Contents of volume XCV, number 3

STUDIA MATHEMATICA

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, telex PL 816112

Correspondence concerning subscriptions and exchange should be addressed to

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, telex PL 816112

PWN–POLISH SCIENTIFIC PUBLISHERS

ISBN 83-01-09606-3 ISSN 0039-3223

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Weak bounds for the maximal function in weighted Orlicz spaces

by

RICHARD J. BAGBY (Las Cruces, N.Mex.)

Abstract. For Φ a Young function and $wE = \int_E w(x)\,dx$ a nontrivial doubling measure on E^n, it is shown that the Hardy–Littlewood maximal function Mf satisfies the weak-type inequality

$$w\{x:\ Mf(x) > \lambda\} \leqslant \int \Phi(c_1|f(x)|/\lambda)w(x)\,dx$$

if and only if the conjugate Young function $\Phi^{\sim}$ satisfies

$$\int_Q \Phi^{\sim}(\varepsilon wQ/|Q|w(x))w(x)\,dx \leqslant wQ$$

for all cubes Q, with some fixed $\varepsilon > 0$. When $\Phi^{\sim}$ is submultiplicative, it is shown that the last inequality can always be strengthened by replacing $\Phi^{\sim}$ by $(\Phi^{\sim})^{1+\delta}$ for some $\delta > 0$. In some weighted Orlicz classes (L^p spaces, for example), this can be used to prove the equivalence of the weak inequality for Mf and the strong inequality

$$\int \Phi(Mf(x))w(x)\,dx \leqslant \int \Phi(c_2|f(x)|)w(x)\,dx,$$

but we show the equivalence is generally false.

1. Introduction. In [6], B. Muckenhoupt found a remarkably simple characterization of the class A_p of weight functions w such that the Hardy–Littlewood maximal function is bounded in $L^p(E^n, w(x)\,dx)$. For $1 < p < \infty$, he proved that $w \in A_p$ if and only if there is a constant c such that

$$\Big(\int_Q w(x)\,dx\Big)\Big(\int_Q w(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant c|Q|^p$$

for all cubes Q in E^n; this inequality is known as the A_p *condition.* Routine arguments show that the A_p condition is equivalent to a weak-type bound

$$\int_{\{x:\, Mf(x) > \lambda\}} w(x)\,dx \leqslant c\lambda^{-p} \int |f(x)|^p\,dx$$

for arbitrary functions f and for all $\lambda > 0$, but in general an inequality of this type is strictly weaker than continuity in $L^p(E^n, w(x)\,dx)$. However, for the maximal function such an inequality does imply continuity in $L^q(E^n, w(x)\,dx)$ for all $q > p$. Muckenhoupt proved that whenever w satisfies the A_p condition, it must also satisfy the $A_{p-\varepsilon}$ condition for sufficiently small $\varepsilon > 0$, thereby proving that the maximal function is continuous in $L^p(E^n, w(x)\,dx)$.

Kerman and Torchinsky [4] then extended Muckenhoupt's results by considering weighted Orlicz spaces. They defined the class A_Φ of weight

---

1980 *Mathematics Subject Classification*: 42B25, 46E30.

functions to be those for which the maximal function satisfies a bound

$$\int \Phi(Mf(x))w(x)\,dx \leqslant \int \Phi(c|f(x)|)w(x)\,dx$$

and proved that $w \in A_\Phi$ if and only if $w \in A_p$ for $1/p$ the upper index of Φ.

Chung, Hunt, and Kurtz [2] found a refinement of Muckenhoupt's result by considering weighted Lorentz space norms. They found that weights satisfying the $A(p, 1)$ condition

$$\int_Q w(x)\,dx \Big/ \int_{Q\cap E} w(x)\,dx \leqslant c(|Q|/|Q\cap E|)^p$$

for arbitrary measurable sets E and cubes Q satisfy the same weak-type inequality as A_p weights provided f is a characteristic function, but not for general $f \in L^p(E^n, w\,dx)$. Consequently, $A(p, 1)$ is a proper subset of A_p which also contains A_q for all $q > p$.

