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Sharp weights and BMO-preserving homeomorphisms

by

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Abstract. Weighted versions of the Fefferman–Stein sharp function inequality hold if and only if the weight is A_∞. A_∞ weights can be characterized by certain restricted L^1 inequalities which yield a new proof of Jones' characterization of the BMO-preserving homeomorphisms of $\mathbf{R}^1$. The dual homeomorphism problem is also considered.

1. Introduction. In 1972, Fefferman and Stein published a good-λ inequality linking the Hardy–Littlewood maximal function $M^*f(x)$ to the sharp function $f^\#(x)$ [4]. For an interval $I\subset \mathbf{R}$, we denote by $I(f)$ the average of f over I, $I(f)=|I|^{-1}\int_I f$. M^*f is defined by

$$M^*f(x)=\sup\{I(|f|):\ x\in I\}$$

and $f^\#$ is given by

$$f^\#(x)=\sup\{I(|f-I(f)|):\ x\in I\}.$$

We also refer to the restricted sharp function $f_J^\#(x)$, where the supremum above is restricted to intervals $I\subset J$. The good-λ inequality leads to the L^p inequalities: If $1<p<\infty$,

$$\int_{\mathbf{R}}|f|^p\,dx\leqslant C_p\int_{\mathbf{R}}(f^\#)^p(x)\,dx \tag{1.1}$$

and

$$\int_I|f-I(f)|^p\,dx\leqslant C_p\int_I(f_I^\#)^p\,dx \tag{1.2}$$

for an appropriate class of functions f (for instance $f\in L^{p_0}(\mathbf{R})$, $p_0\leqslant p$).

A nonnegative weight $w\in A_\infty$ if it satisfies the condition

$$w(E)/w(I)\leqslant C(|E|/|I|)^\delta \tag{A_∞}$$

for all measurable $E\subset I$, an interval, where C and δ are positive constants, and $w(E)$ denotes $\int_E w(x)\,dx$. This condition is equivalent to the reverse Hölder inequality:

$$I(w^p)^{1/p}\leqslant CI(w)\quad \text{for all intervals } I,\ p>1. \tag{RH}$$

See, for instance [3, 6, 10]. Good-λ inequalities are preserved when we change the measure from dx to $w(x)dx$, so long as $w \in A_\infty$, so (1.1) and (1.2) hold with dx replaced by $w(x)dx$. An obvious question arises: Are there other good weights for (1.1) and (1.2)?

DEFINITION 1.3. Let $1 < p < \infty$. w is a *p-sharp weight* if for each interval I and $f \in L^1(I)$,

(1.4) $$\int_I |f - I(f)|^p w(x)\,dx \leqslant C \int_I (f_I^\#)^p(x) w(x)\,dx,$$

and also

(1.5) $$\int_{\mathbf{R}} |f|^p w \leqslant C \int_{\mathbf{R}} (f^\#)^p w$$

for all $f \in L^{p_0}(\mathbf{R})$ for some $p_0 \leqslant p$. w is a *sharp weight* if w is a p-sharp weight for each $1 < p < \infty$ (here the constant C may depend on p), and w is a *weak-sharp weight* if (1.5) holds for each p.

As remarked above, A_∞ weights are sharp weights. That there are no others is the main result of Section 2. The principle tool in the proof is Peter Jones' characterization of the BMO-preserving homeomorphisms of the real line as precisely those homeomorphisms whose derivatives are A_∞ weights [5], stated below as Lemma 2.5. Suppose F is a space of real-valued functions defined on $\mathbf{R}$, and suppose $\varphi\colon \mathbf{R} \to \mathbf{R}$. We say $\varphi \in \operatorname{Hom}(F)$ provided φ is a homeomorphism and whenever $f \in F$, $f \circ \varphi^{-1} \in F$ also. φ is a *dual homeomorphism*, $\varphi \in \operatorname{Hom}^*(F)$, if φ is a homeomorphism with a derivative a.e., and if the operator

$$f(x) \to \varphi'(x) f \circ \varphi(x)$$

maps F into F. If F has a dual space F^* given through an integral representation $\int_{\mathbf{R}} ff^*\,dx$, then a simple change of variables shows that $\operatorname{Hom}^*(F) \subset \operatorname{Hom}(F^*)$, and actually coincides with the differentiable $\varphi \in \operatorname{Hom}(F^*)$, hence the terminology. These classes are closely related to the pointwise multipliers of F: $\varphi \in M(F)$ if whenever $f \in F$, the function $\varphi(x) f(x)$ also belongs to F. For if $\varphi \in \operatorname{Hom}(F) \cap \operatorname{Hom}^*(F)$, the composition of the two operators $f \to f \circ \varphi^{-1}$ and $f \to \varphi' f \circ \varphi$ must map F into F, but this composition is $f \to \varphi' f$, so that $\varphi' \in M(F)$.

