$$= 4 \sup_{\|x\|_E = 1} \sum_{j=1}^{N} a_j |\langle x, (M_j^{*-1}(2/k_i))_{i=1}^n \rangle|$$

$$\leqslant 8 \sup_{\|x\|_E = 1} \sum_{j=1}^{N} a_j \|x\|_{M_j} \|(M_j^{*-1}(2/k_i))_{i=1}^n\|_{M_j^*} \leqslant 32.$$

The last inequality follows from the inequality $\sum_{i=1}^n 1/k_i \leqslant 2$. ∎

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Functional calculi for pseudodifferential operators, III

by

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Abstract. We construct a selfadjoint algebra of L^p-bounded pseudodifferential operators of nonpositive order, acting on functions defined on a compact manifold without boundary. Then, using the Weyl formula, we give a meaning to $f(A_1, \ldots, A_r)$ for functions f with a prescribed finite number of derivatives, when $(A_1, \ldots, A_r)$ is an r-tuple of selfadjoint, commuting or noncommuting operators in the algebra.

§0. Introduction. This is the last paper in a series (cf. [1]–[4]) that studies functions of several commuting and noncommuting selfadjoint pseudodifferential operators of nonpositive order, by means of the Hermann Weyl formula (cf. [14], [11], [12], [5])

$$f(A_1, \ldots, A_r) = \int_{\mathbf{R}^r} e^{-2\pi i(t_1 A_1 + \ldots + t_r A_r)} \hat{f}(t)\, dt. \tag{0.1}$$

The pseudodifferential operators we consider depend on symbols with a finite number of derivatives. No assumption of homogeneity is made. These operators act on functions defined on a space that is either the euclidean space $\mathbf{R}^n$ (cf. [1], [3]), or a compact manifold without boundary of class C^M, for some $M < \infty$ (cf. [2]). We give in each case sufficient differentiability conditions on a function f so that (0.1) defines an operator in the same class. In fact, we prove that this class is a selfadjoint Banach algebra.

The aim of this paper, as stated in [1] and [2], is to extend to L^p-bounded operators, $1 < p < \infty$, the results proved in [2] for $p = 2$.

The key point in dealing with the formula (0.1) is to obtain a "good" estimate of the norm

$$\|e^{-2\pi i t\cdot A}\| \tag{0.2}$$

in terms of $|t|$. Typically, a good estimate is expected to be a polynomial one. In [1] and [2], a roundabout argument is used, by introducing a suitable version of the characteristic operators defined by A. P. Calderón in [6]. Essentially, we make use of the same machinery here, plus a "self-improving" argument, where the Sobolev immersion theorem and the fact that $L^p(X)$ is continuously included in $L^q(X)$ if $p \geqslant q$ play a crucial role (cf. Theorem (3.1) below).

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When the underlying spaces is $\mathbf{R}^n$, it is unknown whether or not (0.2) admits a nontrivial, that is to say, a better than exponential estimate. This problem prevents us from having extended to L^p-bounded operators the functional calculus defined in [1] in the L^2 case.

We divide this paper in 3 sections. Even though the presentation may be slightly different and the hypotheses have been adjusted to fit the L^p case, Sections 1 and 2 do not differ much from what has been done in [2]. Thus, proofs are omitted in most cases, but proper references are supplied. In Section 1 we construct the operator classes $\mathcal{M}_k$ and $\mathcal{R}_k$ on euclidean spaces. It is known (cf. [4]) that "amplitudes" $a(x,y,\zeta)$ in a subclass of the "wrong" class $S^0_{1,1}$ give rise to pseudodifferential operators which are singular integrals in the sense of the Calderón–Zygmund theory. Particularly, they are bounded operators in L^p, $1<p<\infty$. This is the main idea in building up our algebra of L^p-bounded pseudodifferential operators. In doing so, the symbols involved are required to have one derivative in the ζ variables more than in the L^2 case. In Section 1 we also study the pointwise convergence of a truncated oscillatory integral (cf. Lemma (1.3) below), completing [4]. This integral is the main tool in proving that our class of pseudodifferential operators is a selfadjoint Banach algebra.

In Section 2, the definition of our algebras of operators is extended to compact manifolds without boundary. Finally, Section 3 carries the results that deal with the definition of the functional calculus itself.

It should be noted that a remarkable property of the Weyl formula is that once $f(A)$ is properly defined for a single operator, (0.1) can be easily handled without assuming that the operators commute (cf. Remark (3.16) below, [1], p. 57, [2], p. 62). This is not true for the calculi defined in terms of the spectral theorem or the Cauchy integral theorem. However, if the operators $A_1, \ldots, A_r$ commute, (0.1) is the same as the operator defined by the spectral theorem (cf. [12]).

§1. The operator classes $\mathcal{M}_k$ and $\mathcal{R}_k$ on euclidean spaces. Given $0\leqslant\sigma<1$, $k=1,2,\ldots$, let

$$N=\begin{cases}\dfrac{k}{1-\sigma} & \text{if this is an integer,}\\[2mm] \left[\dfrac{k}{1-\sigma}\right]+1 & \text{if not.}\end{cases}$$

(1.1) Definition. For $0\leqslant j+m<N$, S^j_m consists of continuous functions $p(x,\zeta)$ with continuous derivatives in the ζ variable up to order $n+N+2-j$, such that each function $D^\gamma_\zeta p$ has continuous derivatives in x, ζ up to order $2[n/2]+N+k+2-j-m$, satisfying

$$\sup_{\substack{x,\zeta\in\mathbf{R}^n\\ \alpha,\beta,\gamma}}\frac{|D^\alpha_x D^\beta_\zeta D^\gamma_\zeta p(x,\zeta)|}{(1+|\zeta|)^{-j(1-\sigma)+(m+|\alpha|)\sigma-|\beta+\gamma|}}<\infty.$$

With the above supremum as a norm, S^j_m is a Banach space.

We will consider operators K acting on $\mathcal{S}$ as

$$Kf=\sum_{j=0}^{N-1}\int e^{-2\pi i x\cdot\zeta}p_j(x,\zeta)\hat{f}(\zeta)\,d\zeta+Rf, \tag{1.2}$$

where

(i) The function p_j belongs to the class $S^j=S^j_0$.

(ii) For $1<p_0\leqslant 2$ fixed, R is a continuous linear operator from L^p into itself for $p_0\leqslant p\leqslant p_0'=p_0/(p_0-1)$. Furthermore, R and the adjoint R^* are continuous from L^p into L^p_k, where L^p_k denotes the Sobolev space of order k.

We will denote by $\mathcal{M}_k$ and $\mathcal{R}_k$ the classes of operators K and R, respectively. Operators in $\mathcal{R}_k$ will be referred to as *regularizing operators of order* k.

In $\mathcal{R}_k$, we define the norm

$$\|R\|_{\mathcal{R}_k}=\|R\|_{B,B_k}+\|R\|_{C,C_k}+\|R^*\|_{B,B_k}+\|R^*\|_{C,C_k},$$

where $B=L^{p_0}$, $B_k=L^{p_0}_k$, $C=L^{p_0'}$, $C_k=L^{p_0'}_k$, and $\|\ \|_{B_1,B_2}$ denotes the usual operator norm of a continuous linear operator from B_1 into B_2 (Banach spaces).

Under the above conditions, which do not assume homogeneity of the functions p_j, the representation of K given by (1.2) is not unique. So, given $K\in\mathcal{M}_k$, we define

$$\|K\|_{\mathcal{M}_k}=\inf\{\sup_{0\leqslant j\leqslant N-1}\|p_j\|_{S^j}+\|R\|_{\mathcal{R}_k}\},$$

where the infimum is taken over all possible representations of K as in (1.2).

$\mathcal{M}_k$ and $\mathcal{R}_k$ are selfadjoint Banach algebras, with the composition as product (cf. [4]).

The main tool to prove that $\mathcal{M}_k$ is a selfadjoint Banach algebra is to analyze a truncated oscillatory integral (cf. [4], Lemma 2.1). The next result proves the pointwise convergence of that integral, completing [4].