Here we present a significant further refinement of Muckenhoupt's work by considering weak-type inequalities for the Hardy–Littlewood maximal function in weighted Orlicz spaces. We define the class B_Φ to be those weight functions w satisfying

$$\int_{\{x:\, Mf(x)>\lambda\}} w(x)\,dx \leqslant \int \Phi(c|f(x)|/\lambda)w(x)\,dx$$

for all $\lambda > 0$, $f \in L^\Phi(E^n, w(x)dx)$. We prove that B_Φ can be characterized by a natural analogue of the A_p condition, and prove a theorem generalizing Muckenhoupt's result that $w \in A_p$ implies $w \in A_{p-\varepsilon}$. However, this last result requires a hypothesis ruling out choices of Φ such as $\Phi(t) = ct^p(1+\log_+ t)^k$ with $k > 0$. In fact, our characterization of B_Φ shows that these are distinct proper subclasses of A_p containing A_q for all $q > p$.

Our characterization of B_Φ is simple enough that for a given weight w, we may by able to find a nearly optimal Φ for which $w \in B_\Phi$. By using the techniques of J. D. Parsons and the author [1], we can then obtain extremely delicate weighted bounds for the maximal function. For example, $w \in B_\Phi$ with $\Phi(t) = t^2(\log_+ t)^2$ does not imply $w \in A_2$, but it does imply that the Hardy–Littlewood maximal function is bounded from $L^2(\log L)^3(w\,dx)$ to $L^2_{\mathrm{loc}}(w\,dx)$. Such results were not previously obtainable.

2. Preliminaries. Let w be a fixed nonnegative, measurable function on E^n. For $A \subset E^n$ Lebesgue measurable, we shall write $|A|$ for the Lebesgue measure of A and

$$wA = \int_A w(x)\,dx.$$

We shall generally use the letter Q for a cube in E^n, by which we mean the product of n intervals $[a_i, a_i+s]$ with $0 < s < \infty$. For Q such a cube, we write Q^* for the cube having the same center as Q, but edges three times as long. We then say that w defines a *doubling measure* if there is a constant c such that

$$wQ^* \leqslant cwQ \quad \text{for all cubes } Q.$$

We use the version of the Hardy–Littlewood maximal function on E^n defined by

$$Mf(x) = \sup\{|Q|^{-1}\int_Q |f(y)|\,dy:\ x \in Q\}.$$

The simple result below allows us to bound the distribution function of Mf with respect to doubling Borel measures. The proof is quite well known.

(2.1) PROPOSITION. *For K an arbitrary compact subset of $\{x:\ Mf(x) > \lambda\}$, it is possible to choose cubes $Q_1, \ldots, Q_m$ with pairwise disjoint interiors such that*

$$\int_{Q_k} |f(x)|\,dx > \lambda|Q_k|, \quad \textit{for each } Q_k, \textit{ and}$$

$$K \subset \bigcup Q_k^*.$$

Proof. For each $y \in K$, there is a cube Q having x in its interior and $\int_Q |f(x)|\,dx > \lambda|Q|$; by compactness we can choose finitely many such cubes covering K. We may then order this finite collection so that their measures form a nonincreasing sequence.

Let Q_1 be the first cube in this sequence, and let Q_2 be the next cube with interior disjoint from that of Q_1. Having selected $Q_1, \ldots, Q_k$ with pairwise disjoint interiors, choose Q_{k+1} to be the next cube in the cover with this disjointness property.

If Q is not selected, then $|Q\cap Q_i| > 0$ for some selected Q_i with $|Q_i| \geqslant |Q|$. Consequently, $Q \subset Q_i^*$ and the proof is complete.

We shall call a function Φ on $[0, \infty)$ a *Young function* if Φ is continuous, convex, nondecreasing, and satisfies

$$\Phi(0) = 0, \qquad \Phi(t)/t \to \infty \quad \text{as } t \to \infty.$$

The *conjugate Young function* $\tilde{\Phi}$ may be defined by

$$\tilde{\Phi}(t) = \sup\{st - \Phi(s):\ s > 0\};$$

it is the minimal Young function which satisfies $st \leqslant \Phi(s) + \tilde{\Phi}(t)$.

For Φ a Young function, the *Orlicz space* $L^\Phi(d\mu)$ consists of all μ-measurable functions f such that $\Phi(\varepsilon|f|)$ is μ-integrable for some $\varepsilon > 0$. It can be normed by

$$\|f\|_\Phi = \inf\{\lambda > 0:\ \int \Phi(|f|/\lambda)\,d\mu \leqslant 1\}.$$

The Orlicz space version of Hölder's inequality is then

$$\int |fg|\,d\mu \leqslant 2\|f\|_\Phi \|g\|_{\tilde{\Phi}}.$$

Basic facts about Orlicz spaces may be found in Zygmund [7]; an extensive treatment is given by Krasnosel'skii and Rutickii [5].

