Absolutely $p$-summing operators
and Banach spaces
containing all $p_p^n$ uniformly complemented
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Abstract. It is proved that for $p = 1, 2$ and $\infty$ a Banach space $G$ contains uniformly complemented all $p_p^n$'s if (and only if) each operator $T \in L(E,F)$ that satisfies $\|\cdot\|_p \leq \lambda$ for every $p$ such that $1 \leq p \leq \infty$ and $\lambda$ is finite.

1. $S_p$-spaces and $T_{\infty}$-spaces. As usual $p_p^n$ stands for the space $R^n_p$ equipped with the $p$-norm. A real Banach space $G$ is said to be an $S_p$-space if it contains all $p_p^n$ uniformly complemented, i.e., there is a sequence $(G_n)$ of $n$-dimensional subspaces of $G$ and projections $P_n \in L_p(G,G)$ onto $G_n$ such that

$$\sup_n d(G_n, p_p^n) < \infty, \quad \sup_n \|P_n\| < \infty$$

(here as usual $d(\cdot, \cdot)$ denotes the Banach–Mazur distance). Clearly, $G$ is an $S_p$-space if and only if there is a $\lambda \geq 1$ such that for every $n$ there are operators
$I_n \in \mathfrak{L}(\mathbb{L}_p, G)$ and $P_n \in \mathfrak{L}(G, \mathbb{L}_p)$ satisfying

$$\|I_n\| \|P_n\| \leq \lambda.$$  

As an example we mention that every infinite-dimensional $\mathbb{L}_p$-space (in the sense of Lindenstrauss and Pełczyński) is an $S_1$-space. Moreover, it is well known that a Banach space is an $S_1$-space if its dual is an $S_\infty$-space, and every $S_\infty$-space is either $S_1$, $S_2$ or $S_\infty$. Pisier [9] (answering an old question of Lindenstrauss) constructed a class of infinite-dimensional Banach spaces which are not $S_\infty$ for any $1 \leq p \leq \infty$.

We start with a useful characterization of $S_1$- and $S_\infty$-spaces.

1. Proposition. A Banach space $G$ is an $S_1$-space (resp. $S_\infty$-space) if and only if its $l_2$-sum

$$l_2(G) := \{(x_k) \in G^\infty | \|(x_k)\|_2 = \left( \sum_{k=1}^\infty \|x_k\|^2 \right)^{1/2} < \infty \}$$

is an $S_1$-space (resp. $S_\infty$-space).

Proof. Of course, if the complemented subspace $G$ of $l_2(G)$ is an $S_1$-space (resp. $S_\infty$-space) then this also holds for $l_2(G)$ itself. Conversely, assume that $l_2(G)$ is an $S_\infty$-space. A Banach space is an $S_\infty$-space if and only if it contains all $l_\infty$ uniformly, and hence by (a special case of the Maurey–Pisier theorem (see [6], p. 85 or [7], p. 85) a Banach space is $S_\infty$ if it has no finite cotype. But then, since by [6], p. 55, the $l_2$-sum $l_2(G)$ has no finite cotype if and only if $G$ has no finite cotype, $G$ is an $S_\infty$-space. Finally, if $l_2(G)$ is assumed to be $S_1$, then the assertion follows by duality. □

We shall also need a characterization of $S_2$-spaces which appears as a consequence of the following lemma. Let $\mu_{n+1}$ be the normalized rotation invariant Borel measure on the sphere $S_{n+1} := \{x \in \mathbb{R}^{n+2} | \|x\|_2 = 1\}$.

1.2. Lemma. For each $n \in \mathbb{N}$ let $r_n$ be a seminorm on $\mathbb{R}^{n+2}$. Then $r_n(x) \leq \|x\|_2$ for all $x \in \mathbb{R}^{n+2}$. Moreover, assume that

$$\inf_{x \in S_{n+1}} r_n(x) \mu_{n+1}(dx) =: \alpha > 0.$$  

Then there are a constant $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $x \in S_{n+1}$, there exists a subspace $E_n$ of $\mathbb{R}^{n+2}$ satisfying

$$\dim E_n \geq cn$$  

and $c \|x\|_2 \leq r_n(x) \leq \|x\|_2$ for all $x \in E_n$.

