Of course, the same argument applied to \([0, 1/2]\) forces \(\varphi(1/4) = 1/4\), and applied to \([1/2, 1]\) forces \(\varphi(3/4) = 3/4\), and so on. Likewise, if \(\varphi(-1) \neq -1\), this midpoint argument applied to \([-1, 1]\) won’t give \(\varphi(0) = 0\). So we conclude:

\[
\varphi(x) = x \text{ whenever } x \text{ is dyadic.}
\]

Continuity then finishes off the theorem. ■

References

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An interpolation theorem with \(A_p\)-weighted \(L^p\) spaces

by

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Abstract. If \(T \in T^*\), \(uL^p \to BMO\), for all \(u \in A_1\), then \(T: L^p(w) \to L^p(w)\) whenever \(w \in A_p\), \(1 < p < \infty\).

In 1976, Muckenhoupt and Wheeden introduced the weighted bounded mean oscillation spaces, \(BMO_w\) [6]. On \(\mathbb{R}\), \(f \in BMO_w\) provided

\[
|I|^{-1} \int_I f - I(f) \leq C I(u),
\]

for all intervals \(I\), using the notation \(|I| = \text{Lebesgue measure of } I\) and \(I(f) = \int_I f\). These spaces proved to have more than just an intrinsic interest; they turned out to be the solution spaces to some important singly and doubly weighted norm inequalities [1]. The purpose of this note is to show that these spaces arise as interpolation endpoints for \(L^p(w)\), \(w \in A_p\).

A weight \(w \in A_p\) if

\[
I(w) I[w^{-1/(p-1)}]^{p-1} \leq C \quad \text{for all intervals } I,
\]

when \(p > 1\). \(w \in A_1\) if

\[
I(w) \leq C \text{ess inf } w, \quad \text{for all intervals } I.
\]

The \(A_p\) classes are nested, \(A_p \subset A_q\) if \(q > p\), and a weight belongs to some \(A_p\) class if and only if it satisfies a reverse Hölder inequality:

\[
I[w^{p+\delta}]^{1/(p+\delta)} \leq C I(w), \quad \text{for some } \delta > 0 \text{ and all intervals } I.
\]

The Hardy–Littlewood maximal operator and the Hilbert transform are bounded on \(L^p(w)\) if and only if \(w \in A_p\), \(1 < p < \infty\) [2]. Also any \(w \in A_p\) can be factored into the form \(w = uw^{1-p}\), for some \(u\) and \(v \in A_1\), the Jones’ Factorization Theorem [5].

We will use the sharp maximal function of Fefferman and Stein [4],

\[
f^*(x) = \sup \{|I|^{-1} \int_I f - I(f) : x \in I, I \text{ an interval}\}.
\]

If \(1 < p < \infty\), then \(\|f^*\|_p \approx \|f\|_p\), provided \(\|f\|_p\) is finite.
Let $T$ be a linear operator. We say $T: uL^{\infty} \to \text{BMO}_u$ provided
\[ |u|^{-1} \int |T(fu) - I(Tfu)| \leq C(u) \|f\|_u, \]
with $C$ independent of $I$, and we say $T: L^{p}(u) \to L^{p}(u)$ if
\[ \|Tf\|_{L^{p}(u)} \leq C \|f\|_{L^{p}(u)}, \]
for all $f \in L^{p}(u)$.

**Theorem.** If $T$ is a linear operator with adjoint $T^*$ and if $T, T^*: uL^{\infty} \to \text{BMO}_u$ for all $u \in A_1$, then $T: L^{p}(w) \to L^{q}(w)$ for all $w \in A_p$, $1 < p < \infty$.

**Proof.** First take $u = 1$, an $A_1$ weight. This means that $T$ and $T^*$ map $L^{\infty}$ into BMO. If $f \in H^2$ and $g \in L^{\infty}$,
\[ \|Tg\|_{L^{q}(u)} \leq C \|f\|_{L^{p}(u)}, \]
so the operator
\[ u^{-1/2} T(uf^{1/2}) : L^{p} \to L^{p} \]
also.