Note that monotone convergence implies that

$$\int \Phi(|f|/\lambda)\,d\mu \leqslant 1 \quad \text{for } \lambda = \|f\|_\Phi;$$

inequality is possible unless $\Phi(|f|/\lambda)$ is μ-integrable for some $\lambda < \|f\|_\Phi$.

For functions on E^n which are locally in $L^\Phi(wdx)$, the norms

$$\|f\|_{\Phi,w,Q} = \inf\{\lambda > 0\colon \int_Q \Phi(|f|/\lambda)\,d\mu \leqslant wQ\}$$

are quite useful. For $0 < wQ < \infty$, this is the usual norm in $L^\Phi(d\mu)$ with $d\mu$ the restriction to Q of $(w/wQ)dx$.

We conclude this section with a technical lemma relating conjugate Young functions.

(2.2) LEMMA. *For Φ and $\Phi^\sim$ conjugate Young functions, there is a continuous, monotone function g on $[0, \infty)$ such that*

$$\Phi(g(t)) \leqslant tg(t) \leqslant \Phi(2g(t)),$$

$$2\Phi^\sim(t/2) \leqslant tg(t) \leqslant \Phi^\sim(2t).$$

Proof. If we define

$$h(t) = \sup\{s\colon \Phi(s) \leqslant st\},$$

then h is monotone and satisfies $\Phi(h(t)) = th(t)$. Moreover,

$$\Phi^\sim(t) = \sup\{st - \Phi(s)\colon \Phi(s) \leqslant st\} \leqslant th(t),$$

while $th(t) = 2th(t) - \Phi(h(t)) \leqslant \Phi^\sim(2t)$. However, h has a discontinuity if $\Phi(t)/t$ is a nonzero constant near $t = 0$.

Let us define

$$\Phi_0(t) = \begin{cases} \int_0^t (1+s)\Phi'(s)\,ds, & t \leqslant 1, \\ \Phi_0(1) + 2\int_1^t \Phi'(s)\,ds, & t > 1. \end{cases}$$

Then Φ_0 is a Young function with $\Phi(t) \leqslant \Phi_0(t) \leqslant 2\Phi(t)$. Consequently, $2\Phi^\sim(t/2) \leqslant \Phi_0^\sim(t) \leqslant \Phi^\sim(t)$. We then take

$$g(t) = \sup\{s\colon \Phi_0(s) \leqslant st\}.$$

3. The weight class B_Φ. Throughout this section, we assume that w is a nonnegative measurable function on E^n and Φ is a Young function.

We call w a *nontrivial weight* if there is at least one cube Q with $0 < wQ < \infty$. We then define B_Φ to consist of the nontrivial weights w for which there is a constant c with

$$w\{x \in E^n\colon Mf(x) > \lambda\} \leqslant \int \Phi(c|f(x)|/\lambda)w(x)\,dx \tag{3.1}$$

for all $\lambda > 0$ and all $f \in L^\Phi(wdx)$. If c_0 is any value of c for which (3.1) is valid, we say $w \in B_\Phi$ with constant c_0.

Note that if f is the characteristic function of an arbitrary cube Q, then $Mf \geqslant 2^{-n}$ on Q^*. Consequently, whenever $w \in B_\Phi$ we must have $wQ^* \leqslant \Phi(2^n c)wQ$. Thus w must be a doubling measure which is finite and positive on all cubes.

(3.2) LEMMA. *If $w \in B_\Phi$ with constant c_0, then for every $\varepsilon > 0$ we have $(w+\varepsilon) \in B_\Phi$ with constant $2\cdot 3^n c_0$.*

Proof. First note that $\Phi(c_0) \geqslant 1$ since the maximal function of the characteristic function of any cube Q is 1 on Q.