Jones showed that $\operatorname{Hom}(\mathrm{BMO}) = \{\varphi \text{ differentiable with } \varphi' \in A_\infty\}$. Since there are no nonconstant pointwise multipliers of $\mathrm{BMO}(\mathbf{R})$ [1], the analysis above suggests that $\operatorname{Hom}^*(\mathrm{BMO}) = \operatorname{Hom}(H^1)$, and is trivial. That is correct, and is derived in Section 4 along with a new proof of Jones' theorem. Jones' proof was based on the John–Nirenberg distribution inequality for BMO. Our proof characterizes $\operatorname{Hom}^*(H^1)$ via the atomic decomposition of H^1, and should generalize to other atomic settings, where there is no John–Nirenberg type inequality.

Section 3 presents two L^1-type characterizations of A_∞, used in Section 4. Specifically, if T is an operator which is bounded on each $L^p(\mathbf{R})$, $1 < p < \infty$, and if $w \in A_\infty$, the condition (RH) gives for some $p > 1$,

$$|I|^{-1} \int_I T(w\chi_I) \leqslant \left(|I|^{-1} \int_I T(w\chi_I)^p(x)\,dx\right)^{1/p} \quad \text{by Hölder's inequality}$$

$$\leqslant \left(|I|^{-1} \int_{\mathbf{R}} T(w\chi_I)^p\,dx\right)^{1/p} \leqslant C\left(|I|^{-1} \int_I w^p\right)^{1/p} \leqslant CI(w).$$

Here, as will frequently occur in this paper, C denotes a constant that need not stay the same from line to line. The condition derived above,

(1.6) $$I(T(w\chi_I)) \leqslant CI(w) \quad \text{for all intervals } I,$$

completely characterizes A_∞ when T is either the Hardy–Littlewood maximal operator M^* or the Hilbert transform H,

$$Hf(x) = \text{p.v.} \int \frac{f(y)}{x-y}\,dy.$$

If I is an interval, $2I$ will denote the interval concentric with I but of twice the length. A_∞ weights are always *doubling*, that is,

$$w(2I) \leqslant Cw(I).$$

BMO is the space of functions whose sharp function is bounded. We norm BMO modulo the constants by $\|f\|_{\mathrm{BMO}} = \|f^\#\|_\infty$. This space is a Banach space, and is the dual space to H^1. We use two equivalent formulations of H^1: $H^1 = \{f\colon \mathbf{R} \to \mathbf{R}\colon f \text{ and } Hf \in L^1(\mathbf{R})\}$, and also H^1 is the atomic space: a is an *atom* if a is supported in an interval I, $\|a\|_\infty \leqslant 1/|I|$, and $\int a = 0$. $f \in H^1$ if there are atoms $\{a_n\}$ and a sequence $\{\lambda_n\} \in l^1$ with $f = \sum \lambda_n a_n$. The two corresponding norms,

$$\|f\| = \int_{\mathbf{R}} |f| + |Hf| \quad \text{and}$$

$$\|f\| = \inf\{\textstyle\sum |\lambda_n|\colon f = \sum \lambda_n a_n, \text{ the } a_n\text{'s atoms}\}$$

are equivalent. See [2, 4, 10].

2. Sharp weights

THEOREM 2.1. *The following are equivalent*:

(1) *w is a doubling weak-sharp weight.*
(2) *w is a p-sharp weight for some $1 < p < \infty$.*
(3) *w is a sharp weight.*
(4) $w \in A_\infty$.

We will establish several preliminary results first.

