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Absolutely p-summing operators and Banach spaces containing all l_p^n uniformly complemented

by

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Abstract. It is proved that for $p = 1$, 2 and ∞ a Banach space G contains uniformly complemented all l_p^n's if (and only if) each operator $T: E \to F$ such that $\mathrm{id}_G \otimes T: G \otimes_\varepsilon E \to G \otimes_\pi F$ is continuous splits into a product $T = RS$ of an absolutely p-summing operator S and an operator R with an absolutely p'-summing dual.

0. Introduction. In [4] Jarchow conjectured that for a fixed real number $1 \leqslant p \leqslant \infty$ a Banach space G contains all l_p^n uniformly complemented if (and only if) it satisfies the following condition (*): Every operator $T \in \mathscr{L}(E, F)$ such that

$$\mathrm{id}_G \otimes T\colon\ G \otimes_\varepsilon E \to G \otimes_\pi F$$

is continuous can be written as a product RS of two appropriate operators R and S where R' is absolutely p'-summing and S is absolutely p-summing. We give an affirmative answer for $p = 1$, 2 and ∞. For arbitrary $1 \leqslant p \leqslant \infty$ it is proved as a by-product that G satisfies (*) if and only if there is a constant $\lambda \geqslant 1$ such that for every natural number n there are finitely many operators $I_1, \ldots, I_m \in \mathscr{L}(l_p^n, G)$ and $P_1, \ldots, P_m \in \mathscr{L}(G, l_p^n)$ (where m depends on n) with

$$\mathrm{id}_{l_p^n} = \sum_{k=1}^{m} P_k I_k, \qquad \sum_{k=1}^{m} \|P_k\|\,\|I_k\| \leqslant \lambda.$$

Standard notions and notations from Banach space theory are used, as presented in [5]. For the general theory of Banach operator ideals we refer the reader to [8].

1. S_p-spaces and T_p-spaces. As usual l_p^n stands for the space $\boldsymbol{R}^n$ equipped with the l_p-norm. A *real* Banach space G is said to be an S_p-*space* if it contains all l_p^n uniformly complemented, i.e., there is a sequence (G_n) of n-dimensional subspaces of G and projections $P_n \in \mathscr{L}(G, G)$ onto G_n such that

$$\sup_n d(G_n, l_p^n) < \infty, \qquad \sup_n \|P_n\| < \infty$$

(here as usual $d(\cdot, \cdot)$ denotes the Banach–Mazur distance). Clearly, G is an S_p-space if and only if there is a $\lambda \geqslant 1$ such that for every n there are operators

$I_n \in \mathscr{L}(l_p^m, G)$ and $P_n \in \mathscr{L}(G, l_p^m)$ satisfying

$$\mathrm{id}_{l_p^m} = P_n I_n, \qquad \|P_n\| \, \|I_n\| \leqslant \lambda.$$

As an example we mention that every infinite-dimensional $\mathscr{L}_p$-space (in the sense of Lindenstrauss and Pełczyński) is an S_p-space. Moreover, it is well known that a Banach space is an S_p-space iff its dual is an $S_{p'}$-space, and every S_p-space is either S_1, S_2 or S_∞. Pisier [9] (answering an old question of Lindenstrauss) constructed a class of infinite-dimensional Banach spaces which are not S_p for any $1 \leqslant p \leqslant \infty$.

We start with a useful characterization of S_1- and S_∞-spaces.

1.1. Proposition. *A Banach space G is an S_1-space (resp. S_∞-space) if and only if its l_2-sum*

$$l_2(G) := \{(x_k) \in G^N \mid \|(x_k)\|_2 := \Big(\sum_{k=1}^{\infty} \|x_k\|^2\Big)^{1/2} < \infty\}$$

is an S_1-space (resp. S_∞-space).

Proof. Of course, if the complemented subspace G of $l_2(G)$ is an S_1-space (resp. S_∞-space) then this also holds for $l_2(G)$ itself. Conversely, assume that $l_2(G)$ is an S_∞-space. A Banach space is an S_∞-space iff it contains all l_∞^n uniformly, and hence by (a special case of) the Maurey–Pisier theorem (see [6], p. 85 or [7], p. 85) a Banach space is S_∞ iff it has no finite cotype. But then, since by [6], p. 55, the l_2-sum $l_2(G)$ has no finite cotype if and only if G has no finite cotype, G is an S_∞-space. Finally, if $l_2(G)$ is assumed to be S_1 then the assertion follows by duality. ∎

We shall also need a characterization of S_2-spaces which appears as a consequence of the following lemma. Let μ_{n+1} be the normalized rotation invariant Borel measure on the sphere $S_{n+1} := \{x \in \mathbf{R}^{n+2} \mid \|x\|_2 = 1\}$.

