Of course, the same argument applied to $[0, 1/2]$ forces $\varphi(1/4) = 1/4$, and applied to $[1/2, 1]$ forces $\varphi(3/4) = 3/4$, and so on. Likewise, if $\varphi(-1) \neq -1$, this midpoint argument applied to $[-1, 1]$ won't give $\varphi(0) = 0$. So we conclude:

$$\varphi(x) = x \text{ whenever } x \text{ is dyadic.}$$

Continuity then finishes off the theorem. ∎

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An interpolation theorem with A_p-weighted L^p spaces

by

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Abstract. If T and T^*: $uL^\infty \to \mathrm{BMO}_u$ for all $u \in A_1$, then T: $L^p(w) \to L^p(w)$ whenever $w \in A_p$, $1 < p < \infty$.

In 1976, Muckenhoupt and Wheeden introduced the weighted bounded mean oscillation spaces, BMO_u [6]. On $\mathbf{R}$, $f \in \mathrm{BMO}_u$ provided

$$|I|^{-1} \int_I |f - I(f)| \leqslant CI(u),$$

for all intervals I, using the notation $|I| =$ Lebesgue measure of I and $I(f) = |I|^{-1} \int_I f$. These spaces proved to have more than just an intrinsic interest, they turned out to be the solution spaces to some important singly and doubly weighted norm inequalities [1]. The purpose of this note is to show that these spaces arise as interpolation endpoints for $L^p(w)$, $w \in A_p$.

A weight $w \in A_p$ if

$$I(w) I[w^{-1/(p-1)}]^{p-1} \leqslant C \quad \text{for all intervals } I,$$

when $p > 1$. $w \in A_1$ if

$$I(w) \leqslant C \operatorname*{ess\,inf}_I w, \quad \text{for all intervals } I.$$

The A_p classes are nested, $A_p \subset A_q$ if $q > p$, and a weight belongs to some A_p class if and only if it satisfies a reverse Hölder inequality: $I(w^{1+\delta})^{1/(1+\delta)} \leqslant CI(w)$, for some $\delta > 0$ and all intervals I. The Hardy–Littlewood maximal operator and the Hilbert transform are bounded on $L^p(w)$ if and only if $w \in A_p$, $1 < p < \infty$ [2]. Also any $w \in A_p$ can be factored into the form $w = uv^{1-p}$, for some u and $v \in A_1$, the Jones' Factorization Theorem [5].

We will use the sharp maximal function of Fefferman and Stein [4],

$$f^{\#}(x) = \sup\{|I|^{-1} \int_I |f - I(f)| \colon x \in I,\ I \text{ an interval}\}.$$

If $1 < p < \infty$, then $\|f^{\#}\|_p \approx \|f\|_p$, provided $\|f\|_p$ is finite.

Let T be a linear operator. We say $T\colon uL^\infty \to \mathrm{BMO}_u$ provided

$$|I|^{-1}\int_I |T(fu)-I(Tfu)| \leqslant CI(u)\|f\|_\infty,$$

with C independent of I, and we say $T\colon L^p(u)\to L^p(u)$ if

$$\|Tf\|_{L^p(u)} \leqslant C\|f\|_{L^p(u)}.$$

THEOREM. *If T is a linear operator with adjoint T^* and if $T, T^*\colon uL^\infty \to \mathrm{BMO}_u$ for all $u\in A_1$, then $T\colon L^p(w)\to L^p(w)$ for all $w\in A_p$, $1<p<\infty$.*

Proof. First take $u\equiv 1$, an A_1 weight. This means that T and T^* map L^∞ into BMO. If $f\in H^1$ and $g\in L^\infty$,

$$\left|\int (Tf)g\right| = \left|\int fT^*g\right| \leqslant C\|f\|_{H^1}\|g\|_\infty.$$

