STUDIA MATHEMATICA, T. XCV (1989)

# Sharp pointwise estimate for the kernels of the semigroup generated by sums of even powers of vector fields on homogeneous groups

by

WALDEMAR HEBISCH (Wrocław)

**Abstract.** Let $X_1, \ldots, X_m$ be a generating set of the Lie algebra of a homogeneous group. Let

$$L = -\sum_{j=1}^{m} (-1)^{n_j} X_j^{2n_j}.$$

Suppose that $L$ is homogeneous of degree $2n$, $n = \max_{1 \leqslant j \leqslant m} n_j$. It is known that $L$ is the infinitesimal generator of a semigroup $T_t f = f * p_t$, where $p_t$ is a $C^\infty$-function. In this paper we prove that

$$|Y^I p_t(x)| \leqslant C_I t^{-(Q+|I|)/(2n)} \exp\left(-C_1 (|x|^{2n}/t)^{1/(2n-1)}\right),$$

where $Q$ is the homogeneous dimension of the group, $|\cdot|$ is a homogeneous gauge and $Y^I = Y_1^{i_1} \ldots Y_k^{i_k}$, where $Y_j$ is a homogeneous element of the Lie algebra of degree $\alpha(j)$ with $|I| = \sum i_j \alpha(j)$. We do not use the Helffer–Nourrigat theory, instead we prove directly a local subelliptic estimate for $L$.

**1. Introduction.** Let $\mathfrak{g}$ be a graded nilpotent Lie algebra. This means that $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus \ldots \oplus V_p$ with $[V_i, V_j] \subseteq V_{i+j}$ when $i+j \leqslant p$ and $[V_i, V_j] = \{0\}$ if $i+j > p$. Let $G$ be a nilpotent, connected and simply connected Lie group that corresponds to $\mathfrak{g}$ via the exponential map. We equip $\mathfrak{g}$ as well as $G$ with a one-parameter family of dilations by extending $\delta_\lambda(X) = \lambda^j X$, $X \in V_j$, $\lambda > 0$, by linearity to $\mathfrak{g}$ and putting $\delta_\lambda(\exp X) = \exp(\delta_\lambda X)$. Then we define the *homogeneous dimension* $Q = \sum_{j=1}^{p} j \dim V_j$ and choose a *homogeneous norm* on $G$, i.e. a function $x \to |x|$ satisfying:

(i) $x \to |x|$ is continuous on $G$ and $C^\infty$ on $G \setminus \{e\}$;
(ii) $|x| \geqslant 0$ and $|x| = 0$ iff $x = e$;
(iii) $|\delta_\lambda x| = \lambda |x|$, $\lambda > 0$, $x \in G$.

Suppose that elements $X_1, \ldots, X_m$ in $\mathfrak{g}$ generate $\mathfrak{g}$. We consider $X_1, \ldots, X_m$ as left-invariant vector fields on $G$, and we define a differential operator

$$L = -\sum_{j=1}^{m} (-1)^{n_j} X_j^{2n_j} \tag{1.1}$$

acting on $C_c^\infty(G)$. Assume also that $L$ is homogeneous of degree $2n$, $n = \max_{1 \le j \le m} n_j$.

It was observed by Folland and Stein [2] that $L$, being negative and symmetric on $C_c^\infty(G) \subseteq L^2(G)$, is the infinitesimal generator of a one-parameter semigroup $\{T_t\}_{t>0}$ of operators on $L^2(G)$ such that $T_t f = f * p_t$, where the kernels $p_t$, $t > 0$, are $C^\infty$-functions. Moreover, the function

$$p(t, x) = \begin{cases} p_t(x), & t > 0, \\ 0, & t \le 0, \end{cases}$$

is $C^\infty$ on $\boldsymbol{R} \times G \setminus \{(0, e)\}$ and satisfies the "heat equation"