Next note that

$$|\{x\colon Mf(x) > \lambda\}| \leqslant 2\cdot 3^n \int_{\{x:|f(x)|>\lambda/2\}} |f/\lambda|\,dx.$$

This is quite standard; it can be proved by using (2.1) and the observation

$$\{x\colon Mf(x) > \lambda\} \subset \{x\colon Mg(x) > \lambda/2\},$$

where g is the restriction of f to $\{x\colon |f(x)| > \lambda/2\}$. On this set, we have

$$2\cdot 3^n|f(x)|/\lambda \leqslant (2\cdot 3^n|f(x)|/\lambda)\Phi(c_0) \leqslant \Phi(2\cdot 3^n c_0|f(x)|/\lambda).$$

Consequently,

$$(w+\varepsilon)\{x\colon Mf(x) > \lambda\} \leqslant \int \Phi(c_0|f|/\lambda)w\,dx + \varepsilon\int \Phi(2\cdot 3^n c_0|f(x)|/\lambda)\,dx$$
$$\leqslant \int \Phi(2\cdot 3^n c_0|f|/\lambda)(w+\varepsilon)\,dx.$$

Now we are ready to characterize the weight class B_Φ by means of a condition analagous to the A_p condition in the form used by Chung, Hunt, and Kurtz [2].

(3.3) THEOREM. *For w a nontrivial weight, $w \in B_\Phi$ if and only if w defines a doubling measure and there is a constant c such that*

$$\|1/w\|_{\Phi^\sim,w,Q} \leqslant c|Q|/wQ$$

for all cubes Q.

Proof. If Q is any cube such that $\int_Q |f(x)|\,dx > \lambda|Q|$, then

$$\frac{|Q|}{wQ} < \int_Q \frac{1}{w(x)}\cdot\frac{|f(x)|}{\lambda}\cdot\frac{w(x)\,dx}{wQ} \leqslant 2\|1/w\|_{\Phi^\sim,w,Q}\|f/\lambda\|_{\Phi,w,Q}.$$

If also $\|1/w\|_{\Phi^\sim,w,Q} \leqslant c_1|Q|/wQ$, then $\|f/\lambda\|_{\Phi,w,Q} > 1/2c_1$ so that

$$\int_Q \Phi(2c_1|f|/\lambda)w\,dx > wQ.$$

Consequently, for any compact $K \subset \{x\colon Mf(x) > \lambda\}$, we can apply (2.1) to choose a sequence of cubes Q_k with pairwise disjoint interiors such that both $wK \leqslant \sum wQ_k^*$ and

$$wQ_k \leqslant \int_{Q_k} \Phi(2c_1|\lambda)w\,dx.$$

If also w defines a doubling measure, then

$$wK \leqslant \sum c_2 wQ_k \leqslant \sum c_2 \int_{Q_k} \Phi(2c_1|f|/\lambda)w\,dx \leqslant \int \Phi(2c_1c_2|f|/\lambda)w\,dx.$$

Taking the supremum over all compact subsets of $\{x\colon Mf(x) > \lambda\}$ then gives $w \in B_\Phi$ with constant $2c_1c_2$.

Now suppose that $w \in B_\Phi$ with constant c_0. With the additional assumption that $1/w$ is bounded, we prove that

$$\int_Q \Phi^\sim(1/\lambda w)w\,dx \leqslant wQ \quad \text{for } \lambda = cc_0|Q|/wQ.$$

We then use (3.2) to remove the extra assumption.

For g the function given by (2.2), continuity and monotonicity allow us to choose $s > 0$ (depending on Q) such that

$$\int_Q g(1/sw)\,dx = swQ.$$

Since $tg(t) \geqslant 2\Phi^\sim(t/2)$, we then have

$$2\int_Q \Phi^\sim(1/2sw)sw\,dx \leqslant \int_Q g(1/sw)\,dx \leqslant swQ, \quad \text{or}$$

$$\int_Q \Phi^\sim(1/2sw)w\,dx \leqslant wQ/2.$$

For $f(x) = g(1/sw(x))$ on Q and 0 elsewhere, we have $Mf(x) \geqslant swQ/|Q|$ on Q. Thus for $w \in B_\Phi$ with constant c_0,

$$wQ \leqslant \int_Q \Phi\big(c_0|Q|g(1/sw)/swQ\big)w\,dx.$$

But $\int_Q \Phi\big(g(1/sw)\big)w\,dx \leqslant (1/s)\int_Q g(1/sw)\,dx = wQ$, so we must have $1 \leqslant c_0|Q|/swQ$. Thus for $\lambda = 2c_0|Q|/wQ$ we have

$$\int_Q \Phi^\sim(1/\lambda w)w\,dx \leqslant \int_Q \Phi^\sim(1/2sw)w\,dx \leqslant wQ.$$