LEMMA 2.2. *Let w be a p-sharp weight. Then w is not almost everywhere zero on any interval.*

Proof. Otherwise, we may, after a translation, assume that w is zero almost everywhere on the interval $[h, \delta]$ and that $\int_0^h w \neq 0$, where h is substantially smaller than δ. Let $f(x) = \log(1/|x|)$. So $f \in \mathrm{BMO}$. If we put $I = [0, \delta]$, $I(f)$ is much smaller than $\log(1/h)$, and so

$$\int_I |f - I(f)|^p w\,dx \geqslant c\int_0^h (\log(1/x))^p w(x)\,dx \geqslant c(\log(1/h))^p \int_0^h w,$$

while

$$\int_I (f^\#)^p w\,dx \leqslant \|f\|_{\mathrm{BMO}}^p \int_0^h w.$$

These force $\log(1/h) \leqslant c^{-1/p}\|f\|_{\mathrm{BMO}}$, which is absurd as $h \to 0$. ∎

LEMMA 2.3. *Let w be a p-sharp weight. Then on any infinite interval I, $w(I) = \infty$.*

Proof. We give the argument when $I = \mathbf{R}$. Other cases are similar. Suppose instead that $w(\mathbf{R}) < \infty$. For $f \in \mathrm{BMO}$,

$$\int_0^1 |f|^p w \leqslant \int_{-\infty}^{\infty} |f|^p w \leqslant C_p \int (f^\#)^p w \leqslant C_p \|f\|_{\mathrm{BMO}}^p \int_{\mathbf{R}} w.$$

In particular, if $f \equiv n$ on $[0, 1]$,

$$n^p \int_0^1 w \leqslant C_p \|f\|_{\mathrm{BMO}}^p \int_{\mathbf{R}} w.$$

By Lemma 2.2, $\int_0^1 w \neq 0$. On the other hand, it is easy to construct $f_n \equiv n$ on $[0, 1]$ with compact support and with $\|f_n\|_{\mathrm{BMO}} \leqslant 2$, leading to a contradiction. ∎

Now put $\varphi(x) = \int_0^x w(t)\,dt$. The previous two lemmas show that $\varphi\colon \mathbf{R} \to \mathbf{R}$ is one-to-one and onto, whenever w is a p-sharp weight. Let Φ be the operator $\Phi f(x) = f(\varphi^{-1}(x))$.

LEMMA 2.4. *Let w be a p-sharp weight, $1 < p < \infty$, and φ, Φ as above. Then Φ is a bounded operator on* BMO.

Proof. Fix an interval $I = [a, b]$ and let $J = \varphi^{-1}(I) = [\alpha, \beta]$. For $f \in \mathrm{BMO}$,

$$|I|^{-1}\int_I |f(\varphi^{-1}(y)) - J(f)|\,dy = |\varphi(J)|^{-1}\int_\alpha^\beta |f(x) - J(f)|\,w(x)\,dx$$

with the change of variable $y = \varphi(x)$. Using Hölder's inequality, this is bounded by

$$\Big(w(J)^{-1}\int_J |f - J(f)|^p w\Big)^{1/p} \leqslant C_p^{1/p}\Big(w(J)^{-1}\int_J (f_J^\#)^p w\Big)^{1/p} \leqslant C_p^{1/p}\|f\|_{\mathrm{BMO}}.$$

$J(f)$ is not the usual constant, but, as is quite standard, this implies

$$|I|^{-1}\int |\Phi(f) - I[\Phi(f)]| \leqslant 2C_p^{1/p}\|f\|_{\mathrm{BMO}}.$$

Since I was arbitrary, we have $\|\Phi(f)\|_{\mathrm{BMO}} \leqslant 2C_p^{1/p}\|f\|_{\mathrm{BMO}}$. ∎

LEMMA 2.5. *If Φ is a bounded operator on* BMO, *then $w \in A_\infty$.*

This is the hard part of Jones' Theorem. Actually, Jones established the equivalences: $\varphi \in \mathrm{Hom(BMO)}$ iff Φ is bounded on BMO iff $|\varphi'|$ exists a.e. and is A_∞.

Proof of Theorem 2.1. That (4) implies each of the others follows from Fefferman–Stein and the nature of good-λ inequalities. Obviously (3) implies (2), and (2) implying (4) was the essence of the last two lemmas. So (2), (3), and (4) are equivalent and imply (1). We must show that (1) implies any of the others. To this end, we redo Lemma 2.4 using (1) as hypothesis.

