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On a question of A. Wilansky in normed algebras

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Abstract. A. Wilansky asked whether there is a normed Q-algebra which is not inverse closed in its completion. The answer to this question is negative. In fact, we show that every p-normed Q-algebra $(0 < p \leqslant 1)$ is inverse closed in its completion.

By a *p-normed algebra* $(0 < p \leqslant 1)$ we mean an algebra A over $\boldsymbol{K}$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$) endowed with a mapping $\|\cdot\|$ from A to $\boldsymbol{R}_+$ satisfying:

(i) $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda|^p \|x\|$, for all x in A and $\lambda \in \boldsymbol{K}$.
(iii) $\|x+y\| \leqslant \|x\| + \|y\|$, for all x, y in A.
(iv) $\|xy\| \leqslant \|x\| \cdot \|y\|$ for all x, y in A.

A p-normed algebra A is called a *Q-algebra* if its group $G(A)$ of invertible elements is open.

In the following A will always be a unital p-normed algebra and $\hat{A}$ its completion. We denote by $m(A)$ and $m(\hat{A})$ the sets of all nontrivial continuous characters of A and $\hat{A}$ respectively.

Recall that A is *inverse closed* in $\hat{A}$ if, whenever $x \in A$ has an inverse x^{-1} in $\hat{A}$, then x^{-1} is in A.

Let us now state the first proposition, which characterizes the commutative p-normed Q-algebras.

PROPOSITION 1. *If A is commutative, the following assertions are equivalent*:

1° *Every x in A such that $\chi(x) \neq 0$ for every $\chi \in m(A)$ is invertible.*
2° $\operatorname{sp} x = \{\chi(x) \mid \chi \in m(A)\}$ *for every x in A.*
3° *A is inverse closed in $\hat{A}$.*
4° *A is a Q-algebra.*

Proof. 1°⇒2°. Let λ be an element of $\operatorname{sp} x$. Then $\lambda e - x$ is not invertible. By 1°, there exists χ in $m(A)$ such that $\chi(\lambda e - x) = 0$. Hence $\lambda = \chi(x)$. Thus $\operatorname{sp} x \subset \{\chi(x) \mid \chi \in m(A)\}$. But the inverse inclusion is always satisfied, whence 2°.

2°⇒3°. If $\operatorname{sp} x = \{\chi(x) \mid \chi \in m(A)\}$, then $\operatorname{sp} x = \{\chi(x) \mid \chi \in m(\hat{A})\}$ for every element of $m(A)$ can be extended to an element of $m(\hat{A})$. Since $\hat{A}$ is a complete p-normed algebra, $\operatorname{sp}_{\hat{A}} x = \{\chi(x) \mid \chi \in m(\hat{A})\}$ ([3]), so $\operatorname{sp}_A x = \operatorname{sp}_{\hat{A}} x$ for every x in A. This implies 3°.

$3^\circ \Rightarrow 4^\circ$. This is trivial, since $G(A) = G(\hat{A}) \cap A$ and $\hat{A}$ is a Q-algebra ([3]).

$4^\circ \Rightarrow 1^\circ$. Let x be an element of A such that $x \notin \operatorname{Ker}\chi$, for every χ in $m(A)$. Since A is a Q-algebra and the p-normed algebras satisfy the Gelfand–Mazur theorem, the only maximal ideals of A are $\operatorname{Ker}\chi$, $\chi \in m(A)$. Thus x belongs to no maximal ideal of A. Hence x is invertible.

Remark. The four equivalent assertions imply that the Jacobson radical of A is given by $\operatorname{Rad} A = \bigcap_{\chi \in m(A)} \operatorname{Ker}\chi$. But the converse is false. Indeed, if we consider the normed algebra A consisting of all polynomials provided with the norm $\|P\|_\infty = \sup_{x \in [0,1]} |P(x)|$, it is easy to verify that A is not a Q-algebra but $\operatorname{Rad} A = \{0\} = \bigcap_{\chi \in m(A)} \operatorname{Ker}\chi$.

We now obtain another characterization in the noncommutative case.

