STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, W. Orlicz** (*Editor-in-Chief*), **A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

Manuscripts and correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA

Śniadeckich 8, 00-950 Warszawa, Poland

Correspondence concerning exchange should be addressed to

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

Śniadeckich 8, 00-950 Warszawa, Poland

The journal is available at your bookseller or at

ARS POLONA

Krakowskie Przedmieście, 7, 00-068 Warszawa, Poland

ISBN 83-01-09499-0 ISSN 0039-3223

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Positive-definite kernels, length functions on groups and a noncommutative von Neumann inequality

by

MAREK BOŻEJKO (Wrocław)

Abstract. We present a method of construction of a large class of positive-definite operator-valued kernels. As an application we show that the distance function on an $\boldsymbol{R}$-tree is a negative-definite kernel and an abstract length function in the sense of Chiswell on a discrete group is a negative-definite function. Moreover, a noncommutative version of von Neumann's inequality is given.

1. Introduction. Let S be a set and $L(H)$ the space of bounded linear operators on a Hilbert space H. Let N be a *kernel* on S, i.e. a function on $S\times S$ whose values belong to $L(H)$. We say that N is a *negative-definite kernel* if

$$N(x,y) = N(y,x)^*$$

and

$$\sum_{x,y\in S} \langle N(x,y)\,\alpha(x),\,\alpha(y)\rangle \leqslant 0$$

for all finitely supported functions $\alpha\colon S\to H$ such that $\sum_{x\in S}\alpha(x)=0$.

An operator-valued kernel N on a set S with a metric ϱ is called *convex* if for $x,y,z\in S$ such that $\varrho(x,y)=\varrho(x,z)+\varrho(z,y)$ we have $N(x,y)=N(x,z)+N(z,y)$ (in particular, $N(x,x)=0$), $N(x,y)=N(y,x)^*$ and $N(x,y)$ is a dissipative operator.

In the note we show that every convex kernel on an $\boldsymbol{R}$-tree is almost positive-definite (see definition in Section 2) and hence every metric on an $\boldsymbol{R}$-tree is a negative-definite kernel.

In Section 3 we give some properties of positive- and negative-definite kernels connected with the Schur product on $L(H)$.

The main result of this paper is the following theorem concerning a construction of positive-definite kernels:

If K_i $(i=1,2)$ are positive-definite kernels on S_i and $S_1\cap S_2=\{x_0\}$, $K_i(x,x)=I$ for all $x\in S_1\cup S_2$, then the kernel defined by

$$K(s_1,s_2) = K_2(x_0,s_2)\,K_1(s_1,x_0) \quad \text{for } s_i\in S_i.$$

1985 *Mathematics Subject Classification*: Primary 43A65; Secondary 20E05, 22D12, 43A22.

Key words and phrases: positive-definite operator-valued kernels, $\boldsymbol{R}$-trees, abstract length function, Schur product, von Neumann inequality.

$$K(s_2, s_1) = K(s_1, s_2)^*,$$

$$K|_{S_i \times S_i} = K_i \quad (i = 1, 2),$$

is a positive-definite kernel on $S_1 \cup S_2$.

Next, in Section 6 we consider abstract length functions L on a discrete group G introduced and studied by Lyndon [25] and Chiswell [13]; L is a real-valued function on G which satisfies the following properties: $L(e) = 0$, $L(x) = L(x^{-1})$ and $d(x, y) \geq \min(d(x, z), d(z, y))$ for $x, y, z \in G$, where $d(x, y) = \frac{1}{2}(L(x) + L(y) - L(xy^{-1}))$.

We show that each length function on a group G is negative-definite and for each $\lambda \geq 0$, $e^{-\lambda L(x)}$ is a positive-definite function on G.

This is an extension of the theorem of Haagerup [20] who showed the above results in the case of a free group.

In Section 7 we present some theorems on operator-valued positive-definite functions on free products of groups. We prove that if $G = \circledast_{j=1}^{m} G_j$ is the free product group and $u_j\colon G_j \to L(H)$ are positive-definite functions, then the free product function u defined as

$$u(a_{1,j_1} a_{2,j_2} \cdot \ldots \cdot a_{n,j_n}) = \prod_{k=1}^{n} u_{j_k}(a_{k,j_k}),$$

where $a_{k,j_k} \in G_{j_k}$ and $j_k \neq j_{k+1}$, is a positive-definite function on G. This theorem is a generalization of the author's result [9] in the case of scalar positive-definite functions.