The proof is modeled on the proof of Milman’s important theorem [7], 4.2: Fix $n \in \mathbb{N}$ and let $M_n$ denote the median of $r_n$ restricted to $S_{n+1}$, i.e., the unique real number $M_n$ such that

$$\mu_{n+1}[M_n \geq r_n] \geq \frac{1}{2}, \quad \mu_{n+1}[M_n \leq r_n] \geq \frac{1}{2}.$$  

Put $t := \frac{1}{2}$ and $\varepsilon := \frac{1}{2}M_n$. By [7], Theorem 2.4, there is a subspace $E_n$ of $\mathbb{R}^{n+2}$ with

$$\dim E_n \geq \frac{c^2 n}{2 \log 12} = c'M_n^2 n$$

and a $0$-net $N$ in $S_{n+1} \cap E_n$ such that for all $x \in N$ (denote by $\rho$ the geodesic distance on $S_{n+1}$)

$$\rho(x) - M_n \leq \sup \{\rho(x) - \rho(y) | \rho(x, y) \leq \varepsilon\} \leq \rho(x) - \rho(y) \leq \varepsilon \leq \varepsilon.$$  

Now $\ell_2(S_{n+1})$ is applied showing that $\frac{1}{2}M_n \leq r_n(x)$ for all $x \in S_{n+1} \cap E_n$ (we remark that Lemma 4.1 of [7] is only formulated for norms $r_n$, but its proof just uses the fact that $r_n$ is a continuous, convex and homogeneous function). So it remains to prove that for large $n$ the median $M_n$ is larger than or equal to some uniform constant $d > 0$.

For $n \in \mathbb{N}$ put $A_n := \{r_n = M_n\}$ and for $t \geq 0$ let $(A_n)$ be the set of all $x \in S_{n+1}$ such that $\rho(x, y) \leq t$ for some $y \in A_n$. Then for every $x \in S_{n+1}$ and $y \in A_n$

$$\rho(x, y) \leq \rho(x, y) \leq r_n(x) \leq [\rho(x) - M_n],$$

and hence for all $t \geq 0$\n
$$\{x \in S_{n+1} | \rho(x, y) - M_n > t\} \subseteq \text{complement } (A_n).$$

Since by Levy’s lemma (see [7], Corollary 2.3)

$$\mu_{n+1}(A_n) \geq 1 - \sqrt{\pi/2} e^{-\alpha n^2}$$

we obtain

$$\mu_{n+1}(A_n) \geq 1 - \sqrt{\pi/2} e^{-\alpha n^2}$$

and therefore

$$\frac{1}{2} \int_{S_{n+1}} r_n(x) \mu_{n+1}(dx) \leq \frac{1}{2} \int_{S_{n+1}} r_n(x) \mu_{n+1}(dx) \leq -\alpha n^2 \leq \frac{\pi}{2 \sqrt{n}} - \frac{\pi}{2 \sqrt{n}}.$$  

By assumptions this implies for all $n$

$$\alpha \leq \int_{S_{n+1}} r_n(x) \mu_{n+1}(dx) \leq \int_{S_{n+1}} r_n(x) \mu_{n+1}(dx) \leq -\alpha n^2 \leq \frac{\pi}{2 \sqrt{n}},$$

so that for all $n \geq \pi^2/\alpha^2$

$$d := \alpha/2 \leq M_n,$$

which leads to the desired conclusion. □

This lemma implies a useful criterion for $S_2$-spaces.
1.3. PROPOSITION. A Banach space \( G \) is an \( S_p \)-space if and only if there is a constant \( \alpha > 0 \) such that for every \( n \) there are operators \( V \in \mathcal{L}(l_p^{n+2}, G) \) and \( U \in \mathcal{L}(G, l_p^{n+2}) \) satisfying

\[
\int_{s_{n+1}} \| U_n V_n(x) \|_{l_p^{n+1}}(dx) \geq \alpha \| U_n \| \| V_n \| > 0.
\]

Proof. If \( G \) is an \( S_p \)-space then there is \( \lambda \geq 1 \) such that for all \( n \) there are \( I_{n+1} \in \mathcal{L}(l_p^{n+2}, G) \) and \( P_{n+2} \in \mathcal{L}(G, l_p^{n+2}) \) with

\[
\text{id}_{l_p^{n+2}} = P_{n+2} I_{n+1}, \quad \| P_{n+2} \| \| I_{n+1} \| \leq \lambda.
\]

In particular, for \( \alpha := 1/\lambda \), \( V_n := I_{n+1} \) and \( U_n := P_{n+2} \)

\[
\int_{s_{n+1}} \| U_n V_n(x) \|_{l_p^{n+1}}(dx) = 1 \geq \alpha \| U_n \| \| V_n \| > 0.
\]

Conversely, we assume without loss of generality that \( \| V_n \| = \| U_n \| = 1 \).

