On projections on subspaces of codimension one

by

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Abstract. Let \((X, \| \cdot \|)\) be a Banach space. Let \(V\) be a subspace of codimension 1. Let 
\[ \lambda(V, X) = \{ \|P\| : P^2 = P, PX = V \} \]
and let 
\[ H(X) = \sup \{ \lambda(V, X) : \text{codim } V = 1 \} \]
be the hyperplane projection constant. It is shown that 
\[ H(p) \leq H(p^2[0, 1]) \leq 2^{2^p-1}. \]

Let \((X, \| \cdot \|)\) be a Banach space. Let \(Y\) be a subspace of \(X\) of codimension one. By \(\lambda(Y, X)\) we denote the infimum of the norms of continuous linear projections \(P\) onto \(Y\):

\[ \lambda(Y, X) = \inf \{ \|P\| : P^2 = P, PX = Y \}. \]

The hyperplane projection constant is, by definition, the number

\[ H(X) = \sup \{ \lambda(Y, X) : Y \text{ is a subspace of } X \text{ of codimension 1} \}. \]

The aim of the present note is to prove the following

Theorem 1. For all \(1 < p < +\infty\)

\[ H(p) \leq 2^{2^{2^p-1}}. \]

This result is better than other known estimates (cf. [1]). However, the proof goes in a different way and is very simple.

In order to prove Theorem 1 we need the following

Proposition 2. Let \(Y_1, Y_2\) be two subspaces of \(X\) of codimension 1. Let 
\(T\) be an isometry mapping \(X\) onto itself and such that

\[ TY_1 = Y_2. \]

Then

\[ \lambda(Y_1, X) = \lambda(Y_2, X). \]

The proof is obvious.

For an arbitrary continuous linear functional \(f\) on \(X\) of norm one we write

\[ H_f = \{ x : f(x) = 0 \}. \]
COROLLARY 3. Suppose that \( f, g \) are two continuous linear functionals on \( X \) of norm one. If there is an isometry \( T \) mapping \( X \) onto itself such that \( T^* f = g \), then

\[
\lambda(H_f, X) = \lambda(H_g, X).
\]

COROLLARY 4. Suppose that \((X, \| \|)\) is a reflexive Banach space. Suppose that for arbitrary two continuous linear functionals \( f, g \) on \( X \) of norm one there is an isometry \( T^* \) mapping the conjugate space onto itself and such that \( T^* f = g \). Then for all subspaces \( Y \) of \( X \) of codimension one the numbers \( \lambda(Y, X) \) are equal to each other.

One can construct a nonseparable measure \( \mu \) in such a way that the spaces \( L^p(\Omega, \Sigma, \mu) \) satisfy Corollary 4 (cf. [5], Proposition IX.6.7).

By simple calculations \( \lambda(H_f, X) \) is a Lipschitz function with respect to \( f \).

A consequence is

THEOREM 5 (cf. also [4]). In the space \( L^p(0, 1) \), \( 1 < p < +\infty \), for all subspaces \( Y \) of codimension one the numbers \( \lambda(Y, L^p(0, 1)) \) are equal to the hyperplane projection constant \( H(L^p(0, 1)) \).

Proof. For arbitrary \( f, g \in L^p(0, 1)^* = L^p(0, 1) \) of norm one and for an arbitrary \( \epsilon > 0 \) there exist \( h \in L^p(0, 1) \) of norm one and an isometry \( T \) of \( L^p(0, 1) \) onto itself such that

\[
\| T_f h - T_g h \| < \epsilon.
\]

By Proposition 2, \( \lambda(H_f, X) = \lambda(H_h, X) \). The continuity of \( \lambda(H_f, X) \) and (8) together imply that

\[
\lambda(H_f, X) = \lambda(H_g, X). \tag*{\blacksquare}
\]

Theorem 5 and the standard averaging procedure (cf., for instance, [6]) together imply

THEOREM 6. \( H(L^p(0, 1)) = \| P_0 \|_p \), where

\[
P_0 x = x - \frac{1}{|x|} \int_0^1 x(t) \, dt \end{matrix}
\]

and where we denote by \( \| \|_p \) the norm of linear operators acting in \( L^p(0, 1) \).

Proof. By Theorem 5, we can choose as \( Y \) the subspace

\[
Y = \{ x : \frac{1}{0} x(t) \, dt = 0 \}. \tag*{(11)}
\]

In the space \( L^p(0, 1) \) there exists a group of isometries \( T_s \), \( 0 \leq s \leq 1 \). Namely, we can take

\[
T_s x(t) = \begin{cases} x(t+s) & \text{if } t+s \leq 1 \\ x(t+s-1) & \text{if } t+s > 1 \end{cases} \quad (0 \leq s \leq 1).
\]

Observe that \( Y \) is invariant with respect to all \( T_s \). Let \( P_0 \) be a projection onto \( Y \) with minimal norm. By the averaging procedure, we find that the operator

\[
P_0 x = \frac{1}{|x|} \int_0^1 T_s P_1 T_s^{-1} x \, ds \tag*{(12)}
\]

is again a projection with minimal norm. It is easy to verify that \( P_0 \) is of the form (10).

We do not know a formula for the norm \( \| P_0 \|_p \). (Added in proof: Recently C. Franchetti has found one.) However, we can estimate this norm as follows:

THEOREM 7.

\[
H(L^p(0, 1)) = \| P_0 \|_p \leq 2^{\frac{1}{2p-1}}. \tag*{(13)}
\]

Proof. Observe that \( \| P_0 \|_1 = \| P_0 \|_2 = 2 \) and \( \| P_0 \|_1 \) is a convex function of \( s \) on the interval \([0, 1]\). This immediately implies (13).

It is not known if \( H(Z) \leq H(X) \) whenever \( Z \) is a subspace of \( X \). However, it is easy to prove

PROPOSITION 8. If \( Z \) is a subspace of \( X \) then

\[
H(X/Z) \leq H(X). \tag*{(14)}
\]

Proof of Theorem 1. Since \( R \) can be represented as a quotient space of \( L^p(0, 1) \), Proposition 8 immediately implies Theorem 1. \( \blacksquare \)

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References


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