As an application of our main Theorem 4.1 and the free product theorem we get the following noncommutative von Neumann inequality:

If $p = p(x_1, \ldots, x_n)$ is a polynomial in the noncommutative indeterminates $x_1, \ldots, x_n$, then for arbitrary contractions A_i (i.e. $\|A_i\| \leq 1$) on a Hilbert space

$$\|p(A_1, \ldots, A_n)\| \leq \sup\{\|p(U_1, \ldots, U_n)\|\colon U_k \text{ unitary, finite-dimensional}, \; 1 \leq k \leq n\}.$$

2. Positive-definite operator-valued kernels. Let S be a set and H a Hilbert space. Denote by $F(S, H)$ the set of finitely supported functions $\alpha\colon S \to H$. Its subset consisting of functions α such that $\sum_{x \in S} \alpha(x) = 0$ will be denoted by $F_0(S, H)$.

A function $K(x, y)$ defined on the product space $S \times S$ whose values are bounded linear operators on H is called a *positive-definite kernel*, respectively an *almost positive-definite kernel*, if for $x, y \in S$

$$K(x, y) = K(y, x)^*$$

and the inequality

$$\sum_{x,y \in S} \langle K(x, y)\alpha(x), \alpha(y)\rangle \geq 0$$

holds for all $\alpha \in F(S, H)$, respectively $\alpha \in F_0(S, H)$.

A typical example of a positive-definite kernel is the *Gram kernel*, which is of the following form: $K(x, y) = f(y)^* f(x)$ for a mapping $f\colon S \to L(H)$.

In some sense every positive-definite operator-valued kernel is the Gram kernel (see Aronszajn [5] and Kunze [24] for the proof of this fact).

In the sequel we need the following lemma:

LEMMA 2.1. *Let K be a positive-definite operator-valued kernel on S such that $K(x, x) = I$ for each $x \in S$. Then for every $\alpha \in F(S, H)$ and $x_0 \in S$*

$$\sum_{x,y \in S} \langle K(x, y)\alpha(x), \alpha(y)\rangle \geq \Big\|\sum_{y \in S} K(x_0, y)^* \alpha(y)\Big\|^2.$$

Proof. Define a bilinear form $[\cdot,\cdot]$ on the linear space $F(S, H)$ as follows: For $\alpha, \beta \in F(S, H)$ we put

$$[\alpha, \beta] = \sum_{x,y \in S} \langle K(x, y)\alpha(x), \beta(y)\rangle.$$

Since K is a positive-definite kernel, the form $[\cdot,\cdot]$ is a positive-definite scalar product and the Schwarz inequality holds.

Fix $x_0 \in S$ and a vector $h \in H$ and consider the following function from $F(S, H)$:

$$\beta_{x_0,h}(s) = \begin{cases} h & \text{if } s = x_0, \\ 0 & \text{if } s \neq x_0. \end{cases}$$

Then by the Schwarz inequality we get

$$|[\alpha, \beta_{x_0}]|^2 \leq [\alpha, \alpha]\langle K(x_0, x_0)h, h\rangle.$$

On the other hand,

$$|[\alpha, \beta_{x_0}]|^2 = \Big|\Big\langle \sum_{x \in S} K(x, x_0)\alpha(x), h\Big\rangle\Big|^2.$$

If we choose a vector $h \in H$ such that $\|h\| = 1$ and

$$\Big\langle \sum_{x \in S} K(x, x_0)\alpha(x), h\Big\rangle = \Big\|\sum_{x \in S} K(x, x_0)\alpha(x)\Big\|,$$

then the proof of Lemma 2.1 is complete.

The following lemma and corollary are simple generalizations of scalar case results so we omit the proofs (see Donoghue [18] for details).

LEMMA 2.2. *Let N be an almost positive-definite kernel on S and fix z in S. Then the function*

$$K(x, y) = N(x, y) - N(z, y) - N(x, z) + N(z, z)$$

is a positive-definite kernel on S.

COROLLARY 2.3. *Every almost positive-definite kernel N on S is of the following form:*

$$N(x, y) = K(x, y) + V(x) + V(y)^*,$$

where K is positive-definite and $V(x) \in L(H)$ for each $x \in S$.

3. Schur product on $L(H)$. Fix a complete orthonormal basis $\{e_i\}_{i\in I}$ in a Hilbert space H and consider the *Schur product* on $L(H)$ defined as follows:

If $A, B \in L(H)$ and $a_{ij} = \langle Ae_i, e_j\rangle$, $b_{ij} = \langle Be_i, e_j\rangle$, then there exists a unique bounded linear operator C on H such that $\langle Ce_i, e_j\rangle = a_{ij} b_{ij}$ for all $i, j \in I$.

The operator $C = A \circ B$ is called the *Schur product* of A and B.

For a nice explanation and some properties of the Schur product we propose the paper of G. Bennett [6].

For $A \in L(H)$ we define $\bar{A} \in L(H)$ in the following manner: $\langle \bar{A}e_i, e_j\rangle = \langle e_j, Ae_i\rangle$ for $i, j \in I$.

Now we present without proof the main properties of the Schur product.