For general $w \in B_\Phi$ with constant c_0, (3.2) guarantees that $(w+\varepsilon) \in B_\Phi$ with constant $2\cdot 3^n c_0$ for each $\varepsilon > 0$, and of course $1/(w+\varepsilon)$ is bounded. Consequently, for

$$\lambda = 2^2\cdot 3^n c_0|Q|/wQ > 2^2\cdot 3^n c_0|Q|/(wQ+\varepsilon|Q|),$$

we have

$$\int_Q \Phi^\sim\big(1/\lambda(w+\varepsilon)\big)(w+\varepsilon)\,dx \leqslant wQ+\varepsilon|Q|.$$

Letting $\varepsilon \to 0$ gives $\int_Q \Phi^\sim(1/\lambda w)w\,dx \leqslant wQ$ by monotone convergence.

Now we consider inclusion relations among B_Φ classes. As we noted in the proof of (3.2), $w\{x\colon Mf(x) > \lambda\}$ is controlled by the restriction of f to $\{x\colon |f(x)| > \lambda/2\}$. Consequently, $w \in B_\Phi$ means Φ has sufficiently rapid growth at infinity; we always have $B_\Phi \subset B_\Psi$ if $\Phi(t) \leqslant \Psi(ct)$ for all sufficiently large t. On the other hand, Muckenhoupt [6] proved that for $w \in A_p$, we must have $w \in A_q$ for some $q < p$, which reverses the natural inclusion. In the remainder of this section, we develop a generalization of Muckenhoupt's result by adapting the proof given by Chung, Hunt, and Kurtz [2] to a restricted class of Orlicz spaces.

(3.4) LEMMA. *If $w \in B_\Phi$ with constant c_0, then for each cube Q we have*

$$wQ \leqslant \Phi(2c_0)w\{x \in Q\colon w(x) < 2wQ/|Q|\}.$$

Proof. Let $E = \{x \in Q\colon w(x) < 2wQ/|Q|\}$ and $F = Q - E$. Then $2|F|wQ/|Q| \leqslant \int_F w\,dx \leqslant wQ$, so that $|F| \leqslant |Q|/2$ and $|E| \geqslant |Q|/2$. Take $f = \chi_E$; then $Mf \geqslant 1/2$ on Q. Hence

$$wQ \leqslant w\{x\colon Mf(x) \geqslant 1/2\} \leqslant \int \Phi(2c_0|f|)w\,dx = \Phi(2c_0)wE.$$

(3.5) LEMMA. *Suppose $\Phi^\sim(st) \leqslant A\Phi^\sim(s)\Phi^\sim(t)$ for all $s, t > 0$, and $w \in B_\Phi$ with constant c_0. Set $E(\lambda, t, Q) = \{x \in Q\colon \Phi^\sim(1/\lambda w) > t\}$. Then there are constants c_1 and ε (depending only on c_0 and A) such that*

$$\int_{E(\lambda,t,Q)} \Phi^\sim(1/\lambda w)w\,dx \leqslant c_1 twE(\lambda, \varepsilon t, Q)$$

whenever

$$\int_Q \Phi^\sim(1/\lambda w)w\,dx \leqslant twQ.$$

Proof. Since w is a doubling measure, the Calderón–Zygmund decomposition shows that we can select nonoverlapping subcubes Q_i of Q such that

$$twQ_i < \int_{Q_i} \Phi^\sim(1/\lambda w)w\,dx \leqslant \Phi(2^n c_0)twQ_i$$

for each i and $\Phi^\sim(1/\lambda w) \leqslant t$ a.e. in $Q - \bigcup Q_i$. Hence

$$\int_{E(t,\lambda,Q)} \Phi^\sim(1/\lambda w)w\,dx \leqslant \sum \Phi(2^n c_0)twQ_i$$

and

$$wQ_i \leqslant \Phi(2c_0)w\{x \in Q_i\colon w < 2wQ_i/|Q_i|\}$$

by (3.4). We complete the proof by finding $\varepsilon > 0$ such that

$$\{x \in Q_i\colon w < 2wQ_i/|Q_i|\} \subset E(\lambda, \varepsilon t, Q_i),$$

or equivalently, $\Phi^\sim(|Q_i|/2\lambda wQ_i) \geqslant \varepsilon t$.