Define

\[
T_n := U_n V_n \in \mathcal{L}(l_p^{n+2}, l_p^{n+2}), \quad r_n(x) := \| T_n x \|_{l_p^{n+2}} \quad \text{for} \quad x \in l_p^{n+2}.
\]

By the lemma there exists a constant \( c > 0 \) such that for large \( n \) there is a subspace \( E_n \) of \( R^{n+2} \) with \( \dim E_n \geq cn \) and \( c \| x \|_{l_p^{n+2}} \leq r_n(x) \) for all \( x \in E_n \). Now observe that \( T_n \) is injective on \( E_n \) and consider the following commutative diagram:

\[
\begin{array}{ccc}
(E_n, \| \cdot \|_{l_p^{n+2}}) & \xrightarrow{\text{id}_{E_n}} & (E_n, \| \cdot \|_{l_p^{n+2}}) \\
\downarrow{V_n \downarrow l_{E_n}} & & \downarrow{T_n^{-1}} \\
G & \xrightarrow{l_{P_{n+2}}} & (T_n E_n, \| \cdot \|_{l_p^{n+2}})
\end{array}
\]

where \( Q_n \) is the orthogonal projection. With \( J_n := V_n l_{E_n} \) and \( P_n := T_n^{-1} Q_n U_n \), one gets \( \text{id}_{E_n} = P_n J_n \), and, since for all \( x \in T_n E_n \)

\[
\| T_n^{-1} x \|_{l_p^{n+2}} \leq \frac{1}{c} r_n(T_n^{-1} x) = \frac{1}{c} \| x \|_{l_p^{n+2}},
\]

moreover \( \| P_n \| \| I_n \| \leq 1/c \). This proves that "\( G \) contains all \( E_n \) uniformly complemented", and hence is an \( S_p \)-space.

Now, as a technical device, a new class of Banach spaces is defined which a priori is larger than the class of all \( S_p \)-spaces. We call a real Banach space \( G \) a \( T_p \)-space \( (1 < p < \infty) \) if there is a constant \( \lambda \geq 1 \) such that for every \( n \in N \) there are \( m \in N \) and operators \( I_1, \ldots, I_m \in \mathcal{L}(l_p, G) \) and \( P_1, \ldots, P_m \in \mathcal{L}(G, l_p^{m+2}) \) satisfying

\[
\text{id}_G = \sum_{k=1}^m P_k I_k, \quad \sum_{k=1}^m \| P_k \| \| I_k \| \leq \lambda.
\]

A Banach space \( G \) is a \( T_p \)-space if and only if its dual is a \( T_p \)-space (one direction is trivial and the other follows by standard arguments using (a weak form of) the principle of local reflexivity). Obviously, every \( S_p \)-space is a \( T_p \)-space. The following partial converse is the crucial step of the proof of our main result.

1.4. THEOREM. For \( p = 1, 2 \) and \( \infty \) every \( T_p \)-space is an \( S_p \)-space.

Proof. The cases \( p = 1 \) and \( \infty \): Because of the duality of \( T_1 \)- and \( T_\infty \)-spaces as well as \( S_\infty \) and \( S_1 \)-spaces it suffices to prove that every \( T_\infty \)-space is \( S_\infty \). Let \( G \) be a \( T_\infty \)-space. In view of 1.1 we show that the \( l_2 \)-sum of \( G \) is \( S_\infty \). By assumption there is \( \lambda \geq 1 \) (not depending on \( n \)) and there are operators \( I_1, \ldots, I_m \in \mathcal{L}(l_p, G) \) and \( P_1, \ldots, P_m \in \mathcal{L}(G, l_p^{m+2}) \) (where \( m \) depends on \( n \)) such that

\[
\text{id}_{l_p} = \sum_{k=1}^m P_k I_k, \quad \sum_{k=1}^m \| P_k \| \| I_k \| \leq \lambda.
\]