(1.3) Lemma. *Let $p(x,y,\zeta)$ be a continuous function defined in $\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n$. Suppose that given $0\leqslant j\leqslant N-1$, the function $p(x,y,\zeta)$ has continuous derivatives in the variables x,y,ζ up to orders $2[n/2]+N+k+2-j$, $2[n/2]+N+k+2-j$, $n+N+2-j$, respectively. Moreover,*

$$\sup_{\substack{x,y,\zeta\\ \alpha,\beta,\gamma}}\frac{|D^\alpha_x D^\beta_y D^\gamma_\zeta p(x,y,\zeta)|}{(1+|\zeta|)^{-j(1-\sigma)+\sigma(|\alpha|+|\beta|)-|\gamma|}}=B<\infty.$$

Let η be a cut-off function. Then the integral

$$L_\varepsilon f(x)=\int e^{-2\pi i(x-y)\cdot\zeta}p(x,y,\zeta)\,\eta(\varepsilon\zeta)f(y)\,dy\,d\zeta,\quad f\in\mathcal{S}, \tag{1.4}$$

converges pointwise a.e. as $\varepsilon\to 0$, for each $f\in\mathcal{S}$.

Proof. Let

$$p(x,y,\zeta)=\sum_{|\alpha|<N-j}(D_y^\alpha p)(x,x,\zeta)\frac{(y-x)^\alpha}{\alpha!}$$

$$+\sum_{|\alpha|=N-j}(y-x)^\alpha\frac{N-j}{\alpha!}\int_0^1(D_y^\alpha p)(x,x+t(y-x),\zeta)(1-t)^{N-j-1}\,dt.$$

Integrating by parts in (1.4) any of the terms in the first sum, we obtain

$$\left(\frac{i}{2\pi}\right)^{|\alpha|}\frac{1}{\alpha!}\int e^{-2\pi i(x-y)\cdot\zeta}D_\zeta^\alpha[(D_y^\alpha p)(x,x,\zeta)\eta(\varepsilon\zeta)]\,f(y)\,dy\,d\zeta$$

$$=\left(\frac{i}{2\pi}\right)^{|\alpha|}\frac{1}{\alpha!}\int e^{-2\pi ix\cdot\zeta}(D_\zeta^\alpha D_y^\alpha p)\,(x,x,\zeta)\,\eta(\varepsilon\zeta)\hat f(\zeta)\,d\zeta$$

$$+\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\left(\frac{i}{2\pi}\right)^{|\alpha|}\frac{1}{\alpha!}\int e^{-2\pi ix\cdot\zeta}(D_\zeta^{\alpha-\beta}D_y^\alpha p)(x,x,\zeta)$$

$$\times\varepsilon^{|\beta|}(D^\beta\eta)(\varepsilon\zeta)\hat f(\zeta)\,d\zeta.$$

The Lebesgue convergence theorem shows that the first term tends pointwise to

$$\left(\frac{i}{2\pi}\right)^{|\alpha|}\frac{1}{\alpha!}\int e^{-2\pi ix\cdot\zeta}(D_\zeta^\alpha D_y^\alpha p)\,(x,x,\zeta)\hat f(\zeta)\,d\zeta$$

as $\varepsilon\to 0$. For the same reason, any of the terms in the sum converges to zero.

Consider one of the terms in the remainder. Let

$$q_\alpha(x,y,\zeta)=\frac{N-j}{\alpha!}\int_0^1(D_y^\alpha p)(x,x+t(y-x),\zeta)(1-t)^{N-j-1}\,dt,\qquad |\alpha|=N-j.$$

Integrating by parts, we have

$$(1.5)\qquad\left(\frac{i}{2\pi}\right)^{N-j}\int e^{-2\pi i(x-y)\cdot\zeta}D_\zeta^\alpha[q_\alpha(x,y,\zeta)\eta(\varepsilon\zeta)]\,dy\,d\zeta$$

$$=\left(\frac{i}{2\pi}\right)^{N-j}\int e^{-2\pi i(x-y)\cdot\zeta}D_\zeta^\alpha q_\alpha(x,y,\zeta)\eta(\varepsilon\zeta)f(y)\,dy\,d\zeta$$

$$+\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\left(\frac{i}{2\pi}\right)^{N-j}\int e^{-2\pi i(x-y)\cdot\zeta}(D_\zeta^{\alpha-\beta}q_\alpha(x,y,\zeta)$$

$$\times\varepsilon^{|\beta|}(D^\beta\eta)(\varepsilon\zeta)f(y)\,dy\,d\zeta.$$

The amplitude $D_\zeta^\alpha q_\alpha(x,y,\zeta)$ in the first term has continuous derivatives at least up to orders $[n/2]+1$, $[n/2]+1$, $n+2$ in x,y,ζ, respectively. Moreover, each derivative $D_x^\beta D_y^\gamma D_\zeta^\lambda q_\alpha$ can be estimated by

$$C(1+|\zeta|)^{-N(1-\sigma)+\sigma(|\beta|+|\gamma|)-|\lambda|}.$$

Thus, [4], p. 269, applies and one concludes that the operator L_ε^α defined by $D_\zeta^\alpha q_\alpha(x,y,\zeta)$ converges in L^p as $\varepsilon\to 0$ for each $1<p<\infty$, $f\in\mathcal{S}$.

On the other hand, the operator $(\partial/\partial x_j)L_\varepsilon^\alpha$ can be written in terms of the amplitude

$$\left(\frac{i}{2\pi}\right)^{N-j}\left[(-2\pi i\zeta_j)\,D_\zeta^\alpha q_\alpha(x,y,\zeta)+\frac{\partial}{\partial x_j}D_\zeta^\alpha q_\alpha(x,y,\zeta)\right].$$

This function also has continuous derivatives in x,y,ζ up to orders $[n/2]+1$, $[n/2]+1$, $n+2$, respectively. Moreover, each derivative $D_\alpha^\beta D_y^\gamma D_\zeta^\lambda$ can be estimated by

$$C(1+|\zeta|)^{-N(1-\sigma)+1+\sigma(|\beta|+|\gamma|)-|\lambda|}.$$

The choice of the parameter N guarantees that $-N(1-\sigma)+1\leqslant 0$. So, [4], p. 269, applies once more and we conclude that $L_\varepsilon^\alpha f$ converges in the Sobolev space L_1^p as $\varepsilon\to 0$. Then the Sobolev immersion theorem shows that the sequence will also converge pointwise provided $p>n$.

Next, consider any term in the sum that appears in (1.5). We will write the amplitude as

$$(1.6)\qquad \eta(\zeta)D_\zeta^{\alpha-\beta}q_\alpha(x,y,\zeta)\,\varepsilon^{|\beta|}(D^\beta\eta)\,(\varepsilon\zeta)$$

$$+(1-\eta(\zeta))\,D_\zeta^{\alpha-\beta}q_\alpha(x,y,\zeta)\,\varepsilon^{|\beta|}(D^\beta\eta)\,(\varepsilon\zeta).$$

The first term vanishes for $|\zeta|\geqslant 2$. Thus the dominated convergence theorem shows that the corresponding operator converges pointwise a.e. to zero as $\varepsilon\to 0$. The second term vanishes for $|\zeta|\leqslant 1$. Write the corresponding operators as

$$(1.7)\qquad \int{}^{-2\pi i(x-y)\cdot\zeta}(1-\eta(\zeta))\,|\zeta|^{-|\beta|}D_\zeta^{\alpha-\beta}q_\alpha(x,y,\zeta)(\varepsilon|\zeta|)^{|\beta|}(D^\beta\eta)\,(\varepsilon\zeta)f(y)\,dy\,d\zeta.$$

The function $|\zeta|^{|\beta|}(D^\beta\eta)(\zeta)$ can be written as the difference of two cut-off functions. On the other hand, the amplitude $(1-\eta(\zeta))\,|\zeta|^{-|\beta|}D_\zeta^{\alpha-\beta}q_\alpha(x,y,\zeta)$ also defines an operator which converges in L_1^p as $\varepsilon\to 0$. Since the limit does not depend on the cut-off function (cf. [1], p. 4), it has to be zero. So, if $p>n$, (1.7) converges pointwise to zero as $\varepsilon\to 0$.

Finally, we have proved that $L_\varepsilon f(x)$ converges pointwise a.e. as $\varepsilon\to 0$. This completes the proof of Lemma (1.3).