THEOREM (Schur). (1) *If A and B are positive operators in $L(H)$, then so is $A \circ B$.*

(2) *The linear space $L(H)$ with the Schur product and the operator norm is a $*$-commutative Banach algebra. In particular, for $A, B \in L(H)$ we have*

$$\|A \circ B\| \leqslant \|A\|\,\|B\|, \quad (A \circ B)^* = A^* \circ B^*.$$

LEMMA 3.1. (i) *If K_1, K_2 are positive-definite kernels on S, then so is the kernel $K_1(x, y) \circ K_2(x, y)$.*

(ii) *If $A(x) \in L(H)$ for $x \in S$, then the kernel $A(x) \circ \overline{A(y)}$ is positive-definite.*

Proof. (i) Let $\alpha \in F(S, H)$ and $\alpha(x) = \sum_{i\in I} a_i(x) e_i$. Then we have for $K(x, y) = K_1(x, y) \circ K_2(x, y)$ the following equality:

$$P = \sum_{x,y\in S} \langle K(x, y)\,\alpha(x), \alpha(y)\rangle$$

$$= \sum_{x,y\in S} \sum_{i,j\in I} a_i(x)\,\overline{a_j(y)}\,\langle K_1(x, y) e_i, e_j\rangle \langle K_2(x, y) e_i, e_j\rangle.$$

Since K_i $(i = 1, 2)$ is a positive-definite kernel, therefore by the Aronszajn–Kolmogorov–Kunze Theorem there exist a Hilbert space $H_0 \supset H$ and a mapping f_j: $S \to L(H_0)$ such that

$$K_j(x, y) = f_j(y)^* f_j(x) \quad \text{for } j = 1, 2.$$

This implies that

$$P = \sum_{x,y\in S} \sum_{i,j\in J} a_i(x)\,\overline{a_j(y)}\,\langle f_1(x) e_i \otimes f_2(x) e_i, f_1(y) e_j \otimes f_2(y) e_j\rangle$$

$$= \Big\| \sum_{i\in I} a_i(x)\, b_i(x) \Big\|^2,$$

where $b_i(x) = f_1(x) e_i \otimes f_2(x) e_i$.

In order to show (ii) take as before $\alpha(x) = \sum_{i\in I} a_i(x) e_i$. Then one can verify the following formula:

$$\sum_{x,y\in S} \langle (A(x) \circ \overline{A(y)})\,\alpha(x), \alpha(y)\rangle = \sum_{i,j\in J} \Big| \sum_{x\in S} a_i(x) \langle A(x) e_i, e_j\rangle \Big|^2$$

and the proof of Lemma 3.1 is complete.

4. Main construction theorem. In the present section we give a construction of a large class of positive-definite, respectively almost positive-definite kernels.

Let K_i be a kernel on S_i $(i = 1, 2)$ and let $S_1 \cap S_2 = \{x_0\}$ and $K_1(x_0, x_0) = K_2(x_0, x_0)$. We define a kernel K on $S_1 \cup S_2$ in the following way:

(1) $K|_{S_i \times S_i} = K_i$ $(i = 1, 2)$.

(2) (Markov property) For $s_i \in S_i$ $(i = 1, 2)$

$$K(s_1, s_2) = K_2(x_0, s_2) K_1(s_1, x_0), \quad K(s_2, s_1) = K(s_1, s_2)^*.$$

The kernel K is denoted by $K_1 *_{x_0} K_2$ and called the *Markov product* of K_1 and K_2.

Analogously we define the Markov sum: Let N_i be a kernel on S_i and $N_1(x_0, x_0) = N_2(x_0, x_0)$. The kernel N on $S_1 \cup S_2$ such that:

(1') $N|_{S_i \times S_i} = N_i$ $(i = 1, 2)$;

(2') For $s_i \in S_i$ $(i = 1, 2)$

$$N(s_1, s_2) = N_1(s_1, x_0) + N_2(x_0, s_2), \quad N(s_2, s_1) = N(s_1, s_2)^*,$$

is called the *Markov sum* of N_1 and N_2 and is denoted by $N = N_1 +_{x_0} N_2$.

THEOREM 4.1. (i) *If K_1, K_2 are positive-definite kernels on S_1, S_2, respectively, and $K_i(x, x) = I$ for all $x \in S_1 \cup S_2$, then the Markov product $K = K_1 *_{x_0} K_2$ is also a positive-definite kernel.*

(ii) *If N_1, N_2 are almost positive-definite kernels on S_1, S_2, respectively, and $N_i(x_0, x_0) = 0$, then the Markov sum $N = N_1 +_{x_0} N_2$ is almost positive-definite.*

Proof. (i) Since K_i is positive-definite, therefore by Lemma 2.1

$$(*)\qquad \sum_{x,y\in S_i} \langle K_i(x, y)\,\alpha(x), \alpha(y)\rangle \geqslant \Big\| \sum_{x\in S_i} K_i(x, x_0)\,\alpha(x)\Big\|^2$$

for all $\alpha\in F(S_i, H)$.