Without loss of generality we may assume that \( \| P_k \| = \| I_k \| \) for all \( k \). Define

\[
V_k : l_p^m \rightarrow l_2(G), \quad \xi \mapsto (I_1 \xi, \ldots, I_m \xi, 0, \ldots),
\]

\[
U_k : l_2(G) \rightarrow l_\infty^m, \quad (x_k) \mapsto \sum_{k=1}^m P_k x_k.
\]

Then obviously \( \text{id}_{l_\infty} = U_n V_n \) and \( \| V_n \| \leq \sqrt{\lambda}, \| U_n \| \leq \sqrt{\lambda} \), which proves that \( l_2(G) \) contains all \( l_p \)-spaces uniformly complemented.

The case \( p = 2 \): Let \( G \) be a \( T_2 \)-space. In order to show that \( G \) is \( S_2 \) we use criterion 1.3. Let \( n \in N \). Then there are operators \( I_1, \ldots, I_m \in \mathcal{L}(l_2, G) \) and \( P_1, \ldots, P_m \in \mathcal{L}(G, l_p^{m+2}) \) such that

\[
\text{id}_{l_2} = \sum_{k=1}^m P_k I_k, \quad \sum_{k=1}^m \| P_k \| \| I_k \| \leq \lambda,
\]

where the constant \( \lambda \geq 1 \) does not depend on \( n \). Assume that for all \( k = 1, \ldots, m \)

\[
\int_{s_{n+1}} \| P_k I_k(x) \|_{l_2} \| I_k \| (dx) < \frac{1}{\lambda} \| P_k \| \| I_k \|.
\]

Then

\[
1 = \int_{s_{n+1}} \sum_{k=1}^m \| P_k I_k(x) \|_{l_2} \| I_k \| (dx) \leq \sum_{k=1}^m \int_{s_{n+1}} \| P_k I_k(x) \|_{l_2} \| I_k \| (dx) \leq \frac{1}{\lambda} \sum_{k=1}^m \| P_k \| \| I_k \| < 1,
\]

which proves that \( G \) is an \( S_2 \)-space.
holds isometrically. Hence, if as usual \([\mathcal{F}, F]\) stands for the Banach operator ideal of all integral operators then

\[ \mathcal{D}_p(E, F') = (E \otimes_{w_p} F') = (E \otimes_F F') = \mathcal{F}(E, F), \]

provided \(E\) is an \(\mathcal{L}_p\)-space or \(F\) an \(\mathcal{L}_p\)-space.

The following tensor product characterization of \(p\)-dominated operators is a slight extension of results proved in [1] and [4].

2.1. Proposition. For \(1 \leq p \leq \infty\), \(T \in \mathcal{L}(E, F)\) and a Banach space \(G\) consider the following four statements:

1. \(T \in \mathcal{D}_p(E, F)\).
2. \(\text{id}_G \otimes_T G \otimes_{w_p} E \to G \otimes_{w_p} F\) is continuous.
3. \(\text{id}_G \otimes_T G \otimes_{w_p} E \to G \otimes_{w_p} F\) is continuous.
4. \(\text{id}_G \otimes_T G \otimes_{w_p} E \to G \otimes_{w_p} F\) is continuous.

Then (1) \(\Rightarrow\) (3) \(\Rightarrow\) (4) and (1) \(\Rightarrow\) (2) \(\Rightarrow\) (4). Conversely, if \(G\) is a \(T_p\)-space then (1) \(\Rightarrow\) (4) are equivalent. In particular, if \(G\) is an infinite-dimensional \(\mathcal{L}_p\)-space then (1) is equivalent to

5. \(\text{id}_G \otimes_T G \otimes_{w_p} E \to G \otimes_{w_p} F\) is continuous.

We remark that this result in particular implies that Piets' infinite-dimensional Banach spaces \(F\) for which \(P \otimes_{w_p} F = P \otimes F\) holds isomorphically cannot be \(T_p\)-spaces for any \(1 \leq p < \infty\) (for the construction of these spaces see e.g. [9]).