So $T\colon H^1\to L^1$. By Macias interpolation [3], $T\colon L^p\to L^p$, as does T^*. Now for $u\in A_1$ and $f\in L^\infty$,

$$u^{-1}(x)T^*(uf)^\#(x) = \sup_{x\in I}|I|^{-1}\int_I |T^*(uf)(y)-I(T^*uf)|\,dy\cdot u^{-1}(x)$$

$$\leqslant c_1 \sup \|f\|_\infty I(u)u^{-1}(x) \leqslant c_2\|f\|_\infty,$$

by the A_1 condition. So $u^{-1}(T^*u)^\#\colon L^\infty\to L^\infty$. Since also $(T^*)^\#\colon L^2\to L^2$, complex interpolation [8, p. 205] gives, for $1/p+1/q=1$, p near 1,

$$u^{2/q-1}(T^*u^{1-2/q})^\#\colon L^q\to L^q.$$

Thus

$$\int ([T^*u^{1-2/q}f]^\#(x))^q u^{2-q}(x)\,dx \leqslant C\int |f|^q\,dx.$$

Replacing $u^{1-2/q}f$ by g, $|f|^q = |g|^q u^{2-q}$, so this says

$$(T^*)^\#\colon L^q(u^{2-q})\to L^q(u^{2-q}).$$

By the weighted version of Fefferman and Stein's Theorem [9, p. 272],

$$T^*\colon L^q(u^{2-q})\to L^q(u^{2-q}).$$

(We are being a little sloppy here. The theorem actually says: If $\|T^*f\|_{L^q(u^{2-q})}<\infty$, then $\|T^*f\|_{L^q(u^{2-q})} \leqslant C\|(T^*f)^\#\|_{L^q(u^{2-q})}$. Since $T^*\colon L^q\to L^q$ and since A_1 weights are bounded below, on any compact interval,

$$\int (T^*f)^q u^{2-q} \leqslant c_1\int (T^*f)^q \leqslant c_2\int |f|^q < \infty$$

on a dense subspace of $L^q(u^{2-q})$.)

Now let $g\in L^q(u^{2-q})$ and $f\in L^p(u^{2-p})$. By Hölder's inequality,

$$\left|\int (Tf)g\right| = \left|\int (T^*g)u^{2/q-1}fu^{1-2/q}\right| \leqslant \|T^*g\|_{L^q(u^{2-q})}\|f\|_{L^p(u^{2-p})},$$

and duality gives $T\colon L^p(u^{2-p})\to L^p(u^{2-p})$, or $u^{2/p-1}Tu^{1-2/p}\colon L^p\to L^p$.

Next let $f\in L^p$ and $g\in L^q$. Then

$$\left|\int T(fu^{2/q-1})u^{1-2/q}g\right| = \left|\int f[u^{2/q-1}T^*(u^{1-2/q}g)]\right| \leqslant C\|f\|_p\|g\|_q,$$

so the operator

$$u^{1-2/q}T(u^{2/q-1})\colon L^p\to L^p$$

also.

In the arguments above, we could of course interchange T and T^*. So we have shown: If $1/p+1/q=1$ with p near 1 and if u and v belong to A_1, then

$$u^{1-2/q}T(u^{2/q-1})\colon L^p\to L^p \quad\text{and}\quad v^{2/q-1}T(v^{1-2/q})\colon L^q\to L^q.$$

Let

$$\alpha(t) = t-1/q,\qquad \beta(t) = t-1/p.$$

So $\alpha(1/p) = 1-2/q$, $\alpha(1/q) = 0$, $\beta(1/p) = 0$, and $\beta(1/q) = -1+2/q$. So interpolation gives

$$u^{\alpha(t)}v^{\beta(t)}T(u^{-\alpha(t)}v^{-\beta(t)})\colon L^{1/t}\to L^{1/t},\qquad \text{for } 1/q\leqslant t\leqslant 1/p.$$

In particular, taking $t=1/2$ gives $T\colon L^2(w)\to L^2(w)$ whenever w has the form

$$w = u^{2\alpha(1/2)}v^{2\beta(1/2)}.$$

Finally, fix $w\in A_2$. By Jones' Factorization, there exist u and v in A_1 with $w=u/v$. By reverse Hölder, we can find a $\delta>0$ so that u^r and v^r are in A_1 whenever $1\leqslant r\leqslant 1+\delta$. In particular, if we choose p sufficiently near 1, we can force $r = 1/(1-2/q)\leqslant 1+\delta$. But with this r,

$$u^{r2\alpha(1/2)}v^{r2\beta(1/2)} = uv^{-1} = w,$$

so $T\colon L^2(w)\to L^2(w)$.