Fix I and assume that $\int_I f = 0$ and that f is supported in I. So

$$\int_I |f|^p w\,dx = \int_{\mathbf{R}} |f|^p w\,dx \leqslant C_p \int_{\mathbf{R}} (f^\#)^p w\,dx.$$

If $x \in 2I$, then $f^\#(x) = f_{2I}^\#(x)$. Suppose that $x \notin 2I$. For convenience, we take $I = [-\delta/2, \delta/2]$. Then

$$\int_{x \notin 2I} (f^\#)^p w\,dx = \sum_{n=0}^{\infty} \int_{2^n\delta \leqslant |x| \leqslant 2^{n+1}\delta} (f^\#)^p(x) w(x)\,dx.$$

Now if $2^n\delta \leqslant |x| \leqslant 2^{n+1}\delta$, then

$$f^\#(x) = \sup_{x \in J, I \cap J \neq \emptyset} J(|f - J(f)|) \leqslant 2 \sup_{x \in J, I \cap J \neq \emptyset} |J|^{-1}\int_J |f|$$
$$= 2\frac{1}{\mathrm{dist}(x, I)}\int_I |f| \leqslant C\frac{\delta}{|x|} I(|f|).$$

Hence,

$$\int_{x \notin 2I} (f^\#)^p(x) w(x)\,dx \leqslant C^p \delta^p I(|f|)^p \int_{x \notin 2I} \frac{w(x)}{|x|^p}\,dx.$$

Since w is doubling, $w \in B_p$ for p sufficiently large, i.e.

$$\int_{x \notin 2I} \frac{w(x)}{|x|^p}\,dx \leqslant C\frac{w(I)}{\delta^p} \quad [9].$$

For such p,

$$\int_{x\notin 2I} (f^{\#})^p w\,dx \leqslant CI(|f|)^p w(I) \leqslant C\int_I (f_{2I}^{\#})^p(x)w(x)\,dx.$$

And so,

$$\int_I |f|^p w\,dx \leqslant C\int_{2I} (f_{2I}^{\#})^p w\,dx. \tag{2.6}$$

Now suppose that $b\in \mathrm{BMO}$ and that φ is a homeomorphism of $\mathbf{R}$ with derivative w. If $\varphi(J) = I$, we have

$$|I|^{-1}\int_I |\Phi(b)-J(b)|\,dy \leqslant \big(w(J)^{-1}\int_J |b-J(b)|^p w\,dx\big)^{1/p}.$$

By subtracting a constant if need be, we can assume that $J(b) = 0$. We can extend $b\chi_J$ to a function $\tilde{b}\in \mathrm{BMO}(4J)$ with $\int_{2J}\tilde{b} = 0$ and $\tilde{b}$ vanishing off $2J$, for, by Stegenga's Theorem [8], there exists a smooth ψ which multiplies $\mathrm{BMO}(4J)$ pointwise, which is 1 on J, vanishes off $2J$, and can be chosen so that $\int\psi b = 0$. Simply take $\tilde{b} = \psi b$.

Now apply (2.6) with I replaced by $2J$ and f replaced by $\tilde{b}$. That gives

$$\begin{aligned}|I|^{-1}\int_I |\Phi(b)-J(b)| &\leqslant \big(w(J)^{-1}\int_{2J} |\tilde{b}|^p w\,dx\big)^{1/p}\\ &\leqslant C\big(w(J)^{-1}\int_{4J} [(\tilde{b})_{4J}^{\#}]^p w\,dx\big)^{1/p}\\ &\leqslant C\|b\|_{\mathrm{BMO}}[w(4J)/w(J)]^{1/p} \leqslant C\|b\|_{\mathrm{BMO}}\end{aligned}$$

by doubling.

So Lemma 2.4 holds here as well, and by Jones' Lemma 2.5, $w\in A_\infty$, which finishes the proof. ∎

3. Characterizations of A_∞

THEOREM 3.1. *The following are equivalent*:

(1) $w\in A_\infty$.

(2) $I(|H(w\chi_I)|) \leqslant CI(w)$, *for all intervals* I.

(3) $I(M^*(w\chi_I)) \leqslant CI(w)$, *for all intervals* I.

Proof. As we remarked in the introduction, condition (RH) implies both (2) and (3). That (2) implies (1) is contained (somewhat hidden!) in the proof of Theorem 1 in [7]. We must show that (3) implies (1).