Proof. Obviously, (3) \(\Rightarrow\) (4) and (2) \(\Rightarrow\) (4). The proof of (1) \(\Rightarrow\) (2) is easy. Indeed, for \(z = \sum_{i=1}^n x_i \otimes x_i \in G \otimes E\)

\[ \pi(\text{id}_G \otimes_T G \otimes_{w_p} E, F, G) = \sup \left\{ \left( \langle x, u \rangle \right|_{w_p}(u, E', F') \leq 1 \right\}, \]

which as \(\varepsilon\) and \(\pi\) form tensor norms in the sense of Grothendieck, in particular:

\[ \varepsilon \leq w_p, \quad \pi \leq \pi \text{ (see e.g. [2] and [3])}. \]

If \(E\) is an \(\mathcal{L}_p\)-space or \(F\) is an \(\mathcal{L}_p\)-space then isomorphically

\[ E \otimes_{w_p} F = E \otimes_{w_p} F, \quad E \otimes_{w_p} F = E \otimes_{w_p} F. \]

Moreover, for arbitrary Banach spaces \(E\) and \(F\)

\[ \mathcal{D}_p(E, F) = (E \otimes_{w_p} F'), \quad T \mapsto (x \otimes y \mapsto \langle y, Tk \rangle), \]

as usual \(\mathcal{D}^{(p)}(E, F)\) denotes the injective norm on the tensor product \(E \otimes F\) of two Banach spaces and \(\pi^{(p)}(E, F)\) the projective norm. Moreover, for \(1 \leq p \leq \infty\) we shall consider the norms

\[ w_p(z; E, F) = \inf \left\{ w_p(x_i) w_p(y_i) : \sum_{i=1}^n x_i \otimes x_i \right\}, \]

\[ w_p^{(p)}(z; E, F) = \sup \left\{ \langle x, u \rangle : w_p(u, E', F') \leq 1 \right\}, \]

which as \(\varepsilon\) and \(\pi\) form tensor norms in the sense of Grothendieck, in particular:

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\[ w_p(z; E, F) = \inf \left\{ w_p(x_i) w_p(y_i) : \sum_{i=1}^n x_i \otimes x_i \right\}, \]

\[ w_p^{(p)}(z; E, F) = \sup \left\{ \langle x, u \rangle : w_p(u, E', F') \leq 1 \right\}, \]

which as \(\varepsilon\) and \(\pi\) form tensor norms in the sense of Grothendieck, in particular:

\[ w_p \leq w_p, \quad w_p^{(p)} \leq \pi \text{ (see e.g. [2] and [3])}. \]

If \(E\) is an \(\mathcal{L}_p\)-space or \(F\) is an \(\mathcal{L}_p\)-space then isomorphically

\[ E \otimes_{w_p} F = E \otimes_{w_p} F, \quad E \otimes_{w_p} F = E \otimes_{w_p} F. \]
Hence with \( e_j := \text{sgn} \langle Tx_j, y \rangle \) by use of (4)

\[
\sum_{j=1}^{n} \langle Tx_j, y \rangle = \sum_{j=1}^{n} \langle I_k e_j \otimes Tx_j, P_k e_j \otimes y_j \rangle \\
\leq \sum_{j=1}^{n} \left| \langle I_k e_j \otimes Tx_j, P_k e_j \otimes y_j \rangle \right| \\
\leq \sum_{j=1}^{n} \left( \sum_{i=1}^{k} w_i \langle I_k e_i \otimes Tx_i, G F \rangle w_i, \sum_{j=1}^{n} P_k e_j \otimes y_j \rangle; G', F' \right) \\
\leq k \| I_k \| \| I_k \otimes T : G \otimes_w E \to G \otimes \| \| F \| \| w(x_i) \otimes w(y_j),
\]

which as desired proves that \( T \) is \( p \)-dominated. \( \blacksquare \)

As a corollary this proposition implies that an operator \( T \in \mathfrak{L}(E, F) \) is \( p \)-dominated if and only if there is an \( S_p \)-space \( G \) such that \( I_k \otimes T : G \otimes_w E \to G \otimes_w F \) is continuous. In [4] Jarchow asked for a certain converse of this statement: Is a Banach space \( G \) an \( S_p \)-space if every operator \( T \in \mathfrak{L}(E, F) \) such that \( I_k \otimes T : G \otimes_w E \to G \otimes_w F \) is continuous, is already \( p \)-dominated? The following theorem gives a positive answer for \( p = 1, 2 \) and \( \infty \). Moreover, for arbitrary \( 1 \leq p \leq \infty \) it is shown that the answer to Jarchow's problem is positive if one considers the class of all \( S_p \)-spaces instead of the class of all \( S_p \)-spaces.

