\{\Big(\sum_{i=1}^{n} |\langle x, a_i\rangle|^2\Big)^{1/2}: \|x\| \leqslant 1\Big\}.$$

This proves that $\|X|\mathfrak{P}_2'\| \leqslant \|X|\mathfrak{P}_G\|$.

2.7. Finally, we state an inequality which can be obtained by adapting the proof of Lemma 3 in [3]; see also [6, Lemmas 5 and 6].

PROPOSITION. *There exists a constant $c > 0$ such that*

$$(1+\log n)^{1/2} a_n(X) \leqslant c\|X|\mathfrak{P}_G\| \quad \textit{for all } X \in \mathfrak{P}_G(l_2, E).$$

Remark. In the setting of Weyl numbers the preceding result reads as follows:

$$(1+\log n)^{1/2} x_n(T) \leqslant c\|T|\mathfrak{P}_G\| \quad \text{for all } T \in \mathfrak{P}_G(E, F).$$

3. Type and cotype numbers

3.1. Let $2 \leqslant q < \infty$. An operator $T \in \mathfrak{L}(E, F)$ is said to be of (gaussian) *cotype* q if there exists a constant $c \geqslant 0$ such that

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^q\Big)^{1/q} \leqslant c\Big(\int_\Omega \Big\|\sum_{k=1}^{n} g_k(\omega)x_k\Big\|^2 d\mu(\omega)\Big)^{1/2}$$

for all finite families of elements $x_1, \ldots, x_n \in E$. The set of these operators is denoted by $\mathfrak{C}_q(E, F)$. For $T \in \mathfrak{C}_q(E, F)$, we let $\|T|\mathfrak{C}_q\| := \inf c$, where the infimum is taken over all possible constants c. Note that $\mathfrak{C}_q$ is an injective Banach operator ideal.

The following criterion holds (see [19, p. 83]): An operator $T \in \mathfrak{L}(E, F)$ is of cotype q if and only if $TX \in \mathfrak{P}_{q,2}(l_2, F)$ for all $X \in \mathfrak{P}_G(l_2, E)$. In this case,

$$\|T|\mathfrak{C}_q\| = \sup\{\|TX|\mathfrak{P}_{q,2}\|: X \in \mathfrak{P}_G(l_2, E),\ \|X|\mathfrak{P}_G\| \leqslant 1\}.$$

3.2. We now state a fundamental result.

THEOREM. *Let $2 < q < \infty$. Then*

$$(\mathfrak{P}_G)^{(x)}_{q,q} \subseteq \mathfrak{C}_q \subseteq (\mathfrak{P}_G)^{(x)}_{q,\infty}.$$

In the limiting case $q = 2$ we have

$$(\mathfrak{P}_G)^{(x)}_{2,1} \subseteq \mathfrak{C}_2 \subseteq (\mathfrak{P}_G)^{(x)}_{2,\infty}.$$

Proof. The inclusions follows immediately from the preceding criterion and the fact that

$$\mathfrak{L}^{(a)}_{q,q}(l_2, F) \subseteq \mathfrak{P}_{q,2}(l_2, F) \subseteq \mathfrak{L}^{(a)}_{q,\infty}(l_2, F),$$
$$\mathfrak{L}^{(a)}_{2,1}(l_2, F) \subseteq \mathfrak{P}_{2,2}(l_2, F) \subseteq \mathfrak{L}^{(a)}_{2,\infty}(l_2, F);$$

see [EIG, 2.7.4 and 2.7.5].

Remark. As trivial consequences we obtain the inequalities

$$\|T|\mathfrak{C}_{q_0}\| \leqslant cn^{1/q_0 - 1/q_1}\|T|\mathfrak{C}_{q_1}\| \quad \text{whenever } \operatorname{rank}(T) \leqslant n$$

and $2 \leqslant q_0 < q_1 < \infty$ as well as

$$\|T|\mathfrak{C}_q\| \leqslant (1+\log n)^{1/q}\|T|(\mathfrak{P}_G)^{(x)}_{q,\infty}\| \quad \text{whenever } \operatorname{rank}(T) \leqslant n$$

and $2 < q < \infty$; see [4, p. 117], [19, p. 193] and [16, p. 123].