Take $\beta\in F(S_1\cup S_2, H)$. Then we can decompose β as $\beta=\alpha_1+\alpha_2$, where $\mathrm{supp}(\alpha_1)\subset S_i$, $i=1,2$. Moreover, we clearly have

$$Q = \sum_{x,y\in S_1\cup S_2} \langle K(x, y)\,\beta(x), \beta(y)\rangle = \sum_{i,j=1}^{2} A_{ij},$$

where

$$A_{ij} = \sum_{\substack{x\in S_i\\ y\in S_j}} \langle K(x, y)\,\beta(x), \beta(y)\rangle.$$

If we set $A_i = \sum_{x\in S_i} K(x, x_0)\alpha_i(x)$, then by the definition of the Markov product we get $A_{ij} = \langle A_i, A_j\rangle$ for $i\neq j$ and by $(*)$, $A_{ii}\geqslant \langle A_i, A_i\rangle$. Therefore $Q\geqslant \|A_1+A_2\|^2$ and the proof of (i) is complete.

(ii) Choose $\beta\in F_0(S_1\cup S_2, H)$. Since only $x_0\in S_1\cap S_2$, we can decompose $\beta=\alpha_1+\alpha_2$ in such a way that $\mathrm{supp}(\alpha_i)\subset S_i$ and $\alpha_i\in F_0(S_i, H)$, $i=1, 2$. Putting

$$B_{ij} = \sum_{\substack{x\in S_i\\ y\in S_j}} \langle N(x, y)\,\beta(x), \beta(y)\rangle \qquad (i, j = 1, 2)$$

and using the fact that N is the Markov sum we infer that $B_{ij}=0$ for $i\neq j$. Hence we conclude that

$$\sum_{x,y\in S_1\cup S_2} \langle N(x, y)\,\beta(x), \beta(y)\rangle = B_{11}+B_{22}\geqslant 0,$$

since N_1 and N_2 are almost positive-definite kernels.

5. Trees and *R*-trees. By an *arc* in a metric space we mean a subset which is homeomorphic (in the metric topology) to a (possibly degenerate) compact interval in $\boldsymbol{R}$.

An ***R**-tree* is a nonempty metric space in which any two points are joined by a unique arc, and in which every arc is isometric to a compact interval in the real line.

A *tree* is a one-dimensional simply connected simplicial complex. Every tree is a metric space with the metric $\mathrm{dist}(P,Q)$ given by the length of the geodesic from P to Q. We have $\mathrm{dist}(P, Q)=1$ if and only if P and Q are adjacent.

By Theorem II.1.9 of Morgan and Shalen [26], every tree (T, dist) can be embedded isometrically in an $\boldsymbol{R}$-tree $T_{\boldsymbol{R}}$. The $\boldsymbol{R}$-tree $T_{\boldsymbol{R}}$ is the geometric realization of the tree T.

Now we introduce two natural classes of kernels on a metric space.

Let (S, ϱ) be a metric space. An operator-valued kernel $\varphi\colon S\times S\to L(H)$ is called *quasi-multiplicative* if:

(1) $\varphi(x, x) = I$ for all $x\in S$.

(2) $\varphi(x, y) = \varphi(y, x)^*$ for all $x, y\in S$.

(3) $\|\varphi(x, y)\|\leqslant 1$ for all $x, y\in S$.

(4) For all $x, y, z\in S$ such that $\varrho(x, y) = \varrho(x, z)+\varrho(z, y)$ we have $\varphi(x, y) = \varphi(z, y)\,\varphi(x, z)$.

A function $\psi\colon S\times S\to L(H)$ is called *convex* on a metric space (S, ϱ) if:

(1′) $\psi(x, x) = 0$ for $x\in S$.

(2′) $\psi(x, y) = \psi(y, x)^*$ for all $x, y\in S$.

(3′) $\psi(x, y)$ is a dissipative operator for all $x, y\in S$.

(4′) For all $x, y, z\in S$, if $\varrho(x, y) = \varrho(x, z)+\varrho(z, y)$, then $\psi(x, y) = \psi(x, z)+\psi(z, y)$.

EXAMPLE 5.1. Let (S, ϱ) be a metric space and let A be a hermitian operator on a Hilbert space which is a positive contraction. Then the kernel $\Delta_A(x, y) = A^{\varrho(x,y)}$ is quasi-multiplicative on S.