A weight v is an A_1 weight if

$$I(v) \leqslant C\operatorname*{ess\,inf}_I v, \quad \text{for all intervals } I. \tag{3.2}$$

$A_1\subset A_\infty$, and the constants p and C in condition (RH) depend only on the constant C in (3.2).

It will suffice to show that each $M^*(w\chi_J)\in A_1(J)$, in other words (3.2) holds with $v = M^*(w\chi_J)$ and all $I\subset J$, with the constant C independent of J. For then (RH) holds with some universal $p>1$ and C for each $M^*(w\chi_J)$ restricted to J. In particular,

$$\begin{aligned}J(w^p)^{1/p} &\leqslant J\big(M^*(w\chi_J)^p\big)^{1/p} \leqslant CJ\big(M^*(w\chi_J)\big)\\ &\leqslant CJ(w), \quad \text{by (3)}.\end{aligned}$$

So $w\in \mathrm{RH} = A_\infty$.

Now fix J. Let $I\subset J$ and put

$$F = \{x\in I:\ M^*(w\chi_J)(x) = M^*(w\chi_{2I})(x)\}, \qquad G = I\sim F.$$

G will be empty, of course, if I is too large. Now

$$\begin{aligned}|I|^{-1}\int_F M^*(w\chi_J) &= |I|^{-1}\int_F M^*(w\chi_{2I}) \leqslant 2|2I|^{-1}\int_{2I} M^*(w\chi_{2I})\\ &\leqslant C|2I|^{-1}\int_{2I} w \leqslant C\operatorname*{ess\,inf}_I M^*(w\chi_J).\end{aligned}$$

Suppose $x\in G$ and $y\in I$. Let $R\subset J$ be an interval with $x\in R$ and $R\not\subset 2I$. Then there exists an interval $\tilde{R}\subset J$ with $I\cup J\subset \tilde{R}$ and $|\tilde{R}|\leqslant 3|R|$. So

$$|R|^{-1}\int_R w \leqslant 3|\tilde{R}|^{-1}\int_{\tilde{R}} w \leqslant 3\operatorname*{ess\,inf}_I M^*(w\chi_J).$$

Therefore,

$$\begin{aligned}|I|^{-1}\int_I M^*(w\chi_J) &= |I|^{-1}\big[\int_F M^*(w\chi_J) + \int_G M^*(w\chi_J)\big]\\ &\leqslant C\operatorname*{ess\,inf}_I M^*(w\chi_J) + \frac{3|G|}{|I|}\operatorname*{ess\,inf}_I M^*(w\chi_J),\end{aligned}$$

and indeed, $M^*(w\chi_J)\in A_1(J)$. ∎

4. $\mathrm{Hom}(\mathrm{BMO})$ **and** $\mathrm{Hom}^*(\mathrm{BMO})$. We start with a new proof of Lemma 2.5. Assume that $\varphi' = w\geqslant 0$, $\varphi\in \mathrm{Hom}(\mathrm{BMO}) = \mathrm{Hom}^*(H^1)$. We will scale things so that $\varphi(0) = 0$ and $\varphi(1) = 1$. Define α, β and γ by

$$\varphi(\alpha) = 1/2, \quad \varphi(\beta) = 2, \quad \varphi(-\gamma) = -1/2.$$

As a first step, we will bound α below and γ above. Let a and b be the atoms

$$a(x) = \begin{cases}-1 & \text{if } 0\leqslant x\leqslant 1/2,\\ 1 & \text{if } 1/2 < x\leqslant 1,\end{cases} \qquad b(x) = \begin{cases}-1/2 & \text{if } 0\leqslant x\leqslant 1,\\ 1/2 & \text{if } 1 < x\leqslant 2.\end{cases}$$

Since $\varphi\in \mathrm{Hom}^*(H^1)$, $w(a\circ\varphi)$ and $w(b\circ\varphi)\in H^1$, and hence

$$\int |H[(a\circ\varphi)w]| \leqslant C, \quad \int |H[(b\circ\varphi)w]| \leqslant C, \tag{4.1}$$

using the Closed Graph Theorem. Now

$$H[(b\circ\varphi)w](-x) = -\int \frac{w(y)b(\varphi(y))}{x+y}\,dy$$

$$= \tfrac{1}{2}\int_0^1 \frac{w(y)}{x+y}\,dy - \tfrac{1}{2}\int_1^\beta \frac{w(y)}{x+y}\,dy.$$