1.2. Lemma. *For each $n \in N$ let r_n be a seminorm on $\mathbf{R}^{n+2}$ such that $r_n(x) \leqslant \|x\|_2$ for all $x \in \mathbf{R}^{n+2}$. Moreover, assume that*

$$\inf_n \int_{S_{n+1}} r_n(x)\, \mu_{n+1}(dx) =: \alpha > 0.$$

Then there are a constant $c > 0$ and $n_0 \in N$ such that for all $n \geqslant n_0$ there exists a subspace E_n of $\mathbf{R}^{n+2}$ satisfying

$$\dim E_n \geqslant cn \quad \text{and} \quad c\|x\|_2 \leqslant r_n(x) \leqslant \|x\|_2$$

for all $x \in E_n$.

The proof is modeled on the proof of Milman's important theorem [7], 4.2: Fix $n \in N$ and let M_n denote the median of r_n restricted to S_{n+1}, i.e., the unique real number M_n such that

$$\mu_{n+1}[M_n \geqslant r_n] \geqslant \tfrac{1}{2}, \qquad \mu_{n+1}[M_n \leqslant r_n] \geqslant \tfrac{1}{2}.$$

Put $\theta := \frac{1}{3}$ and $\varepsilon := \frac{1}{6}M_n$. By [7], Theorem 2.4, there is a subspace E_n of $\mathbf{R}^{n+2}$ with

$$\dim E_n \geqslant \frac{\varepsilon^2 n}{2\log 12} = c' M_n^2 n$$

and a θ-net N in $S_{n+1} \cap E_n$ such that for all $x \in N$ (denote by ϱ the geodesic distance on S_{n+1})

$$|r_n(x) - M_n| \leqslant \sup\{|r_n(x) - r_n(y)| \mid \varrho(x, y) \leqslant \varepsilon\} \leqslant \sup\{r_n(x-y) \mid \|x-y\|_2 \leqslant \varepsilon\} \leqslant \varepsilon.$$

Now [7], Lemma 4.1, is applied showing that $\frac{1}{4}M_n \leqslant r_n(x)$ for all $x \in S_{n+1} \cap E_n$ (we remark that Lemma 4.1 of [7] is only formulated for *norms* r_n, but its proof just uses the fact that r_n is a continuous, convex and homogeneous function). So it remains to prove that for large n the median M_n is larger than or equal to some uniform constant $d > 0$.

For $n \in N$ put $A_n := [r_n = M_n]$ and for $t \geqslant 0$ let $(A_n)_t$ be the set of all $x \in S_{n+1}$ such that $\varrho(x, y) \leqslant t$ for some $y \in A_n$. Then for every $x \in S_{n+1}$ and $y \in A_n$

$$\varrho(x, y) \geqslant \|x-y\|_2 \geqslant r_n(x-y) \geqslant |r_n(x) - M_n|,$$

and hence for all $t \geqslant 0$

$$\{x \in S_{n+1} \mid |r_n(x) - M_n| > t\} \subseteq \text{complement}\,(A_n)_t.$$

Since by Levy's lemma (see [7], Corollary 2.3)

$$\mu_{n+1}((A_n)_t) \geqslant 1 - \sqrt{\pi/2}\, e^{-t^2 n/2}$$

we obtain

$$\mu_{n+1}[|r_n - M_n| > t] \leqslant \sqrt{\pi/2}\, e^{-t^2 n/2}$$

and therefore

$$\Big|\int_{S_{n+1}} r_n(x)\, \mu_{n+1}(dx) - M_n\Big| \leqslant \int_{S_{n+1}} |r_n(x) - M_n|\, \mu_{n+1}(dx)$$
$$\leqslant \sqrt{\pi/2} \int_0^{\infty} e^{-t^2 n/2}\, dt = \frac{\pi}{2\sqrt{n}}.$$

By assumptions this implies for all n

$$\alpha \leqslant \int_{S_{n+1}} r_n(x)\, \mu_{n+1}(dx) \leqslant M_n + \frac{\pi}{2\sqrt{n}},$$

so that for all $n \geqslant \pi^2/\alpha^2$

$$d := \alpha/2 \leqslant M_n,$$

which leads to the desired conclusion. ∎

This lemma implies a useful criterion for S_2-spaces.