Furthermore, we can conclude that the limit of $\{L_\varepsilon f(x)\}$ as $\varepsilon\to 0$ is

$$Lf(x)=\sum_{|\alpha|<N-j}\left(\frac{i}{2\pi}\right)^{|\alpha|}\frac{1}{\alpha!}\int e^{-2\pi ix\cdot\zeta}(D_\zeta^\alpha D_y^\alpha p)\,(x,x,\zeta)\hat f(\zeta)\,d\zeta$$

$$+\lim_{\varepsilon\to 0}\sum_{|\alpha|=N-j}\left(\frac{i}{2\pi}\right)^{N-j}\int e^{-2\pi i(x-y)\cdot\zeta}(D_\zeta^\alpha q_\alpha)\,(x,y,\zeta)\eta(\varepsilon\zeta)f(y)\,dy\,d\zeta.$$

The limit in the second term defines an operator in the class $\mathcal{R}_k$.

(1.8) Remark. Given two operators in the class $\mathcal{M}_k$,

$$Kf = \sum_{j=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} p_j(x,\zeta)\hat{f}(\zeta)\,d\zeta + R_1 f,$$

$$Hf = \sum_{m=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} q_m(x,\zeta)\hat{f}(\zeta)\,d\zeta + R_2 f,$$

the adjoint K^* and the composition KH can be written as (cf. [3])

$$(1.9)\qquad K^*f = \sum_{j=0}^{N-1} \sum_{|\alpha|<N-j} \left(\frac{i}{2\pi}\right)^{|\alpha|} \frac{1}{\alpha!} \int e^{-2\pi i x\cdot\zeta} D_\zeta^\alpha D_x^\alpha \overline{p_j(x,\zeta)}\hat{f}(\zeta)\,d\zeta + R_3 f,$$

$$(1.10)\qquad KHf = \sum_{\substack{j,m=0\\ j+m<N}}^{N-1} \sum_{|\alpha|<N-j-m} \left(\frac{i}{2\pi}\right)^{|\alpha|} \times \frac{1}{\alpha!} \int^{-2\pi i x\cdot\zeta} D_\zeta^\alpha p_j(x,\zeta) D_x^\alpha q_m(x,\zeta)\hat{f}(\zeta)\,d\zeta + R_4 f.$$

The operators in the class $\mathcal{M}_k$ can be extended to linear continuous operators from L_s^p into itself, for $p_0 \leqslant p \leqslant p_0'$, $|s| \leqslant k$ (cf. [4]).

The class $\mathcal{M}_k$ is defined in the spirit of the classes considered in [6]. However, no assumption of homogeneity is made here, which, as we said before, implies that the symbol $\sum_j p_j(x,\zeta)$ will not necessarily have a unique representation. In spite of this complication, it is still essentially true that once the calculus is properly defined on symbols, it will be possible to extend the definition to the operator class.

But some extra work is necessary to overcome the lack of uniqueness. The basic idea is to consider auxiliary classes, where, essentially, the representation (1.2) is split into a cartesian product. In this new setting, one recovers the uniqueness of the representation. Then, a modification of the characteristic operators introduced in [6] can be used.

We summarize now the definitions and main properties of these auxiliary spaces. We will omit the proofs, which follow very closely those in [1] and [2]. Precise references will be given, though.

(1.11) Definition. $\mathfrak{M}_k$ consists of $(N+1)$-tuples $\mathcal{K} = (p_0, \ldots, p_{N-1}, R)$, where $p_j \in S^j$, $R \in \mathcal{R}_k$.

$\mathfrak{M}_k$ has a natural structure of complex vector space.

Given $(p_0, \ldots, p_{N-1}, R) \in \mathfrak{M}_k$, consider the operator

$$Kf = \sum_{j=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} p_j(x,\zeta)\hat{f}(\zeta)\,d\zeta + Rf.$$

Then $K \in \mathcal{M}_k$ and the map

$$\Lambda\colon \mathfrak{M}_k \to \mathcal{M}_k, \qquad (p_0, \ldots, p_{N-1}, R) \to K,$$

is a homomorphism of complex vector spaces.

Given $(p_0, \ldots, p_{N-1}, R_1), (q_0, \ldots, q_{N-1}, R_2) \in \mathfrak{M}_k$, let $K = \Lambda(p_0, \ldots, p_{N-1}, R_1)$, $H = \Lambda(q_0, \ldots, q_{N-1}, R_2)$. According to (1.9) and (1.10),

$$K^*f = \sum_{h=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} r_h(x,\zeta)\hat{f}(\zeta)\,d\zeta + R_3 f,$$

$$KHf = \sum_{h=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} s_h(x,\zeta)\hat{f}(\zeta)\,d\zeta + R_4 f,$$

where

$$(1.12)\qquad r_h = \sum_{j+|\alpha|=h} \left(\frac{i}{2\pi}\right)^{|\alpha|} \frac{1}{\alpha!} D_\zeta^\alpha D_x^\alpha \overline{p_j(x,\zeta)},$$

$$(1.13)\qquad s_h = \sum_{j+m+|\alpha|=h} \left(\frac{i}{2\pi}\right)^{|\alpha|} \frac{1}{\alpha!} D_\zeta^\alpha p_j D_x^\alpha q_m.$$

It can be shown (cf. [1], p. 46) that the correspondence

$$(p_0, \ldots, p_{N-1}, R_1),\ (q_0, \ldots, q_{N-1}, R_2) \to (s_0, \ldots, s_{N-1}, R_4)$$

is associative. Thus, it defines a product in $\mathfrak{M}_k$. The correspondence

$$(p_0, \ldots, p_{N-1}, R_1) \to (r_0, \ldots, r_{N-1}, R_3)$$

gives a notion of adjoint in $\mathfrak{M}_k$. The space $\mathfrak{M}_k$ becomes a selfadjoint Banach algebra with the norm

$$\|(p_0, \ldots, p_{N-1}, R)\|_{\mathfrak{M}_k} = \sup_{0\leqslant j<N} \|p_j\|_{S^j} + \|R\|_{\mathcal{R}_k}.$$

Moreover, Λ is a continuous homomorphism of algebras that commutes with adjoints.

(1.14) Definition. $\mathfrak{N}_k$ consists of N-tuples $\mathcal{P} = (p_0, \ldots, p_{N-1})$ such that $p_j \in S^j$.

$\mathfrak{N}_k$ is a selfadjoint Banach algebra if we define

$$\mathcal{P}^* = (r_0, \ldots, r_{N-1}), \quad \text{with } r_h \text{ as in (1.12)},$$

$$\mathcal{P}\mathcal{Q} = (s_0, \ldots, s_{N-1}), \quad \text{with } s_h \text{ as in (1.13)},$$

$$\|\mathcal{P}\|_{\mathfrak{N}_k} = \sup_{0\leqslant j<N} \|p_j\|_{S^j}.$$

Given $\mathcal{P} = (p_0, \ldots, p_{N-1})$, we will call p_0 the *principal symbol* of $\mathcal{P}$, $\sigma_p(\mathcal{P})$. We have

$$\sigma_p(\mathcal{P}^*) = \overline{\sigma_p(\mathcal{P})}, \qquad \sigma_p(\mathcal{P}\mathcal{Q}) = \sigma_p(\mathcal{P})\sigma_p(\mathcal{Q}).$$

Thus, if $\mathcal{P}$ is a selfadjoint N-tuple, its principal symbol has to be a real function.

The map

$$\Omega\colon \mathfrak{M}_k \to \mathfrak{N}_k, \quad (p_0, \ldots, p_{N-1}, R) \to (p_0, \ldots, p_{N-1}),$$

is a continuous and surjective homomorphism of algebras. Its kernel is the subalgebra $\{(0, \ldots, 0, R) \mid R \in \mathscr{R}_k\}$, which can be identified with $\mathscr{R}_k$.

Since the product of two $(N+1)$-tuples $(p_0, \ldots, p_{N-1}, 0)$, $(q_0, \ldots, q_{N-1}, 0)$ might have a nonzero regularizing coordinate, $\mathfrak{N}_k$ cannot be identified with a subalgebra of $\mathfrak{M}_k$. Thus, the map

$$\Omega_1\colon \mathfrak{N}_k \to \mathfrak{M}_k, \quad \mathscr{P} \to (\mathscr{P}, 0),$$

which is linear, continuous and satisfies $\Omega\Omega_1 =$ identity, cannot be a homomorphism of algebras.

We will now define a class $\mathfrak{A}_k$ of formal differential operators $\mathscr{U}$. This class generalizes the class of characteristics, introduced by A. P. Calderón in [6].