For $x \geqslant 0$,

$$\int_1^\beta \frac{w(y)}{x+y}\,dy \leqslant \frac{1}{x+1}\int_1^\beta w(y)\,dy = \frac{1}{x+1}\int_0^1 w(y)\,dy \leqslant \int_0^1 \frac{w(y)}{x+y}\,dy,$$

so $H[(b\circ\varphi)w](-x) \geqslant 0$. From (4.1),

$$\tfrac{1}{2}\int_0^1\int_0^1 \frac{w(y)}{x+y}\,dy\,dx \leqslant C + \tfrac{1}{2}\int_0^1\int_1^\beta \frac{w(y)}{x+y}\,dy\,dx$$

$$\leqslant C + \tfrac{1}{2}\int_0^1 \frac{1}{x+1}\int_1^\beta w(y)\,dy\,dx = C + \tfrac{1}{2}\log 2.$$

The same argument with a gives

$$\int_0^1\int_0^\alpha \frac{w(y)}{x+y}\,dy\,dx \leqslant C + \int_0^1\int_\alpha^1 \frac{w(y)}{x+y}\,dy\,dx$$

$$\leqslant C + \int_0^1\int_0^1 \frac{w(y)}{x+y}\,dy\,dx \leqslant 3C + \log 2.$$

But

$$\int_0^1\int_0^\alpha \frac{w(y)}{x+y}\,dy\,dx \geqslant \int_0^1 \frac{1}{x+\alpha}\int_0^\alpha w(y)\,dy\,dx = \tfrac{1}{2}\log\frac{1+\alpha}{\alpha}.$$

So,

$$\log\frac{1+\alpha}{\alpha} \leqslant 6C + 2\log 2,$$

and hence, α is bounded below.

The same argument applied to the interval $[-\gamma, \alpha]$ bounds $\alpha/(\gamma+\alpha)$ below, so that γ is bounded above, also.

Let $I = [-\gamma, 0]$ and $J = [0, \alpha]$. Let c be the atom

$$c(x) = \begin{cases} -1 & \text{for } -1/2 \leqslant x \leqslant 0, \\ 1 & \text{for } 0 < x \leqslant 1/2. \end{cases}$$

Then

$$H[(a\circ\varphi)w](-x) = \int_J \frac{w(y)}{x+y}\,dy - \int_\alpha^1 \frac{w(y)}{x+y}\,dy = -H(w\chi_J)(-x) - \int_\alpha^1 \frac{w(y)}{x+y}\,dy,$$

so that

$$|H(w\chi_J)(-x)| \leqslant |H[(a\circ\varphi)w](-x)| + \int_\alpha^1 \frac{w(y)}{x+y}\,dy.$$

Therefore, by (4.1),

$$\int_I |H(w\chi_J)|\,dx \leqslant C + \int_\alpha^1\int_0^\gamma \frac{w(y)}{x+y}\,dx\,dy \leqslant C + \tfrac{1}{2}\log\left(\frac{1+\gamma}{\alpha}\right),$$

which we know to be bounded. Using (4.1) again with the atom c,

$$H[(c\circ\varphi)w](x) = H(w\chi_J)(x) - H(w\chi_I)(x),$$

so that

$$\int_I |H(w\chi_I)(x)|\,dx \leqslant \int_I |H(w\chi_J)|\,dx + \int_I |H[(c\circ\varphi)w]|\,dx$$

$$\leqslant 2C + \tfrac{1}{2}\log\left(\frac{1+\gamma}{\alpha}\right) = \left[4C + \log\left(\frac{1+\gamma}{\alpha}\right)\right]\int_I w.$$

The constants here are independent of I, and of our scaling, so Theorem 3.1 gives $w \in A_\infty$. ∎

We close this off by characterizing Hom*(BMO) and $\mathrm{Hom}(H^1)$.

THEOREM 4.2. $\mathrm{Hom}^*(\mathrm{BMO}) = \mathrm{Hom}(H^1) = \{\varphi\colon \varphi(x) = a + bx\}$.