$$(\partial_t - L) p(t, x) = \delta_{(0,e)}.$$

Recently, in the case when $n_j = 1$, $j = 1, \ldots, m$, and $X_1, \ldots, X_m \in V_1$ (then $L$, called a *sublaplacian*, is homogeneous of degree 2), D. S. Jerison and A. Sánchez-Calle [4] have established the following pointwise estimate of the heat kernel $p(t, x)$:

$$p(t, x) \le C t^{-Q/2} e^{-c_1 |x|^2/t}. \tag{1.2}$$

Here the positive constants $C$ and $c_1$ depend only on $L$ and the choice of homogeneous norm. Moreover, they proved that for any multiindex $I = (i_1, \ldots, i_l)$ of length $|I| = l$ and any nonnegative integer $s$ one has

$$|\partial_t^s X_I p(t, x)| \le C_{I,s} t^{-s-|I|/2-Q/2} e^{-c_1|x|^2/t}. \tag{1.2'}$$

Here $c_1$ is independent of $I$ and $s$, and $X_I = X_{i_1} \ldots X_{i_l}$.

On the other hand, J. Dziubański and A. Hulanicki [1] have recently obtained the following estimate for the decay of $p_t$ at infinity (with arbitrary $n_j > 0$): for every $\partial$ in the enveloping algebra of $G$ and for every $N > 0$ one has

$$|\partial p_t(x)| \le C_{\partial,t,N} e^{-N|x|}. \tag{1.3}$$

Let $Y_1, \ldots, Y_\nu$ be a basis of the Lie algebra $\mathfrak{g}$ consisting of homogeneous elements, i.e. $Y_j \in V_{\alpha(j)}$ for some $\alpha(j) \in \{1, \ldots, p\}$, $j = 1, \ldots, \nu$. For any multiindex $I = (i_1, \ldots, i_\nu)$ we write $Y^I = Y_1^{i_1} \ldots Y_\nu^{i_\nu}$ and we denote by $|I|$ its homogeneous length $|I| = \sum_{j=1}^{\nu} i_j \alpha(j)$. Since by the homogeneity of $L$, one has

$$(Y^I p_t)(\delta_{s^{-1}} x) = s^{|I|+Q} Y^I p_{s^{2n}t}(x),$$

it is easy to observe that (1.3) immediately gives

$$|Y^I p_t(x)| \le C_I t^{-(Q+|I|)/(2n)} \exp(-|x| t^{-1/(2n)}). \tag{1.4}$$

The aim of this paper is the proof of the following

THEOREM 1. *Let $p(t, x)$ be the "heat kernel" associated to $L = -\sum_{j=1}^{m} (-1)^{n_j} X_j^{2n_j}$ which we assume to be homogeneous of degree $2n$, $n = \max_{1 \le j \le m} n_j$. For every multiindex $I$ and every nonnegative integer $s$ there is a positive constant $C_{I,s}$ such that*

$$|\partial_t^s Y^I p(t, x)| \le C_{I,s} t^{-s-Q/(2n)-|I|/(2n)} \exp\left(-c_1 (|x|^{2n}/t)^{1/(2n-1)}\right)$$

*with a positive constant $c_1$ independent of $I$ and $s$.*



The main idea of the proof comes from [4]. The rate of decay of $p_t$ is investigated by means of the local Gevrey regularity in the $t$ variable. Then a standard homogeneity argument is applied. However, instead of a subelliptic estimate used in [4] (which is a quantitative form of Hörmander's hypoellipticity theorem for $\partial_t - L$, $L$ being the sublaplacian) we apply a subelliptic estimate which we prove directly for the generator $L$ without appeal to the Helffer–Nourrigat theory.

**Acknowledgements.** I am indebted to Andrzej Hulanicki for bringing the problem to my attention and to Krzysztof Stempak for his generous help in preparation of this paper.