More precisely, we consider operators $\mathscr{U}$ which act on C_0^∞ in the following way:

$$(\mathscr{U}f)(x, \zeta) = \sum_{|\alpha| < N} D_\zeta^\alpha [U_\alpha(x, \zeta) f].$$

Each coefficient U_α is an $(N-|\alpha|)$-tuple $(u_{0,\alpha}, \ldots, u_{N-|\alpha|-1,\alpha})$ and each coordinate $u_{j,\alpha}$ belongs to $S^j_{|\alpha|}$ (cf. Definition (1.1)).

Given two monomials $\mathscr{U} = D_\zeta^\alpha(U_\alpha\ )$, $\mathscr{V} = D_\zeta^\beta(V_\beta\ )$, $0 \leqslant |\alpha|, |\beta| < N$, $f, g \in C_0^\infty$, we have formally

$$\begin{aligned}(D_\zeta^\alpha(U_\alpha D_\zeta^\beta V_\beta f), g) &= (-1)^{|\alpha+\beta|}(f, V_\beta D_\zeta^\beta(U_\alpha D_\zeta^\alpha g))\\
&= (-1)^{|\alpha+\beta|}\left(f,\ V_\beta \sum_{0 \leqslant \gamma \leqslant \beta} \binom{\beta}{\gamma} D_\zeta^{\beta-\gamma} U_\alpha D_\zeta^{\alpha+\gamma} g\right)\\
&= \sum_{0 \leqslant \gamma \leqslant \beta} \binom{\beta}{\gamma} (-1)^{|\alpha+\beta|}(-1)^{|\alpha+\gamma|}(D_\zeta^{\alpha+\gamma}(V_\beta D_\zeta^{\beta-\gamma} U_\alpha f), g)\\
&= \sum_{\alpha \leqslant \lambda \leqslant \alpha+\beta} \binom{\beta}{\lambda-\alpha} (-1)^{|\alpha+\beta+\lambda|}(D_\zeta^\lambda(V_\beta D_\zeta^{\alpha+\beta-\lambda} U_\alpha f), g),\end{aligned}$$

where $\lambda = \alpha + \gamma$. Thus, a tentative definition of the product $\mathscr{U}\mathscr{V}$ could be

$$\sum_{\alpha \leqslant \lambda \leqslant \alpha+\beta} \binom{\beta}{\lambda-\alpha} (-1)^{|\alpha+\beta+\lambda|} D_\zeta^\lambda(W_\lambda\ )$$

once we give a sense to $W_\lambda = V_\beta D_\zeta^{\alpha+\beta-\lambda} U_\alpha$.

First, we define $W_\lambda = 0$ if $|\lambda| > N$. If $|\lambda| < N$, W_λ will be an $(N-|\lambda|)$-tuple $(w_{0,\lambda}, \ldots, w_{N-|\lambda|-1,\lambda})$.

Let $V_\beta = (v_{0,\beta}, \ldots, v_{N-|\beta|-1,\beta})$, $U_\alpha = (u_{0,\alpha}, \ldots, u_{N-|\alpha|-1,\alpha})$. It is easy to show that $v_{h,\beta} D^{\alpha+\beta-\lambda} u_{l,\alpha}$ belongs to $S^{h+l+|\alpha+\beta-\lambda|}_{|\lambda|}$, $h+l+|\alpha+\beta-\lambda| < N$. So, for $|\lambda| < N$, $j < N-|\lambda|$, define

$$w_{j,\lambda} = (-1)^{|\alpha+\beta-\lambda|} \binom{\beta}{\lambda-\alpha} \sum_{l+h+|\alpha+\beta-\lambda| = j} v_{h,\beta} D_\zeta^{\alpha+\beta-\lambda} u_{l,\alpha}.$$

This correspondence, extended to $\mathfrak{A}_k$ by linearity, gives an associative product (cf. [1], p. 46). $\mathfrak{A}_k$ is a Banach algebra with the norm

$$\|\mathscr{U}\|_{\mathfrak{A}_k} = \sup_{j,\alpha} \|u_{j,\alpha}\|_{S^j_{|\alpha|}}.$$

Given $\mathscr{P} = (p_0, \ldots, p_{N-1}) \in \mathfrak{N}_k$, we will define the operator

$$\mathscr{U}_{\mathscr{P}} = \sum_{|\alpha| < N} D_\zeta^\alpha(U_\alpha\ ), \quad \text{where}$$

$$U_\alpha = (u_{j,\alpha}), \quad u_{j,\alpha} = \left(\frac{i}{2\pi}\right)^{|\alpha|} \frac{1}{\alpha!} D_x^\alpha p_j.$$

By analogy with [6], we will call $\mathscr{U}_{\mathscr{P}}$ the *characteristic* of $\mathscr{P}$.

Consider the map

$$\pi\colon \mathfrak{N}_k \to \mathfrak{A}_k, \quad \mathscr{P} \to \mathscr{U}_{\mathscr{P}}.$$

π is an injective continuous homomorphism of algebras.

Now, consider the map

$$\pi_1\colon \mathfrak{A}_k \to \mathfrak{N}_k, \quad \mathscr{U} \to U_0.$$

π_1 is linear and continuous but does not commute with products. However, if $\mathscr{U} \in \mathfrak{A}_k$ and $\mathscr{V} \in \pi(\mathfrak{N}_k)$, then $\pi_1(\mathscr{U}\mathscr{V}) = \pi_1(\mathscr{U})\pi_1(\mathscr{V})$. Obviously,

$$\pi_1 \pi = \text{identity}, \quad \pi\pi_1(\mathscr{U}) = \mathscr{U} \quad \text{if } \mathscr{U} \in \pi(\mathfrak{N}_k).$$

Furthermore, $\pi(\mathfrak{N}_k)$ is a closed subalgebra of $\mathfrak{A}_k$.

The map $\pi\Omega$ from $\mathfrak{M}_k$ into $\mathfrak{A}_k$ was denoted in [1] by χ. Given $\mathscr{K} \in \mathfrak{M}_k$, $\chi(\mathscr{K})$ is called the *characteristic* of $\mathscr{K}$.

We also define the map

$$\chi_1\colon \mathfrak{A}_k \to \mathfrak{M}_k, \quad \mathscr{U} \to (u_{0,0}, \ldots, u_{N-1,0}, 0).$$

χ_1 is a linear and continuous map that satisfies

$$\chi\chi_1(\mathscr{U}) = \mathscr{U}, \quad \mathscr{U} \in \chi(\mathfrak{M}_k).$$

We will now state the invariance of $\mathscr{M}_k$ under changes of coordinates in $\boldsymbol{R}^n$. The proof is almost identical to the one of Theorem 0.4 in [2], p. 39, and it will be omitted.

Let U_1, U be open bounded subsets of $\boldsymbol{R}^n$. For a given M, let $\varphi\colon U_1 \to U$ be a diffeomorphism of class C^M. Suppose that φ can be extended to a neighborhood of $\bar{U}_1$.

(1.15) THEOREM. *Let $A \in \mathscr{M}_k$ and let $\varphi\colon U_1 \to U$ be a diffeomorphism as above, with $M = 2[n/2]+n+2N+k+5$. Then:*

(i) *Given a compact set $C_1 \subset U_1$, there exists $A_1 \in \mathscr{M}_k$ such that $A_1(g)(x) = A[g \circ \varphi^{-1}] \circ \varphi(x)$ for $x \in C_1, g \in C_0^\infty(C_1)$.*

(ii) *If*

$$Af = \sum_{j=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} p_j(x,\zeta) \hat{f}(\zeta)\, d\zeta + Rf,$$

$$A_1 f = \sum_{j=0}^{N-1} e^{-2\pi i x\cdot\zeta} q_j(x,\zeta) \hat{f}(\zeta)\, d\zeta + R_1 f,$$

then

$$(1.16)\qquad q_j(x,\zeta) = \sum_{m=0}^{j} \sum_{|\alpha|=j-m} \left(\frac{i}{2\pi}\right)^{|\alpha|} \frac{1}{\alpha!} D_y^\alpha D_\zeta^\alpha [p_m(\varphi(x), J_t^{-1}(x,y)\zeta) \times |\det J(y)| |\det J(x,y)^{-1}|]_{y=x}, \qquad x \in C_1,$$

where J denotes the Jacobian matrix of φ, $J(x,y)$ is the matrix $\int_0^1 J(x+s(y-x))\, ds$ and $J_t(x,y)$ is the transpose of $J(x,y)$.