By (3.3), we can choose a constant c with

$$\int_{Q_i} \Phi^{\sim}(wQ_i/c|Q_i|w)w\,dx \leqslant wQ_i.$$

Since

$$twQ_i \leqslant \int_{Q_i} \Phi^{\sim}(1/\lambda w)w\,dx \leqslant A\int_{Q_i} \Phi^{\sim}(c|Q_i|/\lambda wQ_i)\Phi^{\sim}(wQ_i/c|Q_i|w)w\,dx$$
$$\leqslant A\Phi^{\sim}(c|Q_i|/\lambda wQ_i)wQ_i \leqslant A^2\Phi^{\sim}(2c)\Phi^{\sim}(|Q_i|/2\lambda wQ_i)wQ_i,$$

we may take $1/\varepsilon = A^2\Phi^{\sim}(2c)$.

(3.6) THEOREM. *Suppose $w \in B_\Phi$ with constant c_0. If $\Phi^{\sim}(st) \leqslant A\Phi^{\sim}(s)\Phi^{\sim}(t)$ for all $s, t > 0$, then there is a $\delta > 0$ (depending only on c_0 and A), such that $w \in B_\Psi$ for $\Psi^{\sim}(t) = \Phi^{\sim}(t)^{1+\delta}$.*

Proof. By (3.3) it suffices to find a constant c for which

$$\int_Q \Psi^{\sim}(1/\lambda w)w\,dx \leqslant wQ \quad \text{when } \lambda = c|Q|/wQ;$$

as in the proof of (3.3) it suffices to prove this under the additional assumption that $1/w$ is bounded.

Let us define a measure μ by $d\mu = \Phi^{\sim}(1/\lambda w)w\,dx$; then

$$\int_Q \Psi^{\sim}(1/\lambda w)w\,dx = \int_Q \Phi^{\sim}(1/\lambda w)^{\delta}\,d\mu = \delta\int_0^\infty t^{\delta-1}\mu E(\lambda, t, Q)\,dt$$

where $E(\lambda, t, Q)$ is the set defined in (3.5). For $t \leqslant 1$ we may bound

$$\mu E(\lambda, t, Q) \leqslant \mu Q = \int_Q \Phi^{\sim}(1/\lambda w)w\,dx,$$

while for $t > 1$ and $\lambda \geqslant \|1/w\|_{\Phi^{\sim},w,Q}$ we have $\int_Q \Phi^{\sim}(1/\lambda w)w\,dx < twQ$, and hence $\mu E(\lambda, t, Q) \leqslant c_1 tw(\lambda, \varepsilon t, Q)$ by (3.5). Thus

$$\int_Q \Psi^{\sim}(1/\lambda w)w\,dx \leqslant \mu Q + \delta\int_1^\infty t^{\delta-1}\mu E(\lambda, t, Q)\,dt \leqslant \mu Q + c_1\delta\int_1^\infty t^{\delta}wE(\lambda, \varepsilon t, Q)\,dt$$
$$\leqslant \mu Q + c_1\delta\varepsilon^{-1-\delta}\int_0^\infty t^{\delta}wE(\lambda, t, Q)\,dt$$
$$= \mu Q + c_1\delta\varepsilon^{-1-\delta}(1+\delta)^{-1}\int_Q \Phi^{\sim}(1/\lambda w)^{1+\delta}w\,dx.$$

Choosing δ small enough that $c_1\delta\varepsilon^{-1-\delta}(1+\delta)^{-1} \leqslant 1/2$ gives

$$\int_Q \Psi^{\sim}(1/\lambda w)w\,dx \leqslant 2\mu Q = 2\int_Q \Phi^{\sim}(1/\lambda w)w\,dx \leqslant wQ \quad \text{for } \lambda = 2\|1/w\|_{\Phi^{\sim},w,Q}.$$

The condition $\Phi^{\sim}(st) \leqslant A\Phi^{\sim}(s)\Phi^{\sim}(t)$ excludes quite a few of the standard Young functions. If $\Phi(t) \approx t^p(\log t)^{\alpha p}$ for large t, then $\Phi^{\sim}(t) \approx t^q(\log t)^{-\alpha q}$ for large t, where p and q are Hölder conjugates. Consequently, (3.6) applies when $\alpha \leqslant 0$ but not for $\alpha > 0$. In the next section we show that the conclusion of (3.6) fails when $\Phi(t) = t^2(1+\log_+ t)^2$, so that the condition on $\Phi^{\sim}$ cannot be eliminated from the hypotheses of (3.6).