We conjecture that all inclusions, stated in this subsection, are strict.

3.3. In view of the above theorem, we refer to $x_n(T|\mathfrak{P}_G)$ as the nth *cotype number* of the operator T. The nth cotype number of the identity map of a Banach space E is simply denoted by $x_n(E|\mathfrak{P}_G)$.

3.4. For the sake of completeness, we state the following consequence of Proposition 1.4.

PROPOSITION. *The cotype numbers are injective.*

3.5. Let $1 < p \leqslant 2$. An operator $T \in \mathfrak{L}(E, F)$ is said to be of (gaussian) *type* p if there exists a constant $c \geqslant 0$ such that

$$\Big(\int_\Omega \Big\|\sum_{k=1}^{n} g_k(\omega)Tx_k\Big\|^2 d\mu(\omega)\Big)^{1/2} \leqslant c\Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{1/p}$$

for all finite families of elements $x_1, \ldots, x_n \in E$. The set of these operators is denoted by $\mathfrak{T}_p(E, F)$. For $T \in \mathfrak{T}_p(E, F)$, we let $\|T|\mathfrak{T}_p\| := \inf c$, where the infimum is taken over all possible constants c. Note that $\mathfrak{T}_p$ is an injective and surjective Banach operator ideal.

The following criterion holds (see [19, p. 83]): An operator $T \in \mathfrak{L}(E, F)$ is of type p if and only if $BT \in \mathfrak{P}'_{p',2}(E, F)$ for all $B \in \mathfrak{P}^*_G(F, l_2)$. In this case,

$$\|T|\mathfrak{T}_p\| = \sup\{\|BT|\mathfrak{P}'_{p',2}\|: B \in \mathfrak{P}^*_G(F, l_2),\ \|B|\mathfrak{P}^*_G\| \leqslant 1\}.$$

3.6. The following counterpart of Theorem 3.2 is valid.

THEOREM. *Let $1 < p < 2$. Then*

$$(\mathfrak{P}^*_G)^{(y)}_{p',p'} \subseteq \mathfrak{T}_p \subseteq (\mathfrak{P}^*_G)^{(y)}_{p',\infty}.$$

In the limiting case $p = 2$, we have

$$(\mathfrak{P}^*_G)^{(y)}_{2,1} \subseteq \mathfrak{T}_2 \subseteq (\mathfrak{P}^*_G)^{(y)}_{2,\infty}.$$

Proof. Passing to the dual operators, we may use the same argument as in 3.2.

3.7. In view of the above theorem, we refer to $y_n(T|\mathfrak{P}_G^*)$ as the nth *type number* of the operator T. The nth type number of the identity map of a Banach space E is simply denoted by $y_n(E|\mathfrak{P}_G^*)$.

3.8. PROPOSITION. *The type numbers are injective and surjective.*

Proof. The surjectivity follows immediately from Proposition 1.4, and the injectivity can easily be deduced from the extension property stated in Lemma 3.15.

3.9. Next we describe the duality between the type and cotype numbers.

LEMMA. $x_n(T'|\mathfrak{P}_G) \leqslant y_n(T|\mathfrak{P}_G^*)$ *for all* $T \in \mathfrak{L}(E, F)$.

Proof. By 1.5, we have $x_n(T'|\mathfrak{P}_G) = y_n(T|\mathfrak{P}'_G)$ for all $T \in \mathfrak{L}(E, F)$. The conclusion now follows from the fact that $\|B|\mathfrak{P}_G^*\| \leqslant \|B|\mathfrak{P}'_G\|$ for all $B \in \mathfrak{L}(F, l_2^m)$; see Lemmas 1.3 and 2.5.