(iii) *The map*

$$\varphi^*\colon (p_0, \ldots, p_{N-1}) \to (q_0, \ldots, q_{N-1})$$

is a homomorphism from the algebra $\mathfrak{N}_k$ into itself. Moreover, if $\varphi = \varphi_1 \circ \varphi_2$, then $\varphi^* = \varphi_2^* \circ \varphi_1^*$.

§2. The classes $\mathscr{M}_k$ and $\mathscr{R}_k$ on compact manifolds. Theorem (1.15) is the main ingredient in extending the operator classes $\mathscr{M}_k$ and $\mathscr{R}_k$ to differentiable manifolds.

In the sequel X will stand for an n-dimensional compact differentiable manifold without boundary. We will assume that X is a C^r manifold, $r \geq M$, $M = 2[n/2]+n+2N+k+5$. The reason for this value of M will appear later.

We will also suppose that a measure μ is fixed on X. Moreover, in terms of local coordinates $x = (x_1, \ldots, x_n)$, μ is given by $G(x)\, dx_1 \ldots dx_n$, where $G > 0$ is a function of class C^{r-1}.

Let U_1, U_2 be open bounded subsets of X or $\boldsymbol{R}^n$. Given a C^r diffeomorphism $\varphi\colon U_1 \to U_2$, we will always assume that φ can be extended to a diffeomorphism defined in a neighborhood of $\bar{U}_1$.

Given a function f defined in the ambient space of U, $\varphi^*(f)$ will denote the function defined in the ambient space of U_1 which coincides with $f \circ \varphi$ in U_1 and is zero outside.

Given an operator A acting on functions defined in the ambient space of U_2, $\varphi^*(A)$ will denote the operator acting on functions defined in the ambient space of U_1 as $\varphi^*(A)(f) = \varphi^*[A(\varphi^{-1*}(f))]$.

Given $(p_0, \ldots, p_{N-1}) \in \mathfrak{N}_k$, we will denote by $\varphi^*(p_0, \ldots, p_{N-1})$ the N-tuple that coincides with $(q_0, \ldots, q_{N-1})$, q_j as in (1.16), when $x \in U_1$ and is zero otherwise.

Let $\varphi_1\colon U_1 \to U_2$, $\varphi_2\colon U_1' \to U_2'$ be diffeomorphisms and suppose that U_2, U_1' are nondisjoint subsets of the same ambient space. Then $\varphi_2 \circ \varphi_1$ will be the diffeomorphism that maps $\varphi_1^{-1}(U_2 \cap U_1')$ onto $\varphi_2(U_2 \cap U_1')$. Moreover, $(\varphi_2 \circ \varphi_1)^* = \varphi_1^* \circ \varphi_2^*$.

If A_1 and A_2 are operators which map functions defined in the ambient space of U_2 into functions supported in U_2, then $(\varphi_2 \circ \varphi_1)^*(A_1 \circ A_2) = \varphi_1^*(A_1) \circ \varphi_2^*(A_2)$.

For $1 < p < \infty$, $L^p(X)$ will denote the functions which are integrable to the pth power with respect to μ. For $k \leq r$, $L_k^p(X)$ will denote the functions which in local coordinates admit p-integrable derivatives up to order k. We will identify $L_0^p(X)$ and $L^p(X)$.

A norm can be defined in $L_k^p(X)$ in the following way. Let $\{U_j\}_{1 \leq j \leq H}$ be coordinate neighborhoods in X and let $\varphi_j\colon U_j \to \varphi_j(U_j) \subset \boldsymbol{R}^n$ be C^r diffeomorphisms. For each j, let $\theta_j \in C_0^r(U_j)$ be a function such that $\theta_j \geq 0$, $\sum_j \theta_j > 0$ in X. Then, given $f \in L_k^p(X)$ we define

$$(2.1)\qquad \|f\|_{L_k^p(X)} = \sum_{j=1}^{H} \|\varphi_j^{-1*}(\theta_j f)\|_{L_k^p}.$$

It is not difficult to verify that any two such norms are equivalent.

$L_k^p(X)$ is a Banach space and $L_k^2(X)$ is a Hilbert space.

(2.2) DEFINITION. (a) Given $1 < p_0 \leq 2$, $\mathscr{R}_k(X)$ is the class of linear operators R such that R and R^* map continuously $L^p(X)$ into $L_k^p(X)$ for $p_0 \leq p \leq p_0'$.

(b) Given $1 < p_0 \leq 2$, $\mathscr{M}_k(X)$ denotes those linear operators A that map continuously $L^p(X)$ into itself for $p_0 \leq p \leq p_0'$ and satisfy the following conditions.

(i) If $\varphi_1, \varphi_2 \in C_0^r(X)$ have disjoint supports, then $\varphi_1 A \varphi_2 \in \mathscr{R}_k(X)$.

(ii) Let $U \subset X$ be open and let $\varphi\colon U \to U_1$ be a C^r diffeomorphism, $U_1 \subset \boldsymbol{R}^n$. Suppose that φ can be extended to a neighborhood of $\bar{U}$. Then there exists an operator $A_1 \in \mathscr{M}_k$ such that $\varphi_1 A \varphi_2 = \varphi_1 \varphi^*(A_1) \varphi_2$, $\varphi_1, \varphi_2 \in C_0^r(U)$.

(2.3) THEOREM (cf. [2], p. 44]. *$\mathscr{M}_k(X)$ is a selfadjoint algebra and $\mathscr{R}_k(X)$ is a two-sided ideal of $\mathscr{M}_k(X)$. Given $A \in \mathscr{M}_k(X)$, A is a continuous operator from $L_m^p(X)$ into itself, for $p_0 \leq p \leq p_0'$, $0 \leq m \leq k$.*

We will endow $\mathscr{R}_k(X)$ and $\mathscr{M}_k(X)$ with the norms

$$\|R\|_{\mathscr{R}_k(X)} = \|R\|_{B,B_k} + \|R\|_{C,C_k} + \|R^*\|_{B,B_k} + \|R^*\|_{C,C_k},$$

$$\|A\|_{\mathscr{M}_k(X)} = \sum_{m=0}^{k} [\|A\|_{B_m,B_m} + \|A\|_{C_m,C_m}],$$

where $B = L^{p_0}(X)$, $B_j = L_j^{p_0}(X)$, $C = L^{p_0'}(X)$, $C_j = L_j^{p_0'}(X)$. $\mathscr{R}_k(X)$ is then a selfadjoint Banach algebra and $\mathscr{M}_k(X)$ is a selfadjoint normed algebra, not complete. $\mathscr{R}_k(X)$ is continuously included in $\mathscr{M}_k(X)$.

Now, we will define the counterparts of the auxiliary classes $\mathfrak{M}_k$ and $\mathfrak{N}_k$ on the manifold X. They will depend on the choice of coordinate neighborhoods and a partition of unity in X. So, fix in X coordinate neighborhoods $\{U_j\}$, C^r diffeomorphisms $\varphi_j\colon U_j \to \varphi_j(U_j)$, nonnegative functions $\theta_j \in C_0^r(U_j)$ and a partition of unity $\{\eta_j\}$, under the following assumptions:

(i) $\eta_j \in C_0^r(U_j)$.

(ii) If $\operatorname{supp}(\theta_i) \cap \operatorname{supp}(\theta_j) \neq \emptyset$, then $\operatorname{supp}(\theta_i), \operatorname{supp}(\theta_j) \subset U_i \cap U_j$.

(iii) $\theta_i = 1$ in a neighborhood of $\operatorname{supp}(\eta_j)$ when $i = j$ or when $\operatorname{supp}(\eta_j) \cap \operatorname{supp}(\eta_i) \neq \emptyset$.

Under the above conditions, consider for each j the restriction to $\varphi_j(U_j)$ of an N-tuple $(p_0, \ldots, p_{N-1})$, $p_h \in S^h$. Denote this restriction by $(p_0^{(j)}, \ldots, p_{N-1}^{(j)})$. We define the norm

$$\|(p_0^{(j)}, \ldots, p_{N-1}^{(j)})\| = \sup \frac{|D_x^\alpha D_\xi^\beta D_\zeta^\gamma p_h^{(j)}(x, \zeta)|}{(1+|\zeta|)^{-h(1-\sigma)+\sigma|\alpha|-|\beta+\gamma|}}, \tag{2.4}$$

where the supremum is taken over $x \in \varphi_j(U_j)$, $\zeta \in \mathbf{R}^n$, $|\gamma| \leqslant n+N+2-h$, $|\alpha+\beta| \leqslant 2[n/2]+N+k+2-h$.