In calculations using a given weight, the best possible weighted estimates for Mf would be obtained by finding an optimal Young function Φ for which $w \in B_\Phi$. In such cases, (3.6) says that its conjugate $\Phi^{\sim}$ should not satisfy the condition $\Phi^{\sim}(st) \leqslant A\Phi^{\sim}(s)\Phi^{\sim}(t)$.

4. Applications and examples. By using (5.3) of [1], we can obtain a variety of weighted Orlicz space bounds for the Hardy–Littlewood maximal function when the weight is in B_Φ. We summarize them in (4.1) below.

(4.1) THEOREM. *For $w \in B_\Phi$, the Hardy–Littlewood maximal function is bounded from $L^\Theta(w\,dx)$ to $L^\Psi(w\,dx)$ provided*

$$\int_0^t \Psi'(s)\Phi(t/s)\,ds \leqslant \Theta(ct)$$

for some constant c.

While there are cases (including $\Phi(t) = t^p$, $1 < p < \infty$) where (3.6) and (4.1) can be used to prove that Mf is bounded in $L^\Phi(w\,dx)$ for all $w \in B_\Phi$, this conclusion is generally false. We give two simple examples where B_Φ contains weights not in A_Φ.

(4.2) EXAMPLE. Take

$$\Phi(t) = \begin{cases} t^2 & \text{for } t \leqslant 1, \\ t^3 & \text{for } t > 1. \end{cases}$$

Then $L^\Phi = L^2 \cap L^3$, so that $L^{\Phi^{\sim}} = L^2 + L^3$. It follows easily that $B_\Phi = A_3$ while $A_\Phi = A_2$.

(4.3) EXAMPLE. Let

$$\Phi^{\sim}(t) = \begin{cases} \left(\dfrac{t}{2+t}\right)^2, & 0 \leqslant t \leqslant 1, \\ \left(\dfrac{t}{3+\log t}\right)^2, & t > 1. \end{cases}$$

One may verify that $\Phi(t) \approx (t\log t)^2$ for large t, so that $w \in B_\Phi$ means

$$\int_{\{Mf(x)>\lambda\}} w(x)\,dx \leqslant c^2\int\left(\frac{|f(x)|}{\lambda}\right)^2\left(1+\log_+\frac{c|f(x)|}{\lambda}\right)^2 w(x)\,dx.$$

Let us take $n = 1$ and $w(x) = |x|$; then to prove $w \in B_\Phi$ it is enough to check that

$$\int_0^a \Phi^{\sim}(1/\lambda x)x\,dx \leqslant a^2/2 \quad \text{for } \lambda = c/a,\ a > 0.$$

Choosing $c = 1$ gives

$$\int_0^a \Phi^{\sim}(a/x)x\,dx = \int_0^a \frac{a^2\,dx}{x(3+\log a/x)^2} = a^2/3.$$

On the other hand, for any δ, $\varepsilon > 0$ we have

$$\int_0^a \Phi^{\sim}(\varepsilon/x)^{1+\delta}x\,dx = \infty.$$

The characterization of A_Φ given by Kerman and Torchinsky [4] shows that $A_\Phi = A_2$; in this case $w \in A_p$ for all $p > 2$ but w is not in A_2.

Let us now apply (4.1) with this choice of Φ. Take $\Psi(t) = t^2 - 1$ for $t > 1$ and 0 otherwise. Then L^Ψ consists of the functions whose restriction to every set of finite measure is in L^2. Since

$$\int_1^t (2s)(t/s)^2(1+\log t/s)^2\,ds = (2/3)t^2(1+\log t)^3,$$

we see that the Hardy–Littlewood maximal function is bounded from $L^2(\log L)^3(w\,dx)$ to $L^2_{\rm loc}(w\,dx)$ for all $w \in B_\Phi$.

For the case $w(x) = |x|$ on E^1, slightly better weighted bounds for Mf can be obtained by using the fact that w is in the weight class $A(2, 1)$ of Chung, Hunt, and Kurtz [2]. However, if we modify w by redefining $w(x) = |x|/(\log 2/x)^\varepsilon$ for $|x| < 1$, then $w \notin A(2, 1)$ for $\varepsilon > 0$ but $w \in B_\Phi$ for $\varepsilon > 1$.

References

[1] R. J. Bagby and J. D. Parsons, *Orlicz spaces and rearranged maximal functions*, Math. Nachr. 132 (1987), 15–27.