1.3. PROPOSITION. *A Banach space G is an S_2-space if and only if there is a constant $\alpha > 0$ such that for every n there are operators $V_n \in \mathscr{L}(l_2^{n+2}, G)$ and $U_n \in \mathscr{L}(G, l_2^{n+2})$ satisfying*

$$\int_{S_{n+1}} \|U_n V_n(x)\|_2 \, \mu_{n+1}(dx) \geqslant \alpha \|U_n\| \, \|V_n\| > 0.$$

Proof. If G is an S_2-space then there is $\lambda \geqslant 1$ such that for all n there are $I_{n+2} \in \mathscr{L}(l_2^{n+2}, G)$ and $P_{n+2} \in \mathscr{L}(G, l_2^{n+2})$ with

$$\mathrm{id}_{l_2^{n+2}} = P_{n+2} I_{n+2}, \qquad \|P_{n+2}\| \, \|I_{n+2}\| \leqslant \lambda.$$

In particular, for $\alpha := 1/\lambda$, $V_n := I_{n+2}$ and $U_n := P_{n+2}$

$$\int_{S_{n+1}} \|U_n V_n(x)\|_2 \, \mu_{n+1}(dx) = 1 \geqslant \alpha \|U_n\| \, \|V_n\| > 0.$$

Conversely, we assume without loss of generality that $\|V_n\| = \|U_n\| = 1$. Define

$$T_n := U_n V_n \in \mathscr{L}(l_2^{n+2}, l_2^{n+2}),$$
$$r_n(x) := \|T_n x\|_2 \quad \text{for } x \in \boldsymbol{R}^{n+2}.$$

By the lemma there exists a constant $c > 0$ such that for large n there is a subspace E_n of $\boldsymbol{R}^{n+2}$ with $\dim E_n \geqslant cn$ and $c\|x\|_2 \leqslant r_n(x)$ for all $x \in E_n$. Now observe that T_n is injective on E_n and consider the following commutative diagram:

$$\begin{array}{ccccc}
(E_n, \|\cdot\|_2) & & \xrightarrow{\ \mathrm{id}_{E_n}\ } & & (E_n, \|\cdot\|_2) \\
\big\downarrow {\scriptstyle V_n|_{E_n}} & & & & \big\uparrow {\scriptstyle T_n^{-1}} \\
G & \xrightarrow{\ U_n\ } & l_2^{n+2} & \xrightarrow{\ Q_n\ } & (T_n E_n, \|\cdot\|_2)
\end{array}$$

where Q_n is the orthogonal projection. With $I_n := V_n|_{E_n}$ and $P_n := T_n^{-1} Q_n U_n$ one gets $\mathrm{id}_{E_n} = P_n I_n$ and, since for all $x \in T_n E_n$

$$\|T_n^{-1} x\|_2 \leqslant \frac{1}{c} r_n(T_n^{-1} x) = \frac{1}{c}\|x\|_2,$$

moreover $\|P_n\| \, \|I_n\| \leqslant 1/c$. This proves that "$G$ contains all E_n uniformly complemented", and hence is an S_2-space. ∎

Now, as a technical device, a new class of Banach spaces is defined which a priori is larger than the class of all S_p-spaces. We call a *real* Banach space G a *T_p-space* $(1 \leqslant p \leqslant \infty)$ if there is a constant $\lambda \geqslant 1$ such that for every $n \in \boldsymbol{N}$ there are $m \in \boldsymbol{N}$ and operators $I_1, \ldots, I_m \in \mathscr{L}(l_p^n, G)$ and $P_1, \ldots, P_m \in \mathscr{L}(G, l_p^n)$ satisfying

$$\mathrm{id}_{l_p^n} = \sum_{k=1}^{m} P_k I_k, \qquad \sum_{k=1}^{m} \|P_k\| \, \|I_k\| \leqslant \lambda.$$

A Banach space G is T_p if and only if its dual is $T_{p'}$ (one direction is trivial and the other follows by standard arguments using (a weak form of) the principle of local reflexivity). Obviously, every S_p-space is a T_p-space. The following partial converse is the crucial step of the proof of our main result.