PROPOSITION 2. *The following conditions are equivalent*:

1° *A is inverse closed in its completion $\hat{A}$.*

2° *For every subset of A which contains no invertible element in A, its closure in $\hat{A}$ contains no invertible element in $\hat{A}$.*

3° *The closure in $\hat{A}$ of a (left or right) maximal ideal contains no invertible element in $\hat{A}$.*

4° $\operatorname{sp}_A x = \operatorname{sp}_{\hat{A}} x$, *for every x in A.*

5° *A is a Q-algebra.*

Proof. $1^\circ \Rightarrow 2^\circ$. Let B be a subset of A which contains no invertible element in A. Suppose that the closure $\bar{B}$ of B in $\hat{A}$ contains an element x invertible in $\hat{A}$. Since $\hat{A}$ is a Q-algebra ([3]), there exists a neighborhood U of x such that every element of U is invertible in $\hat{A}$. Since $x \in \bar{B}$, there exists a sequence $(x_n)_n$ in B which converges to x in $\hat{A}$. Hence $x_n \in U$ for large n. Then x_n is invertible in $\hat{A}$ and hence in A, contrary to the hypothesis.

$2^\circ \Rightarrow 3^\circ$. Evident.

$3^\circ \Rightarrow 4^\circ$. The inclusion $\operatorname{sp}_{\hat{A}} x \subset \operatorname{sp}_A x$ is always satisfied. Conversely, if $\lambda \in \operatorname{sp}_A x$, i.e. $\lambda e - x$ is not invertible, then there exists a (left or right) maximal ideal M such that $\lambda e - x \in M \subset \bar{M}$, where $\bar{M}$ is the closure of M in $\hat{A}$. Since A is dense in $\hat{A}$, $\bar{M}$ is a (left or right) ideal in $\hat{A}$ and by 3°, $\bar{M} \neq \hat{A}$. Thus $\lambda e - x$ is not invertible in $\hat{A}$, i.e. $\lambda \in \operatorname{sp}_{\hat{A}} x$.

$4^\circ \Rightarrow 5^\circ$. $G(A) = G(\hat{A}) \cap A$ and $G(\hat{A})$ is open in $\hat{A}$, whence the result.

$5^\circ \Rightarrow 1^\circ$. Let x be an element of A. If $x \notin G(A)$, there exists a maximal ideal M in A such that $x \in M$. Then $x \in \bar{M}$, where $\bar{M}$ is the closure of M in $\hat{A}$. It is a proper ideal in $\hat{A}$. Thus $x \notin G(\hat{A})$.

Remark. The last proposition shows that the answer to Wilansky's question is negative even for p-normed algebras. We point out that a negative answer for normed algebras has been given by Alberto Arosio ([1]).

We now give some examples.

1° Let $A = \boldsymbol{C}^{(N)} = \{(x_n)_n \subset \boldsymbol{C} \mid \exists N \in N\colon x_n = 0 \text{ for } n \geqslant N\}$ provided with the pointwise operations and the p-norm defined by $\|x\| = \sum_{n=0}^{\infty} |x_n|^p$, where $x = (x_n)_n \in A$ and p is a fixed number such that $0 < p < 1$. Then $(A, \|\cdot\|)$ is a p-normed (not normed) algebra. It is not complete. Its completion is

$$\hat{A} = \{(x_n)_n \subset \boldsymbol{C} \mid \sum_{n=0}^{\infty} |x_n| < \infty\}.$$

Consider $A^\# = A \times \boldsymbol{C}$, the unitization of A, with the p-norm $\|(x, \lambda)\| = \|x\| + |\lambda|^p$. Then $A^\#$ is a p-normed Q-algebra which is not normed.

2° Consider the same algebra A with the convolution product:

$$(x_n)_n * (y_m)_m = \Big(\sum_{n+m=k} x_n y_m\Big)_k.$$

The algebra A is not inverse closed in its completion $\hat{A}$. Indeed, the element $x = (2, 1, 0, 0, \ldots)$ of A is invertible in $\hat{A}$. Its inverse is $x^{-1} = (\frac{1}{2}, -\frac{1}{4}, \ldots \ldots, (-1)^{n-1}/2^n, \ldots)$, which is not in A. Thus A is not a Q-algebra.

References

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