Remark. If the Banach space F is *K-convex*, then there exists a constant $c \geqslant 1$ such that $\|B|\mathfrak{P}'_G\| \leqslant c\|B|\mathfrak{P}_G^*\|$ for all $B \in \mathfrak{L}(F, l_2^m)$. This implies a reverse inequality, namely

$$y_n(T|\mathfrak{P}_G^*) \leqslant c x_n(T'|\mathfrak{P}_G) \quad \text{for all } T \in \mathfrak{L}(E, F).$$

3.10. LEMMA. *Let* $T \in \mathfrak{L}(E, F)$. *Then*

$$x_n(T|\mathfrak{P}_G) \leqslant x_n(T|\mathfrak{P}'_2), \quad y_n(T|\mathfrak{P}_G^*) \leqslant y_n(T|\mathfrak{P}_2).$$

Proof. The inequalities follow immediately from the facts that $\|X|\mathfrak{P}'_2\| \leqslant \|X|\mathfrak{P}_G\|$ for all $X \in \mathfrak{L}(l_2^m, E)$ and $\|B|\mathfrak{P}_2\| \leqslant \|B|\mathfrak{P}_G^*\|$ for all $B \in \mathfrak{L}(F, l_2^m)$; see Lemmas 1.3, 2.6 and 2.4.

3.11. We now state an important consequence of Theorem 1.11 and the preceding lemma.

THEOREM. *Let* $0 < r < 2$, $0 < s < \infty$, $0 < w \leqslant \infty$ *and* $1/r = 1/s + 1/2$. *Then*

$$\mathfrak{L}_{r,w}^{(z)} \subseteq (\mathfrak{P}_G)_{s,w}^{(x)} \cap (\mathfrak{P}_G^*)_{s,w}^{(y)}.$$

Remark. This is a generalization as well as a strengthening of an earlier result due to H. König, J. R. Retherford and N. Tomczak-Jaegermann [4, pp. 116–117]; see 3.14.

3.12. LEMMA. *Let* $T \in \mathfrak{L}(E, F)$ *and* $S \in \mathfrak{L}(F, G)$. *Then*

$$z_{4n-3}(ST) \leqslant 4x_n(S|\mathfrak{P}_G)y_n(T|\mathfrak{P}_G^*).$$

Proof. Write $m := 4n-3$, and assume that $z_m(ST) > 0$. Given $\varepsilon > 0$, we choose $X \in \mathfrak{L}(l_2^m, E)$ and $C \in \mathfrak{L}(G, l_2^m)$ such that

$$z_m(ST) \leqslant (1+\varepsilon)a_m(CSTX), \quad \|X|\mathfrak{P}'_2\| \leqslant 1, \quad \|C|\mathfrak{P}_2\| \leqslant 1.$$

Then $M := M(TX)$ is an m-dimensional subspace of F. We now apply a powerful lemma due to D. R. Lewis [5]; see also [19, p. 115]. This yields operators $B_0 \in \mathfrak{L}(M, l_2^m)$ and $Y_0 \in \mathfrak{L}(l_2^m, M)$ such that

$$Y_0 B_0 = I_M, \quad \|B_0|\mathfrak{P}_G^*\| \leqslant m^{1/2}, \quad \|Y_0|\mathfrak{P}_G\| \leqslant m^{1/2}.$$

Let $Y := JY_0$, where J denotes the embedding map from M into F. Recall that the Banach ideal $\mathfrak{P}_G$ is injective. Thus, using the extension property stated in Lemma 3.15, we can find $B \in \mathfrak{L}(F, l_2^m)$ such that $BJ = B_0$ and $\|B|\mathfrak{P}_G^*\| = \|B_0|\mathfrak{P}_G^*\| \leqslant m^{1/2}$.

The preceding construction gives the following diagram:

$$CSTX:\ l_2^m \xrightarrow{X} E \xrightarrow{T} F \xrightarrow{B} l_2^m \xrightarrow{Y} F \xrightarrow{S} G \xrightarrow{C} l_2^m.$$

Therefore

$$a_m(CSTX) = a_{4n-3}(CSYBTX) \leqslant a_{2n-1}(CSY)a_{2n-1}(BTX).$$

Next both factors are treated separately. To this end, we use some results about generalized approximation numbers, which can be found in [14, 6.3]; see also 1.13. We conclude first that

$$n^{1/2}a_{2n-1}(CSY) \leqslant a_n(CSY|\mathfrak{P}_2) \leqslant \|C|\mathfrak{P}_2\| a_n(SY)$$
$$\leqslant x_n(S|\mathfrak{P}_G)\|Y|\mathfrak{P}_G\| \leqslant m^{1/2}x_n(S|\mathfrak{P}_G).$$

Hence $a_{2n-1}(CSY) \leqslant 2x_n(S|\mathfrak{P}_G)$.