EXAMPLE 5.2. Consider the group of integers $\boldsymbol{Z}$ with the natural metric and a contraction A on a Hilbert space. Then the kernel

$$F_A(n, m) = \begin{cases} A^{(n-m)} & \text{if } n\geqslant m,\\ A^{*(m-n)} & \text{if } n<m,\end{cases}$$

is quasi-multiplicative on $\boldsymbol{Z}$.

The following result will follow from what has been proved in the earlier sections.

THEOREM 5.1. *Let (T, ϱ) be an $\boldsymbol{R}$-tree.*

(i) *Every quasi-multiplicative kernel φ on T is positive-definite.*

(ii) *Every convex function ψ on T is an almost positive-definite kernel.*

Proof. (i) It is sufficient to show that φ is a positive-definite kernel on each finite subset of T. We proceed by induction on the cardinality of a finite set E in T.

Fix $a\in E$ and for $x, y\in E$ put

$$(x, y)_a = \tfrac{1}{2}\big(\varrho(x,a)+\varrho(y,a)-\varrho(x, y)\big).$$

Let

$$E_{\max} = \{(x, y) \in E \times E\colon\ x \neq y \text{ and } (x, y)_a = \max\{(u, s)_a\colon\ u \neq s,\ u, s \in E\}\}.$$

Then we have two possibilities:

(I) There exists $(x, y) \in E_{\max}$ which belongs to a unique geodesic coming from the point a, i.e. $E = (E \backslash \{x\}) \cup \{x, y\}$ or $E = (E \backslash \{y\}) \cup \{x, y\}$. Now we can apply Theorem 4.1 and the induction assumption.

(II) There exist $\{x, y\} \in E_{\max}$ and $z \in T \backslash E$ such that $E \cup \{z\} = \tilde{E} \cup \{x, z\} \cup \{y, z\}$, where $|\tilde{E}| = |E| - 1$, $\tilde{E} \cap \{x, z\} = \{z\}$ and $\tilde{E} \cap \{y, z\} = \{z\}$. As before we can apply Theorem 4.1 and the induction hypothesis.

It remains to prove that φ is positive-definite on each two-element set $T_2 = \{\boldsymbol{a}, \boldsymbol{b}\}$. Take $\alpha \in F(T_2, H)$ and set $C = \varphi(\boldsymbol{a}, \boldsymbol{b})$, $\alpha(\boldsymbol{a}) = a$, $\alpha(\boldsymbol{b}) = b$. Consider now

$$W_1 = \sum_{x, y \in T_2} \langle \varphi(x, y)\alpha(x), \alpha(y)\rangle = \|C(a) + b\|^2 + \|a\|^2 - \|C(a)\|^2.$$

Since by the definition of quasi-multiplicative kernel C is a contraction, the proof of part (i) of the theorem is now complete.

The proof of (ii) is exactly the same, we only need to consider the case of the tree $T_2 = \{\boldsymbol{a}, \boldsymbol{b}\}$.

Since the operator $\psi(\boldsymbol{a}, \boldsymbol{b}) = D$ is dissipative and $\psi(x, x) = 0$, we find that for each $\alpha \in F_0(T_2, H)$ we have

$$W_2 = \sum_{x, y \in T_2} \langle \psi(x, y)\,\alpha(x), \alpha(y)\rangle = -2\operatorname{Re}\langle D(a), a\rangle \geqslant 0,$$

where $a = \psi(\boldsymbol{a})$, $b = \psi(\boldsymbol{b})$ and $a + b = 0$.

This completes the proof of Theorem 5.1.

COROLLARY 5.2. *If (T, ϱ) is an $\boldsymbol{R}$-tree, then the metric (x, y) is a negative-definite kernel and for each $\lambda \geqslant 0$ the kernel $e^{-\lambda\varrho(x,y)}$ is positive-definite.*

Remark. (i) In the case of trees the last corollary was obtained by many authors and by different methods (see e.g. Haagerup [20], Alperin [2], Watatani [31], and Julg and Valette [23]).

(ii) Using a theorem of Schoenberg (see Donoghue [18], p. 136) and Corollary 5.2 we find that every $\boldsymbol{R}$-tree can be isometrically embedded in a Hilbert space.

6. Abstract length functions on groups. I. M. Chiswell [13] considered real-valued functions on a group G which satisfy the following three axioms of Lyndon [25]:

(1) $L(e) = 0$.

(2) $L(x^{-1}) = L(x)$ for all $x \in G$.

(3) If $d(x, y) = \frac{1}{2}(L(x) + L(y) - L(xy^{-1}))$, then for all $x, y, z \in G$

$$d(x, y) \geqslant \min(d(x, z), d(z, y)).$$

The function L on a discrete group G is called a *length function* on G. If a group G acts by isometries on an $\boldsymbol{R}$-tree (T, ϱ), then a *based length function* L_p associated to a base point $p \in T$ is defined by $L_p(x) = \varrho(x \cdot p, p)$.