For j fixed, the product of two N-tuples is defined as in (1.10). Namely,

$$(p_0^{(j)}, \ldots, p_{N-1}^{(j)})(q_0^{(j)}, \ldots, q_{N-1}^{(j)}) = (r_0^{(j)}, \ldots, r_{N-1}^{(j)}),$$

where

$$r_h^{(j)} = \sum_{|\alpha|+m+s=h} \left(\frac{i}{2\pi}\right)^{|\alpha|} \frac{1}{\alpha!} D_\zeta^\alpha p_m^{(j)} D_x^\alpha q_s^{(j)}. \tag{2.5}$$

(2.6) Definition. The space $\mathfrak{M}_k(X)$ consists of tuples $\mathscr{K} = \{(p_0^{(j)}, \ldots, p_{N-1}^{(j)}), R\}_j$ with $R \in \mathscr{R}_k(X)$ and $(p_0^{(j)}, \ldots, p_{N-1}^{(j)})$ as above, satisfying the compatibility condition: If $U_i \cap U_j \neq \emptyset$, $\varphi_{ij} = \varphi_j \varphi_i^{-1}$, then

$$(p_0^{(i)}, \ldots, p_{N-1}^{(i)}) = \varphi_{ij}^*(p_0^{(j)}, \ldots, p_{N-1}^{(j)}) \quad \text{in } \varphi_i(U_i \cap U_j). \tag{2.7}$$

$\mathfrak{M}_k(X)$ is a Banach space with the norm

$$\|\mathscr{K}\|_{\mathfrak{M}_k(X)} = \sup_j \|(p_0^{(j)}, \ldots, p_{N-1}^{(j)})\| + \|R\|_{\mathscr{R}_k(X)}.$$

We will now define a map from $\mathfrak{M}_k(X)$ onto $\mathscr{M}_k(X)$, which will play a main role in the construction of a functional calculus on $\mathscr{M}_k(X)$.

Given an N-tuple $(p_0^{(j)}, \ldots, p_{N-1}^{(j)})$, we define an operator $A_j = A(p_0^{(j)}, \ldots, p_{N-1}^{(j)})$ as follows:

$$A_j(f)(x) = \begin{cases} \sum_{h=0}^{N-1} \int e^{-2\pi i x\cdot\zeta} p_h^{(j)}(x,\zeta) \hat{f}(\zeta)\, d\zeta, & x \in \varphi_j(U_j), \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

Now, given $\mathscr{K} \in \mathfrak{M}_k(X)$, we define

$$A_X(\mathscr{K}) = \sum_j \eta_j \varphi_j^*(A_j)\theta_j + R,$$

where $A_j = A(p_0^{(j)}, \ldots, p_{N-1}^{(j)})$.

(2.9) Theorem (cf. [2], p. 53]. *A_X is a linear continuous operator that maps $\mathfrak{M}_k(X)$ onto $\mathscr{M}_k(X)$. Furthermore, suppose that $A \in \mathscr{M}_k(X)$ is selfadjoint. Then $A = A_X\{(p_0^{(j)}, \ldots, p_{N-1}^{(j)}), R\}$, where $p_h^{(j)}$ is a real function for every j.*

A product can be defined on $\mathfrak{M}_k(X)$ in such a way that A_X becomes a continuous homomorphism of Banach algebras (cf. [2], p. 56). *Moreover, the subclass $\mathfrak{N}(X)$ of $\mathfrak{M}_k(X)$ that consists of $\{(0), R\}$ is a closed two-sided ideal. Finally, the element $\{(1, 0, \ldots, 0), 0\}$ is a unit of $\mathfrak{M}_k(X)$.*

(2.10) Definition. The space $\mathfrak{N}_k(X)$ will consist of tuples $\{(p_0^{(j)}, \ldots, p_{N-1}^{(j)})\}_j$, satisfying the compatibility condition (2.7).

With the product (2.5) and the norm $\sup_j$ (2.4), $\mathfrak{N}_k(X)$ is a Banach algebra. It can be identified with the quotient $\mathfrak{M}_k(X)/\mathfrak{N}(X)$.

§3. Definition of a functional calculus on $\mathscr{M}_k(X)$. As mentioned in §1, a roundabout argument is needed in order to overcome the lack of uniqueness in the representation of the operators in $\mathscr{M}_k(X)$. The crucial step is to obtain a polynomial estimate for $\|e^{-2\pi i t\mathscr{K}}\|_{\mathfrak{M}_k(X)}$, $t \in \mathbf{R}$. This is provided by the following result.

(3.1) Theorem. *Let $\mathscr{K} = \{(p_0^{(j)}, \ldots, p_{N-1}^{(j)}), R\}$ be such that the $p_h^{(j)}$ are real functions and the operator $A_X(\mathscr{K})$ is selfadjoint as a bounded operator in $L^2(X)$. Suppose that $1/p_0 - k/n \leqslant 1/2$. Then there exists $C > 0$ such that*

$$\|e^{-2\pi i t\mathscr{K}}\|_{\mathfrak{M}_k(X)} \leqslant C\,[(1+\|\mathscr{K}\|_{\mathfrak{M}_k(X)})(1+|t|)]^{2(\mu+1)}$$

with $\mu = N(N+1)/2 + 2[n/2] + n + k + 5$.

Before proving this theorem, let us conclude the definition of a functional calculus on $\mathscr{M}_k(X)$. Since $\mathscr{M}_k(X)$ is not a complete algebra, given $A \in \mathscr{M}_k(X)$ the exponential $e^{-2\pi i t A}$ will not necessarily belong to $\mathscr{M}_k(X)$. However, since A_X is a continuous homomorphism of Banach algebras from $\mathfrak{M}_k(X)$ onto $\mathscr{M}_k(X)$, the kernel of A_x is a closed two-sided ideal of $\mathfrak{M}_k(X)$. So, if we define

$$|||A||| = \inf\{\|\mathscr{K}\|_{\mathfrak{M}_k(X)} \mid A_X(\mathscr{K}) = A\}$$

it is easy to prove that $(\mathscr{M}_k(X), |||\ |||)$, is a Banach algebra. Moreover, Λ_X is still a continuous homomorphism and given $A \in \mathscr{M}_k(X)$, $\|A\|_{\mathscr{M}_k(X)} \leqslant C\,|||A|||$.

Even though $\mathscr{M}_k(X)$ is a selfadjoint class, there is no notion of adjoint in $\mathfrak{M}_k(X)$.

Now, given $A \in \mathscr{M}_k(X)$, the exponential $e^{-2\pi i t A}$ has a sense in $(\mathscr{M}_k(X), |||\ |||)$. If A is selfadjoint, then Theorem (2.9) asserts that $A = \Lambda_X(\mathscr{H})$, for some $\mathscr{H} = \{(p_0^{(j)}, \ldots, p_{N-1}^{(j)}), R)\}$, where the $p_0^{(j)}$ are real functions. Then, if $1/p_0 - k/n \leqslant 1/2$,

$$|||e^{-2\pi i t A}||| \leqslant C\,[(1+|||A|||)\,(1+|t|)]^{2(\mu+1)}$$

with μ as in Theorem (3.1).

Thus, if $f \in L_s^2$, $s > 2\mu + 5/2$, the Bochner integral $\int_{-\infty}^{\infty} e^{-2\pi i t A}\hat{f}(t)\,dt$ converges in $\mathscr{M}_k(X)$. Moreover,

$$\Big|\Big|\Big| \int_{-\infty}^{\infty} e^{-2\pi i t A}\hat{f}(t)\,dt \Big|\Big|\Big| \leqslant C\,(1+|||A|||)^{2(\mu+1)}\,|||f|||_{L_s^2}.$$

Since A is a linear continuous selfadjoint operator on $L^2(X)$, it admits a resolution of identity $\{E_s\}$. We have (cf. [12], p. 91) the following compatibility:

$$\int_{-\infty}^{\infty} e^{-2\pi i t A}\hat{f}(t)\,dt = \int_{-\infty}^{\infty}\Big[\int_{\sigma(A)} e^{-2\pi i t s}\,dE_s\Big]\hat{f}(t)\,dt$$

$$= \int_{\sigma(A)}\Big[\int_{-\infty}^{\infty} e^{-2\pi i t s}\hat{f}(t)\,dt\Big]\,dE_s = \int_{\sigma(A)} f(s)\,dE_s,$$

where $\sigma(A)$ denotes the spectrum of A.