1.4. THEOREM. *For $p = 1$, 2 and ∞ every T_p-space is an S_p-space.*

Proof. *The cases $p = 1$ and ∞*: Because of the duality of T_1- and T_∞-spaces as well as S_∞- and S_1-spaces it suffices to prove that every T_∞-space is S_∞. Let G be a T_∞-space. In view of 1.1 we show that the l_2-sum of G is S_∞. By assumption there is $\lambda \geqslant 1$ (not depending on n) and there are operators $I_1, \ldots, I_m \in \mathscr{L}(l_\infty^n, G)$ and $P_1, \ldots, P_m \in \mathscr{L}(G, l_\infty^n)$ (where m depends on n) such that

$$\mathrm{id}_{l_\infty^n} = \sum_{k=1}^{m} P_k I_k, \qquad \sum_{k=1}^{m} \|P_k\| \, \|I_k\| \leqslant \lambda.$$

Without loss of generality we may assume that $\|P_k\| = \|I_k\|$ for all k. Define

$$V_n\colon l_\infty^n \to l_2(G), \qquad \xi \rightsquigarrow (I_1 \xi, \ldots, I_m \xi, 0, \ldots),$$
$$U_n\colon l_2(G) \to l_\infty^n, \qquad (x_k) \rightsquigarrow \sum_{k=1}^{m} P_k x_k.$$

Then obviously $\mathrm{id}_{l_\infty^n} = U_n V_n$ and $\|V_n\| \leqslant \sqrt{\lambda}$, $\|U_n\| \leqslant \sqrt{\lambda}$, which proves that $l_2(G)$ contains all l_∞^n uniformly complemented.

The case $p = 2$: Let G be a T_2-space. In order to show that G is S_2 we use criterion 1.3. Let $n \in \boldsymbol{N}$. Then there are operators $I_1, \ldots, I_m \in \mathscr{L}(l_2^{n+2}, G)$ and $P_1, \ldots, P_m \in \mathscr{L}(G, l_2^{n+2})$ such that

$$\mathrm{id}_{l_2^{n+2}} = \sum_{k=1}^{m} P_k I_k, \qquad \sum_{k=1}^{m} \|P_k\| \, \|I_k\| \leqslant \lambda,$$

where the constant $\lambda \geqslant 1$ does not depend on n. Assume that for all $k = 1, \ldots, m$

$$\int_{S_{n+1}} \|P_k I_k(x)\|_2 \, \mu_{n+1}(dx) < \frac{1}{\lambda} \|P_k\| \, \|I_k\|.$$

Then

$$1 = \int_{S_{n+1}} \Big\| \sum_{k=1}^{m} P_k I_k(x) \Big\|_2 \, \mu_{n+1}(dx) \leqslant \int_{S_{n+1}} \sum_{k=1}^{m} \|P_k I_k(x)\|_2 \, \mu_{n+1}(dx)$$
$$< \frac{1}{\lambda} \sum_{k=1}^{m} \|P_k\| \, \|I_k\| \leqslant 1,$$

a contradiction. Hence there is $1 \leqslant k \leqslant m$ such that with $\alpha := 1/\lambda > 0$, $V_n := I_k \in \mathscr{L}(l_2^{n+2}, G)$ and $U_n := P_k \in \mathscr{L}(G, l_2^{n+2})$

$$\int_{S_{n+1}} \|U_n V_n(x)\|_2 \mu_{n+1}(dx) \geqslant \alpha \|U_n\| \|V_n\| > 0,$$

which by criterion 1.3 implies that G is an S_2-space. ∎

It remains unsolved whether the preceding result holds for arbitrary $1 \leqslant p \leqslant \infty$.

2. S_p-spaces and p-dominated operators. Let $1 \leqslant p \leqslant \infty$ and let E be a Banach space. For $x_1, \ldots, x_n \in E$ put

$$w_p(x_i) := \sup\{(\sum_{i=1}^{n} |\langle x_i, x'\rangle|^p)^{1/p} \mid x' \in B_{E'}\}$$

(with the obvious modification for $p = \infty$). Denote by $[\mathscr{D}_p, D_p]$ the Banach operator ideal of all p-dominated operators: $T \in \mathscr{D}_p(E, F)$ if there is a $c \geqslant 0$ such that

$$\sum_{i=1}^{n} |\langle Tx_i, y_i'\rangle| \leqslant c w_p(x_i) w_{p'}(y_i')$$

for all finite families $x_1, \ldots, x_n \in E$ and $y_1', \ldots, y_n' \in F'$. The norm is given by $D_p(T) := \inf c$ where c is as above (see [8], 17.4).