It is not difficult to verify that each based length function L_p on a discrete group G is a length function.

Conversely, given a length function L on a discrete group G, Chiswell [13] has constructed an action of G on an $\boldsymbol{R}$-tree T and a point $p \in T$ such that the based length function L_p is equal to L.

We recall that a function f on a group G whose values are linear operators on a Hilbert space is *positive-definite*, respectively *negative-definite*, if the corresponding kernel $K_f(x, y) = f(y^{-1}x)$ is positive-definite, respectively negative-definite on G.

The classical Gelfand–Naimark–Segal–Nagy theorem (see [27]) says that for every positive-definite function f on a group G such that $f(e) = I$ and $f(x) \in L(H)$ there exists a unitary representation π of G on a Hilbert space $H_0 \supset H$ such that for $s \in G$

$$f(s) = P_{H0}\,\pi(s)|_H,$$

where P_{H0} is the orthogonal projection from H_0 onto H.

Now we can state the following theorem:

THEOREM 6.1. *Every length function L on a discrete group is negative-definite and for each $\lambda \geqslant 0$ the function $\gamma_\lambda(x) = e^{-\lambda L(x)}$ is positive-definite.*

The proof follows immediately from the theorem of Chiswell and Corollary 5.2.

Remark 6.2. If a group G has a length function which is unbounded, then G does not have the Kazhdan property. This follows immediately from the theorem of Akemann and Walter [1] which says that a group has the Kazhdan property if and only if every negative-definite function on G is bounded.

Remark 6.3. In [12] it is shown that the natural length function $l(x)$ on a Coxeter group is also negative-definite (see N. Bourbaki [8] for definitions and main properties of Coxeter groups). It is not clear when the length function $l(x)$ comes from an action of the group G on an $\boldsymbol{R}$-tree.

7. Free product operator-valued functions on a free product group. Using Theorem 5.1 one can obtain a large family of positive-definite functions on a free group or a free product of groups (see also [9], [10], [16]).

Now we extend and give other proofs of the results of the author [9], [10] in the vector-valued case.

Let G_j, $j \in I \subseteq N$, be discrete groups and let $G = \circledast_{j\in I} G_j$ be the free product group. Consider functions $u_j\colon G_j \to L(H)$ such that $u_j(e) = \text{const}$. We define the *free product function* u on G in the following way: If $x \in G$, $x \neq e$ and x has the unique representation as

$$(**) \qquad x = a_{1,i_1} \ldots a_{k,i_k},$$

where $a_{j,i_j} \in G_{i_j}$ and $i_j \neq i_{j+1}$, $j = 1, \ldots, k-1$, then we put

$$u(x) = \prod_{j=1}^{k} u_{i_j}(a_{j,i_j})$$

and $u(e) = u_1(e)$.

The free product function u will be denoted by $\circ_{j\in I} u_j$.

If we introduce the *block length function* $\| \ \|$ on the free product group $G = \circledast_{j\in I} G_j$ putting $\|e\| = 0$ and $\|x\| = k$ if $x \in G$ is of the form (**), then one can verify that $\| \ \|$ is a length function as considered in Section 6 (see also [14]). It comes from the natural action of G on a corresponding tree $T(G)$ (see Serre [29], p. 32).

Next it is easy to observe that for each free product function $u = \circ_{j\in I} u_j$ with $u_j(e) = I$ on the free product group $G = \circledast_{j\in I} G_j$, the kernel $u(y^{-1}x) = K_u(x, y)$ is quasi-multiplicative on the tree $T(G)$.

Therefore as another application of Theorem 5.1 we get a simple proof of the following theorem, which in the scalar case was given in [9], [10]:

THEOREM 7.1. *Let G_j be discrete groups. If $u_j\colon G_j \to L(H)$, $u_j(e) = I$, are positive-definite operator-valued functions, then the free product function $u = \circ_{j\in I} u_j$ is positive-definite on the free product group $G = \circledast_{j\in I} G_j$.*

Remark 7.2. If we consider the amalgamated free product group $G = \circledast_{A, j\in I} G_j$, then by the Baas–Serre theorem G also acts by isometries on a suitable tree $T(G)$ (see [29]), and again by our Theorem 5.1 we can obtain a result of M. A. Picardello [28] about positive-definite functions on the amalgamated group G.

8. Noncommutative von Neumann inequality. If $p = p(x_1, \ldots, x_n)$ is a polynomial in the noncommutative indeterminates $x_1, \ldots, x_n$, then we can consider p as a function on the free group $\boldsymbol{F}_n$ supported by the free semigroup generated by the free generators $x_1, \ldots, x_n$.