Secondly, it follows from

$$n^{1/2}a_{2n-1}(BTX) \leqslant a_n(BTX|\mathfrak{P}'_2) \leqslant a_n(BT)\|X|\mathfrak{P}'_2\|$$
$$\leqslant \|B|\mathfrak{P}_G^*\| y_n(T|\mathfrak{P}_G^*) \leqslant m^{1/2}y_n(T|\mathfrak{P}_G^*)$$

that $a_{2n-1}(BTX) \leqslant 2y_n(T|\mathfrak{P}_G^*)$.

Combining the previous results, we obtain

$$z_m(ST) \leqslant (1+\varepsilon)a_m(CSTX) \leqslant 4(1+\varepsilon)x_n(S|\mathfrak{P}_G)y_n(T|\mathfrak{P}_G^*).$$

Remark. Note that the above proof uses only the fact that $\mathfrak{P}_G$ is an injective Banach operator ideal.

3.13. As an immediate consequence of the preceding lemma we now obtain an important multiplication formula.

THEOREM. *Let* $0 < p, q, r < \infty$ *and* $0 < u, v, w \leqslant \infty$. *If* $1/p + 1/q = 1/r$ *and* $1/u + 1/v = 1/w$, *then*

$$(\mathfrak{P}_G)_{q,v}^{(x)} \circ (\mathfrak{P}_G^*)_{p,u}^{(y)} \subseteq \mathfrak{L}_{r,w}^{(z)}.$$

Remark. It follows from an example given by J. Bourgain that the exponent r defined by $1/r := 1/p + 1/q$ is the best possible; see [1, p. 117] and [19, p. 212]. For the translation of Bourgain's result into the present setting we refer to the next subsection.

3.14. To explain the interplay of our (idealistic) approach with the structure theory of Banach spaces, we use the following notation ($1 < p \leqslant 2$ and $2 \leqslant q < \infty$).

$\mathbf{T}_p$: *the class of Banach spaces with type p,*
$\mathbf{T}_{p,\infty}$: *the class of Banach spaces with weak type p,*
$\mathbf{C}_q$: *the class of Banach spaces with cotype q,*
$\mathbf{C}_{q,\infty}$: *the class of Banach spaces with weak cotype q.*

For the original definitions of the "*weak*" concepts we refer the reader to the recent book of G. Pisier [16, p. 130] and to the thesis of V. Mascioni [7].

Furthermore, for $1 \leqslant r < 2$, we denote by $\mathbf{H}_{r,\infty}$ the class of all Banach spaces E such that $(\lambda_n(S)) \in l_{r,\infty}$ for all nuclear operators $S \in \mathfrak{L}(E)$. Here $(\lambda_n(S))$ is the eigenvalue sequence of S.

The previous results of this chapter (3.2, 3.6, 3.11 and 3.13) combined with Theorems 3.9 and 4.7 of [14] yield the following:

$$\mathbf{H}_{r,\infty} \subset \mathbf{T}_{p,\infty} \subset \mathbf{T}_{p-\varepsilon} \quad \text{for } 1/p' = 1/r - 1/2 \text{ and } \varepsilon > 0,$$

$$\mathbf{H}_{r,\infty} \subset \mathbf{C}_{q,\infty} \subset \mathbf{C}_{q+\varepsilon} \quad \text{for } 1/q = 1/r - 1/2 \text{ and } \varepsilon > 0;$$

see [7, Theorem 4.5]. Conversely, we have

$$\mathbf{T}_{p,\infty} \cap \mathbf{C}_{q,\infty} \subseteq \mathbf{H}_{r,\infty} \quad \text{with } 1/r = 1/p' + 1/q.$$

This inclusion is meaningful only for $1/p - 1/q < 1/2$.