Important particular cases are $\mathscr{D}_1 = \mathscr{P}_1$, the ideal of all absolutely summing operators, $\mathscr{D}_\infty = \mathscr{P}_1^{\text{dual}}$, the ideal of all operators T such that T' is absolutely summing, and $\mathscr{D}_2$, the largest Banach ideal of operators which in Hilbert spaces coincides with the ideal of all nuclear operators. A deep factorization theorem of Kwapień (see [8], 17.4.3) states that an operator T is p-dominated if and only if it can be written as a product RS where R' is absolutely p'-summing and S absolutely p-summing, briefly: $\mathscr{D}_p := \mathscr{P}_{p'}^{\text{dual}} \circ \mathscr{P}_p$ (put $\mathscr{P}_\infty = \mathscr{L}$).

As usual $\varepsilon(\cdot; E, F)$ denotes the injective norm on the tensor product $E \otimes F$ of two Banach spaces and $\pi(\cdot; E, F)$ the projective norm. Moreover, for $1 \leqslant p \leqslant \infty$ we shall consider the norms

$$w_p(z; E, F) := \inf\{w_p(x_i) w_{p'}(y_i) \mid z = \sum_{i=1}^{n} x_i \otimes y_i\}$$

$$w_p^*(z; E, F) := \sup\{|\langle z, u\rangle| \mid w_{p'}(u; E', F') \leqslant 1\},$$

which as ε and π form tensor norms in the sense of Grothendieck, in particular: $\varepsilon \leqslant w_p$, $w_p^* \leqslant \pi$ (see e.g. [2] and [3]). If E is an $\mathscr{L}_{p'}$-space or F is an $\mathscr{L}_p$-space then isomorphically

$$E \otimes_\varepsilon F = E \otimes_{w_p} F, \quad E \otimes_\pi F = E \otimes_{w_p^*} F.$$

Moreover, for arbitrary Banach spaces E and F

$$\mathscr{D}_p(E, F') = (E \otimes_{w_p} F)', \quad T \rightsquigarrow (x \otimes y \rightsquigarrow \langle y, Tx\rangle),$$

holds isometrically. Hence, if as usual $[\mathscr{I}, I]$ stands for the Banach operator ideal of all integral operators then

$$\mathscr{D}_p(E, F') = (E \otimes_{w_p} F)' = (E \otimes_\varepsilon F)' = \mathscr{I}(E, F'),$$

provided E is an $\mathscr{L}_{p'}$-space or F an $\mathscr{L}_p$-space.

The following tensor product characterization of p-dominated operators is a slight extension of results proved in [1] and [4].

2.1. Proposition. *For $1 \leqslant p \leqslant \infty$, $T \in \mathscr{L}(E, F)$ and a Banach space G consider the following four statements:*

(1) $T \in \mathscr{D}_p(E, F)$.
(2) $\mathrm{id}_G \otimes T$: $G \otimes_{w_{p'}} E \to G \otimes_\pi F$ *is continuous.*
(3) $\mathrm{id}_G \otimes T$: $G \otimes_\varepsilon E \to G \otimes_{w_{p'}^*} F$ *is continuous.*
(4) $\mathrm{id}_G \otimes T$: $G \otimes_{w_{p'}} E \to G \otimes_{w_{p'}^*} F$ *is continuous.*

Then (1)$\Rightarrow$(3)$\Rightarrow$(4) *and* (1)$\Rightarrow$(2)$\Rightarrow$(4). *Conversely: If G is a T_p-space then* (1)–(4) *are equivalent. In particular, if G is an infinite-dimensional $\mathscr{L}_p$-space then* (1) *is equivalent to*

(5) $\mathrm{id}_G \otimes T$: $G \otimes_\varepsilon E \to G \otimes_\pi F$ *is continuous.*

We remark that this result in particular implies that Pisier's infinite-dimensional Banach spaces P for which $P \otimes_\varepsilon P = P \otimes_\pi P$ holds isomorphically cannot be T_p-spaces for any $1 \leqslant p \leqslant \infty$ (for the construction of these spaces see e.g. [9]).