THEOREM 8.1. *Let A_k $(k = 1, \ldots, n)$ be contractions on a Hilbert space H and p a polynomial as above. Then*

$$\|p(A_1, \ldots, A_n)\| \leqslant \|p\|_{C^*(\boldsymbol{F}_n)}$$
$$= \sup\{\|p(U_1, \ldots, U_n)\|:\ U_j \text{ unitary finite-dimensional},\ 1 \leqslant j \leqslant n\},$$

where $\|p\|_{C^*(\boldsymbol{F}_n)} = \sup\{\|\sigma(p)\|:\ \sigma$ *a unitary representation of* $\boldsymbol{F}_n\}$.

Proof. Take the group of integers $\boldsymbol{Z}$ (or the tree $\boldsymbol{Z}$). By Example 5.2 and Theorem 5.1 we know that for each contraction operator A on a Hilbert space the function

$$\varphi_A(n) = \begin{cases} A^n & \text{if } n \geqslant 0, \\ A^{*|n|} & \text{if } n < 0, \end{cases}$$

is positive-definite on $\boldsymbol{Z}$ (see also [27] for other proofs of this fact).

By Theorem 7.1 we can form the free product function $\varphi = \circ_{j=1}^{n} \varphi_{A_j}$ on the free group $\boldsymbol{F}_n$ which is a positive-definite function. By the Gelfand–Naimark–Segal–Nagy theorem there exists a unitary representation π of $\boldsymbol{F}_n$ on a Hilbert space $H_0 \supset H$ such that

$$\varphi(x) = P_{H_0} \pi(x)|_H \quad \text{for } x \in \boldsymbol{F}_n,$$

where P_{H_0} is the orthogonal projection from H_0 into H.

Note that if $x = x_{i_1}^{m_1} \cdot \ldots \cdot x_{i_k}^{m_k}$, where $m_j \geqslant 0$, $i_j \neq i_{j+1}$, then

$$\varphi(x) = A_{i_1}^{m_1} \ldots A_{i_k}^{m_k}.$$

Therefore

$$p(A_1, \ldots, A_n) = \sum_{x\in \boldsymbol{F}_n} p(x)\, \varphi(x) = P_{H_0}\Big(\sum_{x\in \boldsymbol{F}_n} p(x)\, \pi(x)\Big).$$

Hence we get

$$\|p(A_1, \ldots, A_n)\| \leqslant \Big\|\sum_{x\in \boldsymbol{F}_n} p(x)\, \pi(x)\Big\| = \|\pi(p)\| \leqslant \|p\|_{C^*(\boldsymbol{F}_n)}.$$

On the other hand, the free group is maximally almost periodic (see [22], (22.22)(d)). This means that for all functions f with finite support on $\boldsymbol{F}_n$

$$\|f\|_{C^*(\boldsymbol{F}_n)} = \sup\{\|\omega(f)\|:\ \omega \text{ unitary finite-dimensional}\}.$$

This proves the theorem.

References

[1] C. A. Akemann and M. E. Walter, *Unbounded negative definite functions*, Canad. J. Math. 33 (1981), 862–871.
[2] R. C. Alperin, *Locally compact groups acting on trees and property T*, Monatsh. Math. 93 (1982), 261–265.
[3] R. C. Alperin and H. Bass, *Length functions of group actions on Λ-trees*, to appear.
[4] R. C. Alperin and K. Moss, *Real-valued Archimedean length functions in groups*, to appear.
[5] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. 68 (1950), 337–404.
[6] G. Bennett, *Schur multipliers*, Duke Math. J. 44 (1977), 603–639.
[7] C. Berg and F. Forst, *Potential Theory on Locally Compact Abelian Groups*, Ergeb. Math. Grenzgeb. 87, Springer, New York 1975.