We will summarize the results above in the following theorem.

(3.2) THEOREM. *Suppose* $1/p_0 - k/n \leqslant 1/2$. *Let* $\mu = N(N+1)/2 + 2[n/2] + n + k + 5$. *Let* $A \in \mathscr{M}_k(X)$ *be a selfadjoint operator. Given* $f \in L_s^2$, $s > 2\mu + 5/2$, *the Bochner integral* $\int_{-\infty}^{\infty} e^{-2\pi i t A}\hat{f}(t)\,dt$ *converges in* $\mathscr{M}_k(X)$ *to an operator which coincides with the value of* $f(A)$ *given by* $\int_{\sigma(A)} f(s)\,dE_s$. *Moreover,*

$$|||f(A)||| \leqslant C\,(1+|||A|||)^{2(\mu+1)}\,\|f\|_{L_s^2}.$$

A main ingredient in the proof of Theorem (3.1) is to have at hand a good estimate for $\|e^{-2\pi i t \mathscr{H}}\|_{\mathfrak{N}_k(X)}$.

First of all, since $\mathfrak{N}_k(X)$ is a Banach algebra, given $\mathscr{H} = \{(p_0^{(j)}, \ldots, p_{N-1}^{(j)})\} \in \mathfrak{N}_k(X)$, the exponential $e^{-2\pi i t \mathscr{H}}$ has a meaning for any complex number t. However, when $p_0^{(j)}$ is a real function for every j, $\|e^{-2\pi i t \mathscr{H}}\|_{\mathfrak{N}_k(X)}$ has polynomial growth as a function of $t \in \boldsymbol{R}$. More precisely:

(3.3) LEMMA. *Let* $\mathscr{H} = \{(p_0^{(j)}, \ldots, p_{N-1}^{(j)})\} \in \mathfrak{N}_k(X)$. *Suppose that* $p_0^{(j)}$ *is a real function for every* j. *Then*

$$e^{-2\pi i t \mathscr{H}} = \{(r_0^{(j)}(t), \ldots, r_{N-1}^{(j)}(t))\}, \quad t \in \boldsymbol{C},$$

where

$$r_h^{(j)}(t) = \sum_{s=0}^{h(h+3)/2} C_{s,h}^{(j)} \frac{(-2\pi i t)^s}{s!} e^{-2\pi i t p_0^{(j)}} \tag{3.4}$$

and $C_{s,h}^{(j)}$ *is a linear combination of* $\prod_{m=1}^{s} D_x^{\alpha^m} D_\xi^{\beta^m} p_{k_m}^{(j)}$, *with* $\sum \alpha^m = \sum \beta^m$, $\sum k_m + \sum |\alpha^m| = h$, $C_{0,0}^{(j)} = 1$. *Moreover, if* $t \in \boldsymbol{R}$, *then*

$$\|e^{-2\pi i t \mathscr{H}}\|_{\mathfrak{N}_k(X)} \leqslant C\,[(1+\|\mathscr{H}\|_{\mathfrak{N}_k(X)})\,(1+|t|)]^{\mu}, \tag{3.5}$$

where $\mu = N(N+1)/2 + 2[n/2] + n + k + 5$ *and the constant* C *does not depend on* t *or* $\mathscr{H}$.

It is in proving Lemma (3.3) that the characteristics play a crucial role. Let us briefly recall how (cf. [1], p. 50).

Given $\mathscr{U} \in \mathfrak{A}_k$, let $\mathscr{U} = a + \mathscr{B}$, where $a = (u_{0,0}, 0, \ldots, 0)$. We define inductively a sequence $\{\mathscr{B}_s\}$ of operators in $\mathfrak{A}_k$ as (see [1])

$$\mathscr{B}_0 = 1 = (1, 0, \ldots, 0), \quad \mathscr{B}_1 = \mathscr{B},$$

$$\mathscr{B}_{s+1} = \mathscr{B}\mathscr{B}_s + [a, \mathscr{B}_s], \quad s = 1, 2, \ldots,$$

where, as usual, $[a, \mathscr{B}_s]$ denotes the commutator $a\mathscr{B}_s - \mathscr{B}_s a$.

It is easy to prove by induction that given $m = 1, 2, \ldots$,

$$\mathscr{U}^m = \sum_{s=0}^{m} \binom{m}{s} \mathscr{B}_s a^{m-s}.$$

Now, if P is a polynomial,

$$P(\mathscr{U}) = \sum_{s \geqslant 0} \mathscr{B}_s P^{(s)}(u_{0,0})/s!. \tag{3.6}$$

The main fact concerning (3.6) is that the number of terms has a bound independent of the degree of P. Indeed, it was proved in [1], p. 52, that $\mathscr{B}_s = 0$ for $s \geqslant N(N+1)/2$.

Now, let $f\colon \boldsymbol{R} \to \boldsymbol{C}$ be an analytic function. If $u_{0,0}$ is a real function, we claim that

$$f(\mathscr{U}) = \sum_{s=0}^{N(N+1)/2-1} \mathscr{B}_s f^{(s)}(u_{0,0})/s!. \tag{3.7}$$

Indeed, since $\mathfrak{A}_k$ is a Banach algebra, $f(\mathscr{U})$ has a meaning. The right-hand side of (3.7) can be obtained by approaching f in the C^∞ topology with a sequence of polynomials. Furthermore, denote by $u_{h,\alpha}^{(s)}$ the coordinates of the coefficients in $\mathscr{B}_s$. It is proved in [1] that given $s = 1, 2, \ldots$, if $j > r(r+1)/2$, then $u_{h,\alpha}^{(s)} = 0$ for

$h+|\alpha| < r$. So, for a fixed s, let r be the largest integer such that $s \geqslant r(r+1)/2$. The coordinates of $\mathscr{B}_s f^{(s)}(u_{0,0}^{\prime})$ will belong to the required space if f has continuous derivatives up to order $2[n/2]+n+k+2(N-r)+s+4$. When s varies from 0 to $N(N+1)/2-1$, the maximum is attained at $s = N(N+1)/2-1$, $r = N-1$.

Finally, we can conclude that the right-hand side of (3.7) belongs to $\mathfrak{A}_k$ provided that f is continuously differentiable up to order $N(N+1)/2+ +2[n/2]+n+k+5$.

Given $\mathscr{P} = (p_0, \ldots, p_{N-1})$, p_0 a real function, we set

$$f(\mathscr{P}) = \pi_1 f(\mathscr{U}_{\mathscr{P}}) = \sum_{s=0}^{N(N+1)/2-1} \pi_1(\mathscr{B}_s) f^{(s)}(p_0)/s! . \tag{3.8}$$

Let $\pi_1 f(\mathscr{U}_{\mathscr{P}}) = (r_0, \ldots, r_{N-1})$. It is shown in [2], p. 36, that

$$r_h = \sum_{s=0}^{h(h+3)/2} C_{s,h} f^{(s)}(p_0)/s!, \tag{3.9}$$

where $C_{s,h}$ is a linear combination of $\prod_{m=1}^{s} D_x^{\alpha^m} D_\xi^{\beta^m} p_{k_m}$, $\Sigma\alpha^m = \Sigma\beta^m$, $\Sigma k_m + \Sigma|\alpha^m| = h$ and $C_{0,0} = 1$.

When $h = 0$, we get the expected relation between principal symbols:

$$\sigma_{\mathrm{p}}(f(\mathscr{P})) = f(\sigma_{\mathrm{p}}(\mathscr{P})).$$

Proof of Lemma (3.3). The expression (3.4) comes from (3.9). Since $(d/dx)^s(e^{-2\pi itx}) = (-2\pi it)^s e^{-2\pi itx}$ and $s \leqslant N(N+1)/2+2[n/2]+n+k+5$, we obtain (3.5). The estimate of the constant is deduced from the value of the coefficient $C_{s,h}^{(j)}$ in (3.4). This completes the proof of the lemma.

Proof of Theorem (3.1). Let $A = \Lambda_X(\mathscr{H}) \in \mathscr{M}_k(X)$, $\mathscr{K} = \Omega(\mathscr{H}) = \{(p_0^{(j)}, \ldots, p_{N-1}^{(j)})\} \in \mathfrak{N}_k(X)$. We claim that $e^{-2\pi it\mathscr{H}} - \Omega_1(e^{-2\pi it\mathscr{K}})$ is of the form $\{(0), R(t)\}$.