Proof. Obviously, (3)$\Rightarrow$(4) and (2)$\Rightarrow$(4). The proof of (1)$\Rightarrow$(2) is easy: Indeed, for $z = \sum_{i=1}^n g_i \otimes x_i \in G \otimes E$

$$\pi(\mathrm{id}_G \otimes T(z); G, F) = \sup\{|\langle \mathrm{id}_G \otimes T(z), \varphi\rangle| \mid \varphi \in B_{(G \otimes_\pi F)'}\}$$

$$= \sup\{|\sum_{i=1}^{n} \langle Tx_i, L_\varphi g_i\rangle| \mid \varphi \in B_{(G \otimes_\pi F)'}\} \leqslant D_p(T) w_p(x_i) w_{p'}(g_i)$$

(here $L_\varphi \in \mathscr{L}(G, F')$ denotes the canonical operator associated with φ), and hence

$$\pi(\mathrm{id}_G \otimes T(z); G, F) \leqslant D_p(T) w_{p'}(z; G, E).$$

The implication (1)$\Rightarrow$(3) follows from (1)$\Rightarrow$(2) by duality, since $T \in \mathscr{D}_p(E, F)$ iff $T' \in \mathscr{D}_{p'}(F', E')$ and the embeddings

$$G \otimes_\varepsilon E \hookrightarrow (G' \otimes_\pi E')', \quad G \otimes_{w_{p'}^*} F \hookrightarrow (G' \otimes_{w_p} F')'$$

are isometries. So it remains to prove (4)$\Rightarrow$(1) provided that G is a T_p-space. Fix $x_1, \ldots, x_n \in E$ and $y_1', \ldots, y_n' \in F'$. Then there is a constant $\lambda \geqslant 1$ (independent of n) and there are operators $I_1, \ldots, I_m \in \mathscr{L}(l_p^n, G)$ and $P_1, \ldots, P_m \in \mathscr{L}(G, l_p^n)$ (m depending on n) such that

$$\mathrm{id}_{l_p^n} = \sum_{k=1}^{m} P_k I_k, \quad \sum_{k=1}^{m} \|P_k\| \|I_k\| \leqslant \lambda.$$

Hence with $\varepsilon_j := \operatorname{sgn}\langle Tx_j, y'_j\rangle$ by use of (4)

$$\begin{aligned}\sum_{i=1}^{n} |\langle Tx_i, y'_i\rangle| &= \sum_{i,j}\sum_{k} \langle I_k e_i\otimes Tx_i,\ P'_k e_j\otimes \varepsilon_j y'_j\rangle \\ &\leqslant \sum_{k} \Big|\Big\langle \sum_{i} I_k e_i\otimes Tx_i,\ \sum_{j} P'_k e_j\otimes\varepsilon_j y'_j\Big\rangle\Big| \\ &\leqslant \sum_{k} w^*_{p'}\Big(\sum_{i} I_k e_i\otimes Tx_i;\ G, F\Big) w_p\Big(\sum_{j} P'_k e_j\otimes\varepsilon_j y'_j;\ G', F'\Big) \\ &\leqslant \lambda \|\mathrm{id}_G\otimes T\colon\ G\otimes_{w_{p'}} E\to G\otimes_{w^*_{p'}} F\|\, w_p(x_i)\, w_{p'}(y'_j),\end{aligned}$$

which as desired proves that T is p-dominated. ■

As a corollary this proposition implies that an operator $T\in\mathscr{L}(E, F)$ is p-dominated if and only if there is an S_p-space G such that $\mathrm{id}_G\otimes T\colon G\otimes_\varepsilon E\to G\otimes_\pi F$ is continuous. In [4] Jarchow asked for a certain converse of this statement: *Is a Banach space G an S_p-space if every operator $T\in\mathscr{L}(E, F)$ such that $\mathrm{id}_G\otimes T\colon G\otimes_\varepsilon E\to G\otimes_\pi F$ is continuous, is already p-dominated*? The following theorem gives a positive answer for $p = 1, 2$ and ∞. Moreover, for arbitrary $1\leqslant p\leqslant\infty$ it is shown that the answer to Jarchow's problem is positive if one considers the class of all T_p-spaces instead of the class of all S_p-spaces.

2.2. THEOREM. *Let $1\leqslant p\leqslant\infty$ and let G be a Banach space. Then the following are equivalent*:

(1) *Every $T\in\mathscr{L}(E, F)$ such that $\mathrm{id}_G\otimes T\colon G\otimes_\varepsilon E\to G\otimes_\pi F$ is continuous, is p-dominated.*

(2) *There is a constant $c\geqslant 0$ such that for each $T\in\mathscr{L}(l^n_{p'}, l^n_{p'})$*

$$I(T)\leqslant c\,\|\mathrm{id}_G\otimes T\colon\ G\otimes_\varepsilon l^n_{p'}\to G\otimes_\pi l^n_{p'}\|$$

($I(T)$ is the integral = nuclear norm of T).

