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On projections on subspaces of codimension one

by

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Abstract. Let $(X, \| \|)$ be a Banach space. Let V be a subspace of codimension 1. Let $\lambda(V, X) = \{\|P\|: P^2 = P, PX = V\}$ and let $H(X) = \sup\{\lambda(V, X): \text{codim } V = 1\}$ be the hyperplane projection constant. It is shown that $H(l^p) \leqslant H(L^p[0, 1]) \leqslant 2^{|2/p-1|}$.

Let $(X, \| \|)$ be a Banach space. Let Y be a subspace of X of codimension one. By $\lambda(Y, X)$ we denote the infimum of the norms of continuous linear projections P onto Y:

(1) $$\lambda(Y, X) = \inf\{\|P\|: P^2 = P, PX = Y\}.$$

The *hyperplane projection constant* is, by definition, the number

(2) $$H(X) = \sup\{\lambda(Y, X): Y \text{ is a subspace of } X \text{ of codimension } 1\}.$$

The aim of the present note is to prove the following

THEOREM 1. *For all* $1 < p < +\infty$

(3) $$H(l^p) \leqslant 2^{|2/p-1|}.$$

This result is better than other known estimates (cf. [1]). However, the proof goes in a different way and is very simple.

In order to prove Theorem 1 we need the following

PROPOSITION 2. *Let* Y_1, Y_2 *be two subspaces of* X *of codimension* 1. *Let* T *be an isometry mapping* X *onto itself and such that*

(4) $$TY_1 = Y_2.$$

Then

(5) $$\lambda(Y_1, X) = \lambda(Y_2, X).$$

The proof is obvious.

For an arbitrary continuous linear functional f on X of norm one we write

(6) $$H_f = \{x: f(x) = 0\},$$

COROLLARY 3. *Suppose that f, g are two continuous linear functionals on X of norm one. If there is an isometry T mapping X onto itself such that $T^*f = g$, then*

(7) $$\lambda(H_f, X) = \lambda(H_g, X).$$

COROLLARY 4. *Suppose that $(X, \|\ \|)$ is a reflexive Banach space. Suppose that for arbitrary two continuous linear functionals f, g on X of norm one there is an isometry T^* mapping the conjugate space onto itself and such that $T^*f = g$. Then for all subspaces Y of X of codimension 1 the numbers $\lambda(Y, X)$ are equal to each other.*

One can construct a nonseparable measure μ in such a way that the spaces $L^p(\Omega, \Sigma, \mu)$ satisfy Corollary 4 (cf. [5], Proposition IX.6.7).

By simple calculations $\lambda(H_f, X)$ is a Lipschitz function with respect to f. A consequence is

THEOREM 5 (cf. also [4]). *In the space $L^p[0, 1]$, $1 < p < +\infty$, for all subspaces Y of codimension 1 the numbers $\lambda(Y, L^p[0, 1])$ are equal to the hyperplane projection constant $H(L^p[0, 1])$.*

Proof. For arbitrary $f, g \in (L^p[0, 1])^* = L^q[0, 1]$ of norm one and for an arbitrary $\varepsilon > 0$ there exist $h_\varepsilon \in L^q[0, 1]$ of norm one and an isometry T_ε of $L^q[0, 1]$ onto itself such that

(8) $$\|T_\varepsilon f - T_\varepsilon h_\varepsilon\| < \varepsilon.$$

By Proposition 2, $\lambda(H_f, X) = \lambda(H_{h_\varepsilon}, X)$. The continuity of $\lambda(H_f, X)$ and (8) together imply that

(9) $$\lambda(H_f, X) = \lambda(H_g, X). \blacksquare$$

Theorem 5 and the standard averaging procedure (cf., for instance, [6]) together imply

THEOREM 6. *$H(L^p[0, 1]) = \|P_0\|_p$, where*

(10) $$P_0 x = x - \int_0^1 x(t)\,dt \cdot 1$$

and where we denote by $\|\ \|_p$ the norm of linear operators acting in $L^p[0, 1]$.

Proof. By Theorem 5, we can choose as Y the subspace

(11) $$Y = \left\{x\colon \int_0^1 x(t)\,dt = 0\right\}.$$

In the space $L^p[0, 1]$ there exists a group of isometries T_s, $0 \leqslant s \leqslant 1$. Namely, we can take

$$T_s x(t) = \begin{cases} x(t+s) & \text{if } t+s \leqslant 1 \\ x(t+s-1) & \text{if } t+s > 1 \end{cases} \quad (0 \leqslant s \leqslant 1).$$

Observe that Y is invariant with respect to all T_s. Let P_1 be a projection onto Y with minimal norm. By the averaging procedure, we find that the operator

(12) $$P_0 x = \int_0^1 T_s P_1 T_s^{-1} x\,ds$$

is again a projection with minimal norm. It is easy to verify that P_0 is of the form (10). $\blacksquare$

We do not know a formula for the norm $\|P_0\|_p$. (Added in proof: Recently C. Franchetti has found one.) However, we can estimate this norm as follows:

THEOREM 7.

(13) $$H(L^p[0, 1]) = \|P_0\|_p \leqslant 2^{|2/p-1|}.$$

Proof. Observe that $\|P_0\|_1 = \|P_0\|_\infty = 2$ and $\|P_0\|_2 = 1$. By the M. Riesz interpolation theorem, $\log \|P_0\|_{1/s}$ is a convex function of s on the interval $[0, 1]$. This immediately implies (13). $\blacksquare$

It is not known if $H(Z) \leqslant H(X)$ whenever Z is a subspace of X. However, it is easy to prove

PROPOSITION 8. *If Z is a subspace of X then*

(14) $$H(X/Z) \leqslant H(X).$$

Proof of Theorem 1. Since l^p can be represented as a quotient space of $L^p[0, 1]$, Proposition 8 immediately implies Theorem 1. $\blacksquare$

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