Indeed, since Ω is a continuous homomorphism of algebras and Ω_1 is a right inverse of Ω, we have

$$\Omega\left(e^{-2\pi it\mathscr{H}} - \Omega_1(e^{-2\pi it\mathscr{K}})\right) = e^{-2\pi it\mathscr{K}} - \Omega\Omega_1(e^{-2\pi it\mathscr{K}}) = 0.$$

Furthermore, Ω_1 is a continuous map from $\mathfrak{N}_k(X)$ into $\mathfrak{M}_k(X)$ and Lemma (3.3) gives an estimate of the exponential in $\mathfrak{N}_k(X)$. Thus, it suffices to estimate the norm of $\{(0), R(t)\}$ in $\mathfrak{M}_k(X)$, which coincides with $\|R(t)\|_{\mathscr{R}_k(X)}$.

Since Λ_X is a continuous homomorphism of algebras,

$$R(t) = \Lambda_X[e^{-2\pi it\mathscr{H}} - \Omega_1(e^{-2\pi it\mathscr{K}})] = e^{-2\pi itA} - \Lambda_X \Omega_1(e^{-2\pi it\mathscr{K}}).$$

$R(t)$ is a differentiable function from $\boldsymbol{R}$ into $\mathscr{R}_k(X)$, and

$$\frac{d}{dt} R(t) = e^{-2\pi itA}(-2\pi iA) - \Lambda_X \Omega_1(e^{-2\pi it\mathscr{K}}(-2\pi i\mathscr{K})) \tag{3.10}$$

or

$$\begin{aligned}\frac{d}{dt} R(t) &= [e^{-2\pi itA} - \Lambda_X \Omega_1(e^{-2\pi it\mathscr{K}})](-2\pi iA) \\ &\quad + \Lambda_X[\Omega_1(e^{-2\pi it\mathscr{K}})(-2\pi i\mathscr{H}) - \Omega_1(e^{-2\pi it\mathscr{K}}(-2\pi i\mathscr{K}))] \\ &= R(t)(-2\pi iA) + B_1(t).\end{aligned} \tag{3.11}$$

We claim that $B_1(t) \in \mathscr{R}_k(X)$, for each t. Indeed,

$$\Omega[\Omega_1(e^{-2\pi it\mathscr{K}})(-2\pi i\mathscr{H}) - \Omega_1(e^{-2\pi it\mathscr{K}}(-2\pi i\mathscr{K}))] = 0.$$

Moreover,

$$\begin{aligned}\|B_1(t)\|_{\mathscr{R}_k(X)} &= \|\Omega_1(e^{-2\pi it\mathscr{K}})(-2\pi i\mathscr{H}) - \Omega_1(e^{-2\pi it\mathscr{K}}(-2\pi i\mathscr{K}))\|_{\mathfrak{M}_k(X)} \\ &\leqslant C(1+|t|)^{\mu}.\end{aligned}$$

Since $R(0) = 0$, from (3.11) we have

$$R(t) = \int_0^t B_1(s) e^{-2\pi i(t-s)A} ds. \tag{3.12}$$

Suppose now that $p_0 = 2$. Thus,

$$\|R(t)\|_{L^2(X), L_k^2(X)} \leqslant C(1+|t|)^{\mu+1}.$$

We can also write

$$\frac{d}{dt} R(t) = (-2\pi iA) e^{-2\pi itA} - \Lambda_X \Omega_1((-2\pi i\mathscr{K}) e^{-2\pi it\mathscr{K}}) = (-2\pi iA) R(t) + B_2(t).$$

$B_2(t)$ also belongs to $\mathscr{R}_k(X)$ for each t and $\|B_2(t)\|_{\mathscr{R}_k(X)} \leqslant C(1+|t|)^{\mu}$. Thus,

$$R(t) = \int_0^t e^{-2\pi i(t-s)A} B_2(s) ds \tag{3.13}$$

and taking adjoints,

$$R^*(t) = \int_0^t B_2^*(s) e^{2\pi i(t-s)A} ds. \tag{3.14}$$

$R^*(t)$ is also a linear bounded operator from $L^2(X)$ into $L_k^2(X)$ and

$$\|R^*(t)\|_{L^2(X), L_k^2(X)} \leqslant C(1+|t|)^{\mu+1}.$$

This concludes the proof when $p_0 = 2$.

Suppose now that $1 < p_0 < 2$ and also that we have the estimate

$$\|e^{-2\pi itA}\|_{L^{p_0}(X), L^{p_0}(X)} \leqslant C(1+|t|)^{\mu+1}. \tag{3.15}$$

Since $(e^{-2\pi itA})^* = e^{2\pi itA}$, we will also have

$$\|e^{-2\pi itA}\|_{L^{p'_0}(X),L^{p'_0}(X)} \leqslant C(1+|t|)^{\mu+1}.$$

From (3.12) and (3.14) we conclude that

$$\|R(t)\|_{\mathscr{M}_k(X)} \leqslant C(1+|t|)^{2(\mu+1)}.$$

Now, let us prove (3.15). Since $R(t) = e^{-2\pi itA} - \Lambda_X \Omega_1 (e^{-2\pi it\mathscr{X}})$ and

$$\|\Lambda_X \Omega_1 (e^{-2\pi it\mathscr{X}})\|_{L^{p_0}(X),L^{p_0}(X)} \leqslant C\|\Lambda_X \Omega_1 (e^{-2\pi it\mathscr{X}})\|_{\mathscr{M}_k(X)} \leqslant C\|e^{-2\pi it\mathscr{X}}\|_{\mathfrak{M}_k(X)} \leqslant C(1+|t|)^{\mu},$$

it suffices to show that

$$\|R(t)\|_{L^{p_0}(X),L^{p_0}(X)} \leqslant C(1+|t|)^{\mu+1}.$$

The Sobolev immersion theorem shows that $L_k^{p_0}(X)$ is continuously included in $L^q(X)$ if $1/q = 1/p_0 - k/n$. Moreover, since X is a compact manifold, $L^q(X)$ is continuously included in $L^2(X)$, provided that $q \geqslant 2$. For the same reason, $L^2(X)$ is continuously included in $L^{p_0}(X)$, since $p_0 < 2$. Then from (3.13) we get the desired estimate.

Finally, that the constant involved in the estimate is of the form $C(1+\|\mathscr{H}\|_{\mathfrak{M}_k(X)})^{2(\mu+1)}$ comes from Lemma (3.3) and the proof above. This concludes the proof of the theorem.

(3.16) Remark. This theorem was proved in [2] for $p_0 = 2$. It is clear that the crucial step in the proof above is to majorize $\|e^{-2\pi itA}\|_{L^{p_0}(X),L^{p_0}(X)}$ by a polynomial in t. This is obvious when $p_0 = 2$. When $p_0 < 2$ the proof relies heavily on the Sobolev immersion theorem and the fact that X is a compact manifold. So, the condition $1/p_0 - k/n \leqslant 1/2$ has to be imposed for technical reasons. We do not know if (3.15) remains true without this assumption.

It is also an open problem to obtain a polynomial estimate of

$$\|e^{-2\pi itA}\|_{L^p(\mathbf{R}^n),L^p(\mathbf{R}^n)}.$$

A positive answer would permit us to apply the results in [4] to construct a functional calculus in the euclidean case.

The Weyl formula can be used to construct a functional calculus over r-tuples of selfadjoint operators, without additional work. Indeed, let $A_1, \ldots, A_r$ be r selfadjoint operators in $\mathscr{M}_k(X)$. Consider the Bochner integral

$$\int_{\mathbf{R}^r} e^{-2\pi it\cdot A} \hat{f}(t)\,dt,$$

where $t\cdot A = t_1A_1 + \ldots + t_rA_r$. We can split this integral into two terms, namely, $\int_{|t|\leqslant 1} + \int_{|t|\geqslant 1}$. The first one can be easily handled. Now,

$$\int_{|t|\geqslant 1} e^{-2\pi it\cdot A} \hat{f}(t)\,dt = \int_{|t|\geqslant 1} e^{-2\pi i|t|(t/|t|\cdot A)} \hat{f}(t)\,dt,$$

and it reduces to the 1-dimensional case (cf. [1], p. 58, and [2], p. 62, for details).

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