(3) *G is a T_p-space.*

Moreover, for $p = 1, 2$ and ∞ these statements are equivalent to

(4) *G is an S_p-space.*

Proof. Obviously, (1)⇒(2) is a consequence of the closed graph theorem and the fact that $\mathscr{D}_p(l_{p'}, l_{p'}) = \mathscr{I}(l_{p'}, l_{p'})$ (see the preliminary remarks of this section). Moreover, (3)⇒(1) follows directly from 2.1, and (3)⇔(4) (if $p = 1, 2$ or ∞) was already stated in 1.4. Hence it remains to prove that (2) implies (3): For $n\in N$ consider the linear surjection

$$c_{\mathrm{tr}}\colon\ (l^n_p\otimes G')\otimes(G\otimes l^n_{p'})\to l^n_p\otimes l^n_{p'},\qquad (\eta\otimes g')\otimes(g\otimes\xi)\rightsquigarrow\langle g, g'\rangle\,\eta\otimes\xi,$$

and define the quotient norm

$$\delta(z;\ l^n_{p'}, l^n_{p'}) := \inf\{\pi(w;\ l^n_p\otimes_\varepsilon G', G\otimes_\varepsilon l^n_{p'})\mid c_{\mathrm{tr}}(w) = z\}$$

on $l^n_p\otimes l^n_{p'}$.

Step 1: We prove that under the hypothesis of (2) for all n

$$\delta(\cdot;\ l^n_p, l^n_{p'})\leqslant c\varepsilon(\cdot;\ l^n_p, l^n_{p'})$$

or dually: For $u = \sum_i x_i\otimes y_i\in l^n_{p'}\otimes l^n_p$

$$\pi(u;\ l^n_{p'}, l^n_p)\leqslant c\delta'(u;\ l^n_{p'}, l^n_p)$$

where

$$\delta'(u;\ l^n_{p'}, l^n_p) := \sup\{|\langle u, v\rangle|\mid \delta(v;\ l^n_p, l^n_{p'})\leqslant 1\}.$$

Consider the operator $T_u := \sum_i y_i\otimes x_i\in\mathscr{L}(l^n_{p'}, l^n_{p'})$. Then

$$\|\mathrm{id}_G\otimes T_u\colon\ G\otimes_\varepsilon l^n_{p'}\to G\otimes_\pi l^n_{p'}\|\leqslant\delta'(u;\ l^n_{p'}, l^n_p).$$

Indeed, for $z = \sum_i g_i\otimes\xi_i\in G\otimes l^n_{p'}$ choose

$$w = \sum_j g'_j\otimes\eta_j\in G'\otimes_\varepsilon l^n_p = (G\otimes_\pi l^n_{p'})'$$

with $\varepsilon(w;\ G', l^n_p)\leqslant 1$, $\pi(\mathrm{id}_G\otimes T_u(z);\ G, l^n_{p'}) = |\langle\mathrm{id}_G\otimes T_u(z), w\rangle|$, and check

$$\begin{aligned}\pi(\mathrm{id}_G\otimes T_u(z);\ G, l^n_{p'}) &= \Big|\Big\langle\sum_i g_i\otimes T_u(\xi_i),\ \sum_j g'_j\otimes\eta_j\Big\rangle\Big| \\ &= \Big|\Big\langle u,\ c_{\mathrm{tr}}\Big(\Big(\sum_j\eta_j\otimes g'_j\Big)\otimes\Big(\sum_i g_i\otimes\xi_i\Big)\Big)\Big\rangle\Big| \\ &\leqslant\delta'(u;\ l^n_{p'}, l^n_p)\,\delta(c_{\mathrm{tr}}(\ldots);\ l^n_p, l^n_{p'})\leqslant\delta'(u;\ l^n_{p'}, l^n_p)\,\varepsilon(z;\ G, l^n_{p'}).\end{aligned}$$

Hence by (2)

$$\pi(u;\ l^n_{p'}, l^n_p) = I(T_u\colon\ l^n_{p'}\to l^n_{p'})\leqslant c\delta'(u;\ l^n_{p'}, l^n_p).$$

Step 2: Let us now prove that G is a T_p-space. For all n the following diagram commutes:

$$\begin{array}{ccc}(l^n_p\otimes_\varepsilon G')\otimes_\pi(G\otimes_\varepsilon l^n_{p'}) & \overset{c_{\mathrm{tr}}}{\twoheadrightarrow} & l^n_p\otimes_\delta l^n_{p'} \\ \| & & \| \\ \mathscr{L}(l^n_p, G)\otimes_\pi\mathscr{L}(G, l^n_p) & \overset{\Psi}{\to} & \mathscr{L}(l^n_p, l^n_p) \\ & S\otimes T\rightsquigarrow TS. & \end{array}$$