**2.2. Theorem.** Let \( 1 \leq p \leq \infty \) and let \( G \) be a Banach space. Then the following are equivalent:

1. Every \( T \in \mathfrak{L}(E, F) \) such that \( I_k \otimes T : G \otimes_w E \to G \otimes_w F \) is continuous, is \( p \)-dominated.
2. There is a constant \( c > 0 \) such that for each \( T \in \mathfrak{L}(l_p^m, l_p^m) \)

\[
1(T) \leq c \| I_k \otimes T : G \otimes_w l_p^m \to G \otimes_w l_p^m \|
\]

\( I(T) \) is the integral (nuclear norm) of \( T \).
3. \( G \) is a \( T_p \)-space.

Moreover, for \( p = 1, 2 \) and \( \infty \) these statements are equivalent to

4. \( G \) is an \( S_p \)-space.

**Proof.** Obviously, (1) \( \Rightarrow \) (2) is a consequence of the closed graph theorem and the fact that \( \mathfrak{L}(l_p^m, l_p) = \mathfrak{L}(l_p, l_p) \) (see the preliminary remarks of this section). Moreover, (3) \( \Rightarrow \) (1) follows directly from 2.1, and (3) \( \Rightarrow \) (4) (if \( p = 1, 2 \) or \( \infty \)) was already stated in 1.4. Hence it remains to prove that (2) implies (3). For \( n \in \mathbb{N} \) consider the linear surjection

\[
c_n : (l_p^m \otimes G) \otimes (G \otimes l_p^m) \to l_p^m \otimes l_p^m, \quad (\eta \otimes g') \otimes (g \otimes \xi) \mapsto \langle g, g' \rangle \eta \otimes \xi,
\]

and define the quotient norm

\[
\delta(z; l_p^m, l_p) := \text{inf} \{ \pi(w; l_p^m \otimes G', G \otimes l_p^m) | c_n(w) = z \}
\]

on \( l_p^m \otimes l_p \).

**Step 1:** We prove that under the hypothesis of (2) for all \( n \)

\[
\delta(z; l_p^m, l_p) \leq c \delta(z; l_p^m, l_p)
\]

or dually: For \( u = \sum x_i \otimes y_i \in l_p^m \otimes l_p \)

\[
\pi(u; l_p^m, l_p) \leq c \delta'(u; l_p^m, l_p)
\]

where

\[
\delta'(u; l_p^m, l_p) := \sup \{ \langle u, v \rangle | \delta(u; l_p^m, l_p) \leq 1 \}
\]

Consider the operator \( T_u := \sum (x_i \otimes y_i \in l_p^m \otimes l_p) \)

\[
\| I_k \otimes T_u : G \otimes l_p^m \to G \otimes l_p^m \| \leq \delta(u; l_p^m, l_p).
\]

Indeed, for \( u = \sum x_i \otimes y_i \in G \otimes l_p^m \), choose

\[
w = \sum_j (x_j \otimes y_j \in G \otimes l_p^m)
\]

with \( w \in l_p^m \), \( G \), \( l_p^m \) \( \leq 1 \), \( \pi(I_k \otimes T_u(z); G, l_p^m) = \langle I_k \otimes T_u(z), w \rangle \), and check

\[
\pi(I_k \otimes T_u(z); G, l_p^m) = \langle \sum_j x_j \otimes T_u(z_j), \sum_j x_j \otimes y_j \rangle \]

\[
= \langle u, c_n((\sum_j x_j \otimes y_j) \otimes (\sum_j x_j \otimes y_j)) \rangle \]

\[
\leq \delta(u; l_p^m, l_p) \delta(c_n(...; l_p^m, l_p)) \leq \delta(u; l_p^m, l_p) \delta(z; l_p^m, l_p).
\]

Hence by (2)

\[
\pi(u; l_p^m, l_p) = I(T_u) \otimes l_p^m \leq \delta(u; l_p^m, l_p).
\]