For exponents other than 2, the argument just given can be modified appropriately, or if one prefers, we can appeal to Rubio de Francias' Extrapolation Theorem [7], and the proof is complete.

As an application, we derive Hunt, Muckenhoupt and Wheeden's theorem, that the Hilbert transform $H\colon L^p(w)\to L^p(w)$ whenever $w\in A_p$ and $1<p<\infty$. $Hf(x) = \text{p.v.}\int \frac{f(y)}{x-y}\,dy$, so $H^* = -H$. To apply the theorem, we must show that $H\colon uL^\infty\to\mathrm{BMO}_u$ whenever $u\in A_1$. It's well known that $H\colon L^\infty\to\mathrm{BMO}$. The point of the theorem is that whatever holds for BMO invariably holds for BMO_u when $u\in A_1$ and can be proven without so much as a break in stride. Yet that minuscule bit of generality lets one interpolate to $L^p(w)$ for all $w\in A_p$.

Let $u\in A_1$. Let's mimic the old proofs and show $H\colon uL^\infty\to\mathrm{BMO}_u$. Let $f\in L^\infty$, fix an interval I and put $f_1 = f\chi_{2I}$ and $f_2 = f-f_1$, where $2I$ denotes the interval concentric with I of twice the length. Let $p > 1$ be near enough to 1 so that reverse Hölder holds for u with exponent p. Since $H\colon L^p \to L^p$, we have

$$\begin{aligned}
|I|^{-1}\int_I |H(uf_1)| &\leqslant \big(|I|^{-1}\int_{\mathbf{R}} |H(uf_1)|^p\big)^{1/p} \leqslant c_1\big(|I|^{-1}\int_{2I} |f|^p u^p\big)^{1/p} \\
&\leqslant c_1 2^{1/p}\|f\|_\infty \big(\tfrac{1}{2}|I|^{-1}\int_{2I} u^p\big)^{1/p} \\
&\leqslant c_2\|f\|_\infty \big(\tfrac{1}{2}|I|^{-1}\int_{2I} u\big), \quad \text{by reverse Hölder,} \\
&\leqslant c_3\|f\|_\infty \operatorname*{ess\,inf}_I u, \quad \text{by } A_1, \\
&\leqslant c_3\|f\|_\infty I(u).
\end{aligned}$$

Let x_0 be the center of I and $\delta = |I|$. Then

$$\begin{aligned}
|H(uf_2)(x) - H(uf_2)(x_0)| &= \left|\int_{y\notin 2I}\left[\frac{1}{x-y} - \frac{1}{x_0-y}\right]u(y)f(y)\,dy\right| \\
&\leqslant \|f\|_\infty \delta \int_{y\notin 2I} \frac{1}{|x-y||x_0-y|}u(y)\,dy \\
&\leqslant c_4\|f\|_\infty \delta \sum_{n=1}^{\infty} 2^{-2n}\delta^{-2}\int_{|y-x_0|\leqslant 2^n\delta} u(y)\,dy \\
&\leqslant c_5\|f\|_\infty \sum_{n=1}^{\infty} 2^{-n} \operatorname*{ess\,inf}_{|y-x_0|\leqslant 2^n\delta} u \\
&\leqslant \tfrac{1}{2}c_5\|f\|_\infty \operatorname*{ess\,inf}_I u \leqslant \tfrac{1}{2}c_5\|f\|_\infty I(u).
\end{aligned}$$

Integrating, we have

$$|I|^{-1}\int_I |Hu(f_1+f_2)(x) - Huf_2(x_0)| \leqslant (c_3 + \tfrac{1}{2}c_5)\|f\|_\infty I(u),$$

so that $Huf = Hu(f_1+f_2) \in \mathrm{BMO}_u$.

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