Let $E \in \mathbf{C}_{2,\infty}$ and $1/p' = 1/r - 1/2$. Then

$$E \in \mathbf{H}_{r,\infty} \quad \text{if and only if} \quad E \in \mathbf{T}_{p,\infty}.$$

Let $E \in \mathbf{T}_{2,\infty}$ and $1/q = 1/r - 1/2$. Then

$$E \in \mathbf{H}_{r,\infty} \quad \text{if and only if} \quad E \in \mathbf{C}_{q,\infty};$$

see [7, Theorem 2.10].

In the limiting case $p = q = 2$ we obtain $\mathbf{T}_{2,\infty} \cap \mathbf{C}_{2,\infty} = \mathbf{H}_{1,\infty}$. This is just G. Pisier's definition of the class of weak Hilbert spaces; see [15, 16].

3.15. For the convenience of the reader, we add a well-known auxiliary result; see [2, p. 157] and [10, 15.5].

LEMMA. *Suppose that the Banach operator ideal $\mathfrak{A}$ is injective. Let M be a subspace of the Banach space E, and denote the associated embedding map by J. Let F be a finite-dimensional Banach space. Then every operator $T_0 \in \mathfrak{L}(M, F)$ admits an $\mathfrak{A}$-norm preserving extension $T \in \mathfrak{L}(E, F)$. This means that $TJ = T_0$ and $\|T|\mathfrak{A}\| = \|T_0|\mathfrak{A}\|$.*

Proof. Apply the Hahn–Banach theorem via trace duality.

4. Examples

4.1. We begin with an immediate consequence of Dvoretzky's theorem on almost spherical sections; see [10, 5.8] or [16, Theorem 4.3].

PROPOSITION. *If the Banach space E is infinite-dimensional, then*

$$x_n(E|\mathfrak{P}_G) \geqslant n^{-1/2}, \quad y_n(E|\mathfrak{P}_G^*) \geqslant n^{-1/2}.$$

Proof. It is well known that, for every operator between finite-dimensional Hilbert spaces, we have $\|T|\mathfrak{P}_G\| = \|T|\mathfrak{P}_G^*\| = \|T|\mathfrak{S}_2\|$, where $\|T|\mathfrak{S}_2\|$ denotes the Hilbert–Schmidt norm of T. Using this fact, it can easily be seen that

$$x_n(l_2^n|\mathfrak{P}_G) = y_n(l_2^n|\mathfrak{P}_G^*) = n^{-1/2},$$

which implies the required estimate.

4.2. LEMMA. *Let $2 \leqslant p < \infty$. Then*

$$y_n(l_{p'}^{2n}|\mathfrak{P}_G^*) \geqslant x_n(l_p^{2n}|\mathfrak{P}_G) \geqslant cn^{-1/p},$$

where the constant $c > 0$ depends only on p.

Proof. It follows from

$$a_n(I: l_2^{2n} \to l_\infty^{2n}) \leqslant a_n(I: l_2^{2n} \to l_p^{2n}) \|I: l_p^{2n} \to l_\infty^{2n}\|$$

and [OPI, 11.11.8]

$$a_n(I: l_1^m \to l_2^m) = \left(\frac{m-n+1}{m}\right)^{1/2} \quad \text{for } n = 1, \ldots, m$$

that $a_n(I: l_2^{2n} \to l_p^{2n}) \geqslant \sqrt{1/2}$. On the other hand, we have

$$\|I: l_2^{2n} \to l_p^{2n}|\mathfrak{P}_G\| \leqslant \|I: l_2^{2n} \to l_p^{2n}|\mathfrak{P}_2\| = (2n)^{1/p}.$$

This implies the desired lower estimate for the cotype numbers. An application of Lemma 3.9 completes the proof.