**Step 2:** Let us now prove that \( G \) is a \( T_p \)-space. For all \( n \) the following diagram commutes:

\[
\begin{array}{ccc}
(l_p^m \otimes G') \otimes (G \otimes l_p^m) & \xrightarrow{\text{nat}} & l_p^m \otimes l_p^m \\
\| & & \|
\end{array}
\]

\[
\begin{array}{ccc}
\mathfrak{L}(l_p^m, G) \otimes l_p^m & \xrightarrow{\text{nat}} & \mathfrak{L}(l_p^m, l_p^m) \\
\| & & \|
\end{array}
\]

\[
S \otimes T = TS.
\]

Therefore by step 1 there is \( w \in l_p^m \otimes (G \otimes l_p^m) \) such that \( \pi(w; l_p^m \otimes (G \otimes l_p^m) \leq (1 + c_0)w \)

which by the definition of the \( \pi \)-norm implies the assertion: For every \( n \) there is \( m \) and there are operators \( I_1, \ldots, I_m \in \mathfrak{L}(l_p^m, G) \) and \( P_1, \ldots, P_m \in \mathfrak{L}(G, l_p^m) \) satisfying

\[
\pi(I_k \otimes T_k : G \otimes l_p^m \to G \otimes l_p^m) \leq (1 + c_0)w.
\]

\( \blacksquare \)
Since $\mathcal{S}_p \subseteq \mathcal{S}_p$ for $1 < p < \infty$ (see [8], 17.4.5) one gets as an immediate consequence that every $T_p$-space ($1 \leq p \leq \infty$) is either $S_1$ or $S_2$ or $S_\infty$.

We remark that the theorem holds in the complex sense also (for the definition of complex $S_p$- and $T_p$-spaces use the complex $l_p^n$'s instead of the real ones). Indeed, it can be checked easily that the proof of the equivalences $(1)\iff(2)\iff(3)$ does not depend on the scalar field. Moreover, for $p = 1, 2$ and $\infty$ every complex $T_p$-space contains all complex $l_p^n$ uniformly complemented. For $p = 1, \infty$ the proof is exactly that of 1.4, and for $p = 2$ the argument is as follows: Let $G$ be a complex $T_2$-space. Then it is immediate that $G$ considered as a real Banach space is $T_2$, and hence $S_2$ by 1.4. But, if a complex Banach space considered as a real one contains all real $l_2^n$ uniformly complemented, then it also contains all complex $l_2^n$ uniformly complemented (this was pointed out to me by Pisier; his argument is based on the facts that Theorem 5.11 of [10] is valid in the complex case also and moreover its converse is essentially true).

Finally, we state some simple reformulations of the theorem. It follows from 2.1 that for $G = l_p$ every $T \in \mathcal{L}(l_p, l_p)$ is integral ($i = p$-dominated) if and only if $\text{id}_G \otimes T : G \otimes l_p \rightarrow G \otimes l_p$ is continuous (see also [2], 5.2). By the theorem and the closed graph theorem the following partial converse holds: For $p = 1, 2, \infty$ the fact that every $T \in \mathcal{L}(l_p, l_p)$ is integral if $\text{id}_G \otimes T : G \otimes l_p \rightarrow G \otimes l_p$ is continuous, implies that $G$ is an $S_p$-space.

Purely formulated in terms of operators this means

2.3. COROLLARY. Let $p = 1, 2$ or $\infty$ and let $G$ be a Banach space. Then the following are equivalent:

1. Every $T \in \mathcal{L}(l_p, l_p)$ such that $T \in \mathcal{L}(G, l_p)$ is integral for every $S \in \mathcal{L}(G, l_p)$ is integral itself.

2. There is a constant $c > 0$ such that for all $T \in \mathcal{L}(l_p, l_p)$

$$I(T) \leq c \sup \{I(TS) : S \in \mathcal{L}(G, l_p) \}.$$ 

3. $G$ is an $S_p$-space.

The equivalence of (1) and (2) follows by standard arguments (using the closed graph theorem). The proof of (2)$\iff$(3) is a direct consequence of 2.2 and the commutativity of the following diagram:

$$\begin{array}{c}
G \otimes l_p \xrightarrow{\text{id}_G \otimes T} G \otimes l_p \\
\| \quad \| \\
\mathcal{L}(G, l_p) \xrightarrow{I} I(G, l_p)
\end{array}$$

We remark that $l_p$ in 2.3(1) can be replaced by any infinite-dimensional $\mathcal{L}_p$-space.

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