[8] N. Bourbaki, *Groupes et algèbres de Lie*, Chap. IV, *Groupes de Coxeter et systèmes de Tits*, Hermann, Paris 1968.
[9] M. Bożejko, *Positive definite functions on the free group and the noncommutative Riesz product*, Boll. Un. Mat. Ital. (6) 5-A (1986), 13–21.
[10] —, *Uniformly bounded representations of free groups*, J. Reine Angew. Math. 377 (1987), 170–186.
[11] —, *Positive definite bounded matrices and a characterization of amenable groups*, Proc. Amer. Math. Soc. 95 (1985), 357–360.
[12] M. Bożejko, T. Januszkiewicz and R. Spatzier, *An infinite Coxeter group is not Kazhdan*, J. Operator Theory 19 (1988), 63–68.
[13] I. M. Chiswell, *Abstract length functions in groups*, Math. Proc. Cambridge Philos. Soc. 80 (1976), 451–463.
[14] —, *Length functions and free products of groups*, Proc. London Math. Soc. 43 (1981), 42–58.
[15] M. Culler and J. W. Morgan, *Group actions on R-trees*, to appear.
[16] L. De Michele and A. Figà-Talamanca, *Positive definite functions on free groups*, Amer. J. Math. 102 (1980), 503–509.
[17] P. G. Dixon and S. W. Drury, *Unitary dilations, polynomial identities and the von Neumann inequality*, Math. Proc. Cambridge Philos. Soc. 99 (1986), 115–122.
[18] W. F. Donoghue, Jr., *Monotone Matrix Functions and Analytic Continuation*, Springer, Berlin 1974.
[19] A. Figà-Talamanca and M. A. Picardello, *Harmonic Analysis on Free Groups*, Lecture Notes in Pure and Appl. Math. 87, Dekker, New York 1983.
[20] U. Haagerup, *An example of a non nuclear C^*-algebra, which has the metric approximation property*, Invent. Math. 50 (1979), 279–293.
[21] N. Harrison, *Real length functions in groups*, Trans. Amer. Math. Soc. 174 (1972), 77–106.
[22] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol. I, Springer, Berlin 1963.
[23] P. Julg and A. Valette, *K-theoretic amenability for $SL_2(Q_p)$, and the action on the associated tree*, J. Funct. Anal. 58 (1984), 194–215.
[24] R. A. Kunze, *Positive definite operator-valued kernels and unitary representations*, in: Functional Analysis, Proc. Conf., Irvine 1966, Thompson and Academic Press, 1967, 235–247.
[25] R. C. Lyndon, *Length functions in groups*, Math. Scand. 12 (1963), 209–234.
[26] J. W. Morgan and P. B. Shalen, *Valuations, trees, and degenerations of hyperbolic structures, I*, Ann. of Math. 120 (1984), 401–476.
[27] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, 1970.
[28] M. A. Picardello, *Positive definite functions and L^p convolution operators on amalgams*, Pacific J. Math., 123 (1) (1986), 209–221.
[29] J.-P. Serre, *Trees*, Springer, Berlin 1980.
[30] J. Tits, *A 'theorem of Lie–Kolchin' for trees*, in: Contributions to Algebra: A Collection of Papers Dedicated to Ellis Kolchin, Academic Press, New York 1977, 377–388.
[31] Y. Watatani, *Property (T) of Kazhdan implies property (FA) of Serre*, Math. Japon. 27 (1981), 97–103.

INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Received November 10, 1986
Revised version October 27, 1988
(2240)

STUDIA MATHEMATICA, T. XCV (1989)

Incomplete normed algebra norms on Banach algebras

by

BOHDAN J. TOMIUK (Ottawa, Ont.) and BERTRAM YOOD (University Park, Pa.)

Abstract. Let A be a semisimple Banach algebra. In various analytic situations one considers (incomplete) normed algebra norms on A and the completions of A in these norms. A study is made of all possible normed algebra norms and completions for classes of semisimple Banach algebras.

1. Introduction. An original impulse for this investigation came from the theory of generalized almost periodic functions. Let $A = \mathrm{AP}(G)$ be the set of all almost periodic functions on a topological group G considered as a Banach algebra under the sup norm, pointwise addition and convolution multiplication. For the classical case $G = \boldsymbol{R}$, the reals, and $1 \leqslant p < \infty$ the Stepanov S^p-almost periodic functions can be considered as the completion of $\mathrm{AP}(\boldsymbol{R})$ in an incomplete normed algebra norm $\|f\|_{S^p}$ on $\mathrm{AP}(\boldsymbol{R})$. Likewise the Besicovitch B^p-almost periodic functions arise in this way. Similarly we may consider completions in the noncommutative situation of $\mathrm{AP}(G)$. The question naturally arises whether there are any interesting properties shared by all possible completions of $\mathrm{AP}(G)$ in all possible normed algebra norms. Of course the same question occurs for the completions of other Banach algebras.

It is easy (see §4) to exhibit a commutative semisimple Banach algebra B with no nonzero idempotent and a normed algebra norm on B where the completion of B contains such an idempotent. For $\mathrm{AP}(G)$, or more generally any semisimple annihilator Banach algebra A, any idempotent in the completion of A in a normed algebra topology must already be in A.

Let $|x|_1$ and $|x|_2$ be two normed algebra norms on a semisimple Banach algebra A. We say that these norms are consistent if $|x_n - x|_1 \to 0$ and $|x_n - y|_2 \to 0$ imply that $x = y$ (where all the elements are in A). In view of the closed graph theorem the uniqueness of the norm theorem [4, Theorem 9, p. 130] can be expressed as saying that any two complete norms $|x|_1$ and $|x|_2$ are consistent.

On the other hand (see §3), it is easy to find two incomplete normed algebra norms on the disc algebra which are not consistent. Nevertheless, for many of the usual examples of semisimple Banach algebras, any two normed algebra norms are consistent. This is the case for C^*-algebras, annihilator algebras, regular commutative Banach algebras and other instances as shown in §3.