4.3. LEMMA. $y_n(l_1^{2n}|\mathfrak{P}_G^*) \geqslant c > 0$.

Proof. Since l_1 is of cotype 2, we have $\mathfrak{P}_2(l_2, l_1) = \mathfrak{P}_G(l_2, l_1)$, by [6, Theorem 7]. Passing to the adjoint Banach operator ideals yields $\mathfrak{P}_2(l_1, l_2) = \mathfrak{P}_G^*(l_1, l_2)$. Using the well-known fact that the identity map from l_1 into l_2 is 2-summing, we obtain

$$\|I: l_1^m \to l_2^m|\mathfrak{P}_G^*\| \leqslant c_0 := \|I: l_1 \to l_2|\mathfrak{P}_G^*\|.$$

Recall that

$$a_n(I: l_1^m \to l_2^m) = \left(\frac{m-n+1}{m}\right)^{1/2} \quad \text{for } n = 1, \ldots, m.$$

Hence $y_n(l_1^{2n}|\mathfrak{P}_G^*) \geqslant c > 0$.

4.4. We now establish the main result of this chapter.

EXAMPLE. *Let $1 \leqslant p \leqslant \infty$. If the Banach function space L_p is infinite-dimensional, then*

$$x_n(L_p|\mathfrak{P}_G) \asymp \begin{cases} n^{-1/2} & \text{for } 1 \leqslant p \leqslant 2, \\ n^{-1/p} & \text{for } 2 \leqslant p < \infty, \\ (1+\log n)^{-1/2} & \text{for } p = \infty, \end{cases}$$

$$y_n(L_p|\mathfrak{P}_G^*) \asymp \begin{cases} n^{-1/p'} & \text{for } 1 \leqslant p \leqslant 2, \\ n^{-1/2} & \text{for } 2 \leqslant p < \infty, \\ 1 & \text{for } p = \infty. \end{cases}$$

Proof. First, we consider the case $1 < p < \infty$. Since L_p is of type p for $1 < p \leqslant 2$ and of type 2 for $2 \leqslant p < \infty$, it follows from Theorem 3.6 and Lemma 3.9 that

$$x_n(L_{p'}|\mathfrak{P}_G) \leqslant y_n(L_p|\mathfrak{P}_G^*) \prec \begin{cases} n^{-1/p'} & \text{for } 1 < p \leqslant 2, \\ n^{-1/2} & \text{for } 2 \leqslant p < \infty. \end{cases}$$

Recalling the trivial fact that $L_{p'}$ contains the spaces $l_{p'}^n$ uniformly, for $1 < p \leqslant 2$, the lower estimate can be deduced from Lemma 4.2. The case $2 \leqslant p < \infty$ is obvious, by Proposition 4.1.

Next we conclude from Lemma 4.3 that the sequence $(y_n(L_1|\mathfrak{P}_G^*))$ does not converge to zero. The same holds for $(y_n(L_\infty|\mathfrak{P}_G^*))$, since L_1 can be embedded into L_∞.

Since L_1 is of cotype 2, we have $x_n(L_1|\mathfrak{P}_G) \leqslant cn^{-1/2}$. The lower estimate follows from Proposition 4.1.

Finally, we observe that the asymptotic behaviour of the sequence $(x_n(L_\infty|\mathfrak{P}_G))$ can be derived from Proposition 2.7 and [6, Lemma 6].

Remark. The preceding results show that the type and cotype numbers cannot be used to distinguish Banach spaces of the ordinary type or cotype from those of weak type or weak cotype, respectively.

4.5. At first glance, the next result is quite surprising.

PROPOSITION. *The space ideals*

$$\mathbf{C}_{q,w} := \{E\colon I_E \in (\mathfrak{P}_G)^{(x)}_{q,w}\} \quad \text{with } 2 \leqslant q < \infty$$

and

$$\mathbf{T}_{p,w} := \{E\colon I_E \in (\mathfrak{P}_G^*)^{(y)}_{p',w}\} \quad \text{with } 1 < p \leqslant 2$$

do not depend on the finite parameter w for $0 < w < \infty$.

Proof. Assume that there exists a Banach space E such that $E \in \mathbf{C}_{q,w} \backslash \mathbf{C}_{q,w_0}$ with $0 < w_0 < w < \infty$. Then we have $q = \inf\{s\colon E \text{ is of cotype } s\}$. Hence, by the famous Maurey–Pisier theorem, E contains the spaces l_q^n uniformly; see [10, Chap. 13]. Thus we deduce from Lemma 4.2 that $x_n(E|\mathfrak{P}_G) \geqslant cn^{-1/q}$, which in turn yields $E \notin \mathbf{C}_{q,w}$. This contradiction proves the assertion in the "cotype" case. The "type" case can be treated analogously.