In the arguments above, we could of course interchange $T$ and $T^*$. So we have shown: if $1/p + 1/q = 1$ with $p$ near 1 and if $u$ and $v$ belong to $A_1$, then
\[ u^{-1/2} T(uf^{1/2}) : L^{p} \to L^{p} \quad \text{and} \quad v^{2/3 - 1/2} T(u^{1/2}v^{1/2}) : L^{p} \to L^{p}. \]
Let
\[ \alpha(t) = t - 1/q, \quad \beta(t) = t - 1/p. \]
So \( \alpha(1/p) = 1 - 2/q, \alpha(1/q) = 0, \beta(1/p) = 0, \text{ and } \beta(1/q) = -1 + 2/q. \) So interpolation gives
\[ u^{2/3 - 1/2} T(uf^{1/2}) : L^{p} \to L^{p}, \]
for $1/q \leq t \leq 1/p$. In particular, taking $t = 1/2$ gives $T: L^2(w) \to L^2(w)$ whenever $w$ has the form
\[ w = u^{2/3 + 1/2}(x, y) \phi(1/2). \]
Finally, fix $w \in A_2$. By Jones’ Factorization, there exist $u$ and $v$ in $A_1$ with $w = uv$. By reverse Hölder, we can find a $\delta > 0$ so that $u^\delta$ and $v^\delta$ are in $A_1$ whenever $1 \leq \delta \leq 1 + \delta$. In particular, if we choose $p$ sufficiently near 1, we can force $r = 1/(1 - 2/q) \leq 1 + \delta$. But with this $r$,
\[ u^{2s/3 + 1/2}(x, y) v^{2r/(1/2)} = uv^{-1} = w, \]
so $T: L^2(w) \to L^2(w)$.

For exponents other than 2, the argument just given can be modified appropriately, or if one prefers, we can appeal to Rubio de Francia’s Extrapolation Theorem [7], and the proof is complete.

As an application, we derive Hunt, Muckenhoupt and Wheeden’s theorem, that the Hilbert transform $H: L^p(w) \to L^p(w)$ whenever $w \in A_p$ and $1 < p < \infty$. $Hf(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x - y} \, dy$, so $H^* = -H$. To apply the theorem, we must show that $H: uL^{\infty} \to \text{BMO}_u$ whenever $\in A_1$. It’s well known that $H: L^p \to \text{BMO}$. The point of the theorem is that whatever holds for BMO invariably holds for \text{BMO}$_u$ when $u \in A_1$, and can be proven without so much as a break in style. Yet that minuscule bit of generality lets one interpolate to $L^p(w)$ for all $w \in A_p$.

Let $w \in A_1$. Let’s mimic the old proofs and show $H: uL^{p} \to \text{BMO}_u$. Let $f \in L^{p}$, fix an interval $I$ and put $f_I = f|x_I$ and $f_{2I} = f - f_I$, where $2I$ denotes
the interval concentric with \( I \) of twice the length. Let \( p > 1 \) be near enough to 1 so that reverse Hölder holds for \( u \) with exponent \( p \). Since \( H: L^p \to L^p \), we have
\[
|I|^{-1} \int_I |H(uf_i)(x)| \leq |I|^{-1} \int |H(uf_i)|^p \leq c_1 |I|^{-1} \int |f|^p u^p \leq c_2 \|f\|_\infty \|u\|_\infty \frac{|I|^{-1} \int u^p}{|I|}.
\]
by reverse Hölder,
\[
|I|^{-1} \int_I |H(uf_i)(x)| \leq c_3 \|f\|_\infty \operatorname{ess inf} u, \quad \text{by } A_1,
\]
\[
|I|^{-1} \int_I |H(uf_i)(x)| \leq c_4 \|f\|_\infty \|I(u)\|.
\]
Let \( x_0 \) be the center of \( I \) and \( \delta = |I|/2 \). Then
\[
|H(uf_i)(x) - H(uf_i)(x_0)| = \left| \int_{\mathbb{R}^n} \frac{1}{|x - y|} \frac{1}{|x_0 - y|} u(y)f(y)dy \right|
\]
\[
\leq \|f\|_\infty \delta \int_{\mathbb{R}^n} \frac{1}{|x - y| |x_0 - y|} u(y)dy
\]
\[
\leq c_4 \|f\|_\infty \delta \sum_{n=1}^{\infty} 2^{-n} \delta^{-2} \int_{|x - x_0| < 2^n} u(y)dy
\]
\[
\leq c_5 \|f\|_\infty \sum_{n=1}^{\infty} 2^{-sn} \operatorname{ess inf} u_{|x - x_0| < 2^n}
\]
\[
\leq \frac{1}{2} c_5 \|f\|_\infty \operatorname{ess inf} u \leq \frac{1}{2} c_5 \|f\|_\infty |I(u)|.
\]
Integrating, we have
\[
|I|^{-1} \int_I |Hu(uf_i + uf_i)(x) - Hu(f_i)(x_0)| \leq (c_5 + \frac{1}{2} c_5) \|f\|_\infty |I(u)|,
\]
so that \( Hu(uf_i + uf_i) \in BMO \).

References