[2] H.-M. Chung, R. A. Hunt, and D. S. Kurtz, *The Hardy–Littlewood maximal function on $L(p, q)$ spaces with weights*, Indiana Univ. Math. J. 31(1982), 109–120.

[3] M. de Guzmán, *Differentiation of Integrals in R^n*, Lecture Notes in Math. 481, Springer, 1975.

[4] R. A. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. 71 (1981/82), 277–284.

[5] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen 1961.

[6] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[7] A. Zygmund, *Trigonometric Series*, Vol. I, 2nd ed., Cambridge Univ. Press, Cambridge 1959.

DEPARTMENT OF MATHEMATICAL SCIENCES
NEW MEXICO STATE UNIVERSITY
Las Cruces, New Mexico 88003, U.S.A.

Received March 16, 1988 (2422)

STUDIA MATHEMATICA, T. XCV (1990)

H^p spaces over open subsets of R^n

by

AKIHIKO MIYACHI (Tokyo)

Abstract. Part of the theory of H^p spaces over R^n, originated by C. Fefferman and E. M. Stein [4], is generalized to the case of arbitrary open subsets of R^n. The following subjects are treated: (1) Definition of $H^p(\Omega)$, where Ω is an open subset of R^n, by means of maximal functions; (2) Atomic decomposition for $H^p(\Omega)$; (3) Identification of the duals of $H^p(\Omega)$ with certain function spaces over Ω; (4) The complex method of interpolation for $H^p(\Omega)$ and $L^p(\Omega)$; (5) Extension of a distribution in $H^p(\Omega)$ to a distribution in $H^p(R^n)$. All the results are given in the situation that R^n has a parabolic metric.

1. Introduction. In this paper, we introduce H^p spaces over arbitrary open subsets of R^n by means of certain maximal functions and show that they have some properties similar to the H^p spaces over R^n (for the H^p spaces over R^n, see Calderón–Torchinsky [1], [2] or Torchinsky's book [10; Chapt. XIV]).

We briefly review our results.

Let φ be a function in $C_0^\infty(R^n)$ such that $\operatorname{supp}\varphi \subset \{x \in R^n \mid |x| < 1\}$ (if $x \in R^n$, then $|x|$ denotes the usual Euclidean norm of x) and $\int \varphi(x)\,dx = 1$. For $t > 0$, we define $(\varphi)_t$ by $(\varphi)_t(x) = t^{-n}\varphi(t^{-1}x)$ (we shall modify this definition afterwards; see the next to the last paragraph in this section). Let Ω be an open subset of R^n. For $f \in \mathcal{D}'(\Omega)$, we define the *radial maximal function* $f^+_{\varphi,\Omega}(x)$, $x \in \Omega$, by

$$f^+_{\varphi,\Omega}(x) = \sup\{|\langle f, (\varphi)_t(x-\cdot)\rangle| \mid 0 < t < \operatorname{dis}(x, \Omega^c)\},$$

where Ω^c denotes the complement of Ω (throughout this paper, $\mathcal{D}'(\Omega)$ denotes the set of distributions on Ω and $\langle f, \psi\rangle$, where $f \in \mathcal{D}'(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, means $f(\psi)$; we use the same notation $\langle f, \psi\rangle$ if f is a distribution with compact support and ψ is a smooth function on R^n). For p with $0 < p \leqslant 1$, we define $H^p(\Omega)$ as the set of those $f \in \mathcal{D}'(\Omega)$ for which $f^+_{\varphi,\Omega}$ belongs to $L^p(\Omega)$. We consider $H^p(\Omega)$ a quasinormed linear space by defining the quasinorm of $f \in H^p(\Omega)$ to be equal to the $L^p(\Omega)$-norm of $f^+_{\varphi,\Omega}$. (By a *quasinorm* we mean a function σ on a linear space X which has the following properties: (i) $\sigma(x) > 0$ if $x \neq 0$ and $\sigma(0) = 0$; (ii) $\sigma(\lambda x) = |\lambda|\sigma(x)$ for all scalars λ and all $x \in X$; (iii) there exists a positive constant k such that $\sigma(x+y) \leqslant k(\sigma(x)+\sigma(y))$ for all $x, y \in X$.) Then the maximal inequality given by the author [8] shows that the above definition

1980 *Mathematics Subject Classification*: Primary 42B30; Secondary 46F05.
Key words and phrases: H^p spaces, maximal function, atomic decomposition, Lipschitz spaces, BMO, complex method of interpolation, parabolic metric.