Remark. For $2 < q < \infty$ and $1 < p < 2$, we even have

$$\mathbf{C}_{q,w} = \bigcup_{2<r<q} \mathbf{C}_{r,\infty} \quad \text{and} \quad \mathbf{T}_{p,w} = \bigcup_{p<r<2} \mathbf{T}_{r,\infty}$$

provided that $0 < w < \infty$.

4.6. The preceding proposition implies that there does not exist any Banach space E such that

$$x_n(E|\mathfrak{P}_G) \asymp \frac{1}{n^{1/q}(1+\log n)^\alpha},$$

where $2 \leqslant q < \infty$ and $\alpha > 0$. Thus one could expect that, for every Banach space E not containing the spaces l_∞^n uniformly, the sequence $(x_n(E|\mathfrak{P}_G))$ behaves like $(n^{-1/q})$ for some exponent q with $2 \leqslant q < \infty$. The next example shows that this is not so.

Let (E_k) be any sequence of Banach spaces. We denote by $(\bigoplus_{k=1}^\infty E_k)_{l_2}$ the Banach space of all sequences (x_k) with $x_k \in E_k$ for which the expression

$$\|(x_k)|l_2\| := \Big(\sum_{k=1}^\infty \|x_k\|^2\Big)^{1/2}$$

is finite.

In what follows we use logarithms to the base 2.

EXAMPLE. *Let $2 \leqslant q < \infty$. Put*

$$E := \Big(\bigoplus_{k=1}^\infty l_{q_k}^{2^k}\Big)_{l_2} \quad \text{with } 1/q_k = 1/q - \alpha\frac{1+\log k}{k} \text{ and } \alpha > 0.$$

Then

$$x_n(E|\mathfrak{P}_G) \asymp \frac{(1+\log n)^\alpha}{n^{1/q}}.$$

Proof. We may proceed in the same way as in the proof of Example 7.3 in [14]. There it is shown that

$$z_n(E) \asymp \frac{(1+\log n)^\alpha}{n^{1/r}} \quad \text{with } 1/r = 1/q + 1/2.$$

Since E is of type 2, we have $y_n(E|\mathfrak{P}_G^*) \leqslant cn^{-1/2}$. Hence, by means of Lemmas 1.10, 3.10 and 3.12, the asymptotic behaviour of $(x_n(E|\mathfrak{P}_G))$ can be derived from that of $(z_n(E))$, and conversely.

5. Appendix. Let $x_n(E|\mathfrak{P}_2')$ and $y_n(E|\mathfrak{P}_2)$ denote the nth $\mathfrak{P}_2'$-Weyl number and the n-th $\mathfrak{P}_2$-Chang number of the identity map of a Banach space E, respectively. We stress the fact that $y_n(E|\mathfrak{P}_2)$ was already introduced (as an auxiliary quantity) by G. Pisier [17]. Taking into account his famous K-convexity theorem [16, Theorem 2.4], it is possible to rephrase [17, Théorème 7 and Remarque 8] in our terminology; see also Lemmas 1.10 and 3.10 as well as [14, Lemmas 1.13 and 2.11]. Concerning the definitions of $\Gamma_n(E)$ and $z_n(E)$ we refer the reader to [14, 1.1 and 2.1].

THEOREM. *For every Banach space E, the following properties are equivalent:*

(1) *E is K-convex.* (2) $\lim_n x_n(E|\mathfrak{P}_2') = 0$.
(3) $\lim_n y_n(E|\mathfrak{P}_2) = 0$. (4) $\lim_n y_n(E|\mathfrak{P}_G) = 0$.
(5) $\lim_n n^{1/2} z_n(E) = 0$. (6) $\lim_n n^{-1/2} \Gamma_n(E) = 0$.

Remark. Note that $\lim_n x_n(E|\mathfrak{P}_G) = 0$ for all Banach spaces E; see Prop. 2.7.

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SEKTION MATHEMATIK
FRIEDRICH-SCHILLER-UNIVERSITÄT
DDR-6900 Jena, G.D.R.

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