Therefore by step 1 there is $w\in\mathscr{L}(l^n_p, G)\otimes\mathscr{L}(G, l^n_p)$ such that $\Psi(w) = \mathrm{id}_{l^n_p}$ and

$$\pi(w;\ \mathscr{L}(l^n_p, G), \mathscr{L}(G, l^n_p))\leqslant(1+\varepsilon)c,$$

which by the definition of the π-norm implies the assertion: For every n there is m and there are operators $I_1, \ldots, I_m\in\mathscr{L}(l^n_p, G)$ and $P_1, \ldots, P_m\in\mathscr{L}(G, l^n_p)$ satisfying

$$\mathrm{id}_{l^n_p} = \sum_{k=1}^{m} P_k I_k,\qquad \sum_{k=1}^{m}\|P_k\|\,\|I_k\|\leqslant(1+\varepsilon)^2 c.\ ■$$

Since $\mathscr{D}_p \subseteq \mathscr{D}_2$ for $1 < p < \infty$ (see [8], 17.4.5) one gets as an immediate consequence that every T_p-space $(1 \leqslant p \leqslant \infty)$ is either S_1 or S_2 or S_∞.

We remark that the theorem holds in the complex sense also (for the definition of complex S_p- and T_p-spaces use the complex l_p^n's instead of the real ones). Indeed, it can be checked easily that the proof of the equivalences (1)⇔(2)⇔(3) does not depend on the scalar field. Moreover, for $p = 1$, 2 and ∞ every complex T_p-space contains all complex l_p^n uniformly complemented. For $p = 1$, ∞ the proof is exactly that of 1.4, and for $p = 2$ the argument is as follows: Let G be a complex T_2-space. Then it is immediate that G considered as a real Banach space is T_2, and hence S_2 by 1.4. But, if a complex Banach space considered as a real one contains all real l_2^n uniformly complemented, then it also contains all complex l_2^n uniformly complemented (this was pointed out to me by Pisier; his argument is based on the facts that Theorem 5.11 of [10] is valid in the complex case also and moreover its converse is essentially true).

Finally, we state some simple reformulations of the theorem. It follows from 2.1 that for $G = l_{p'}$ every $T \in \mathscr{L}(l_p, l_p)$ is integral ($= p'$-dominated) if and only if $\mathrm{id}_G \otimes T\colon G \otimes_\varepsilon l_p \to G \otimes_\pi l_p$ is continuous (see also [2], 5.2). By the theorem and the closed graph theorem the following partial converse holds: For $p = 1$, 2, ∞ the fact that every $T \in \mathscr{L}(l_p, l_p)$ is integral if $\mathrm{id}_G \otimes T\colon G \otimes_\varepsilon E \to G \otimes_\pi F$ is continuous, implies that G is an $S_{p'}$-space.

Purely formulated in terms of operators this means

2.3. Corollary. *Let $p = 1$, 2 or ∞ and let G be a Banach space. Then the following are equivalent*:

(1) *Every $T \in \mathscr{L}(l_p, l_p)$ such that $TS \in \mathscr{L}(G, l_p)$ is integral for every $S \in \mathscr{L}(G, l_p)$, is integral itself.*

(2) *There is a constant $c \geqslant 0$ such that for all $T \in \mathscr{L}(l_p^n, l_p^n)$*

$$I(T) \leqslant c \sup\{I(TS) \mid \|S\colon G \to l_p^n\| \leqslant 1\}.$$

(3) *G is an S_p-space.*

The equivalence of (1) and (2) follows by standard arguments (using the closed graph theorem). The proof of (2)⇔(3) is a direct consequence of 2.2 and the commutativity of the following diagram:

$$\begin{array}{ccc} G' \otimes_\varepsilon l_p^n & \xrightarrow{\mathrm{id}_{G'} \otimes T} & G' \otimes_\pi l_p^n \\ \| & & \| \\ \mathscr{L}(G, l_p^n) & \longrightarrow & I(G, l_p^n) \\ & S \rightsquigarrow TS. & \end{array}$$

We remark that l_p in 2.3(1) can be replaced by any infinite-dimensional $\mathscr{L}_p$-space.

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