Proof. By rescaling, we can assume that $\varphi(0) = 0$ and $\varphi(1) = 1$. The theorem asserts that such $\varphi \in \mathrm{Hom}^*(\mathrm{BMO})$ and $\in \mathrm{Hom}(H^1)$ if and only if $\varphi(x) = x$. Clearly that φ is a good homeomorphism. Since $\mathrm{Hom}^*(\mathrm{BMO}) = \{\varphi\colon \varphi$ is differentiable a.e. and $\varphi \in \mathrm{Hom}(H^1)\}$, it suffices to show that the only such homeomorphism belonging to $\mathrm{Hom}(H^1)$ is the identity homeomorphism.

So assume that such a $\varphi \in \mathrm{Hom}(H^1)$. Let $\alpha = \varphi(1/2)$, and consider the atom a defined above. By our assumption, $a\circ\varphi^{-1} \in H^1$, so that $H(a\circ\varphi^{-1}) \in L^1$. Let $x \geqslant 2$. Then

$$|H(a\circ\varphi^{-1})(x)| = \left|\int \frac{a(\varphi^{-1}(y))\,dy}{x-y}\right| = \left|-\int_0^\alpha \frac{1}{x-y}\,dy + \int_\alpha^1 \frac{1}{x-y}\,dy\right|$$

$$= \left|\log\frac{(x-\alpha)^2}{x(x-1)}\right| = \left|\log\left(1 + \frac{1-2\alpha}{x-1} + \frac{\alpha^2}{x^2-x}\right)\right|.$$

For large x, this behaves like $\dfrac{|1-2\alpha|}{x-1} + \dfrac{\alpha^2}{x^2-x}$, which is not in L^1 unless $\alpha = 1/2$.

Of course, the same argument applied to $[0, 1/2]$ forces $\varphi(1/4) = 1/4$, and applied to $[1/2, 1]$ forces $\varphi(3/4) = 3/4$, and so on. Likewise, if $\varphi(-1) \neq -1$, this midpoint argument applied to $[-1, 1]$ won't give $\varphi(0) = 0$. So we conclude:

$$\varphi(x) = x \text{ whenever } x \text{ is dyadic.}$$

Continuity then finishes off the theorem. ∎

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An interpolation theorem with A_p-weighted L^p spaces

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Abstract. If T and T^*: $uL^\infty \to \mathrm{BMO}_u$ for all $u \in A_1$, then T: $L^p(w) \to L^p(w)$ whenever $w \in A_p$, $1 < p < \infty$.

In 1976, Muckenhoupt and Wheeden introduced the weighted bounded mean oscillation spaces, BMO_u [6]. On $\mathbb{R}$, $f \in \mathrm{BMO}_u$ provided

$$|I|^{-1} \int_I |f - I(f)| \leqslant C I(u),$$

for all intervals I, using the notation $|I|$ = Lebesgue measure of I and $I(f) = |I|^{-1} \int_I f$. These spaces proved to have more than just an intrinsic interest, they turned out to be the solution spaces to some important singly and doubly weighted norm inequalities [1]. The purpose of this note is to show that these spaces arise as interpolation endpoints for $L^p(w)$, $w \in A_p$.

A weight $w \in A_p$ if

$$I(w) I[w^{-1/(p-1)}]^{p-1} \leqslant C \quad \text{for all intervals } I,$$

when $p > 1$. $w \in A_1$ if

$$I(w) \leqslant C \operatorname*{ess\,inf}_I w, \quad \text{for all intervals } I.$$

The A_p classes are nested, $A_p \subset A_q$ if $q > p$, and a weight belongs to some A_p class if and only if it satisfies a reverse Hölder inequality: $I(w^{1+\delta})^{1/(1+\delta)} \leqslant C I(w)$, for some $\delta > 0$ and all intervals I. The Hardy–Littlewood maximal operator and the Hilbert transform are bounded on $L^p(w)$ if and only if $w \in A_p$, $1 < p < \infty$ [2]. Also any $w \in A_p$ can be factored into the form $w = uv^{1-p}$, for some u and $v \in A_1$, the Jones' Factorization Theorem [5].

We will use the sharp maximal function of Fefferman and Stein [4],

$$f^{\#}(x) = \sup\{|I|^{-1} \int_I |f - I(f)|: x \in I, I \text{ an interval}\}.$$

If $1 < p < \infty$, then $\|f^{\#}\|_p \approx \|f\|_p$, provided $\|f\|_p$ is finite.