**2. Gevrey regularity.** We identify the group $G$ with a Euclidean space $\boldsymbol{R}^\nu$ (thus $\nu = \sum_{j=1}^{p} \dim V_j$). Then we fix the Euclidean distance in $\boldsymbol{R}^{\nu+1}$: $\|(t, x)\| = (t^2 + x_1^2 + \ldots + x_\nu^2)^{1/2}$ and set

$$B(r) = \{(t, x)\colon \|(t, x)\| < r\}.$$

For any functions $f$, $g$ on an open subset $\Omega \subseteq \boldsymbol{R}^{\nu+1}$ we write

$$\|f\|_{L^2(\Omega)} = \Big(\iint_\Omega |f(t, x)|^2 \, dt \, dx\Big)^{1/2}$$

and, when $\Omega = \boldsymbol{R}^{\nu+1}$,

$$\langle f, g \rangle = \iint_\Omega f(t, x) g(x, t) \, dt \, dx.$$

We will also abbreviate $\|f\|_{L^2(\boldsymbol{R}^{\nu+1})}$ by $\|f\|$.

Throughout the paper functions are supposed to be real-valued and $c$, $c'$, $c_1, \ldots$, will denote constants that may vary from line to line.

We will also use the folowing notation. By $\mathcal{O}(M, J)$, $M > 0$, $J = 1, 2, \ldots$, we denote the set of all $C^\infty$-functions $\varphi$ on $\boldsymbol{R}^{\nu+1}$ satisfying: $\operatorname{supp} \varphi \subseteq B(1)$, $0 \le \varphi \le 1$, and for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_{\nu+1})$ with $|\alpha| \le J$

$$\|\partial^\alpha \varphi\|_\infty \le M^{|\alpha|},$$

$|\alpha| = \sum_{i=1}^{\nu+1} \alpha_i$ being the Euclidean length of $\alpha$.

It is clear that for any $X \in \mathfrak{g}$, considered now as a left-invariant vector field on $\boldsymbol{R}^\nu$, one has

$$\langle Xf, g \rangle = -\langle f, Xg \rangle, \quad f, g \in C_0^\infty(\boldsymbol{R}^\nu).$$

Let $L$ be the differential operator on $\boldsymbol{R}^\nu$ defined by (1.1). Recall that $n = \max_{1 \leqslant j \leqslant m} n_j$. The main result of this section is the following

PROPOSITION 1. *There is a constant $R > 0$ such that for any function $u$ satisfying $(\partial_t - L)u(t, x) = 0$ on $B(1)$ we have*

$$\|\partial_t^k u\|_{L^2(B(1/2))} \leqslant R^k (k!)^{2n} \|u\|_{L^2(B(1))}.$$

The proof of Proposition 1 requires several lemmas.

LEMMA 1. *Let $X \in \mathfrak{g}$ be a left-invariant vector field on $\boldsymbol{R}^\nu$ and let $k, r$ be positive integers. Then for every $a > 0$ there is a positive constant $C = C(X, k, r, a)$ such that for every $M > 0$, $\varphi \in \mathcal{O}(M, 1)$ and $u \in C^\infty(B(1))$*

$$\|\varphi^{r+k} X^k u\|^2 \leqslant \frac{a}{M^2} \|\varphi^{r+k+1} X^{k+1} u\|^2 + CM^2 \|\varphi^{r+k-1} X^{k-1} u\|^2. \tag{2.1}$$

Proof. We have

$$\begin{aligned}
&\|\varphi^{r+k} X^k u\|^2 \\
&\quad = \langle \varphi^{2(r+k)} X^k u, X^k u \rangle = -\langle X(\varphi^{2(r+k)} X^k u), X^{k-1} u \rangle \\
&\quad = -2(r+k)\langle \varphi^{2(r+k)-1} X\varphi \cdot X^k u, X^{k-1} u \rangle - \langle \varphi^{2(r+k)} X^{k+1} u, X^{k-1} u \rangle \\
&\quad = -2(r+k)\langle \varphi^{r+k} X\varphi \cdot X^k u, \varphi^{r+k-1} X^{k-1} u \rangle \\
&\qquad -\langle \varphi^{r+k+1} X^{k+1} u, \varphi^{r+k-1} X^{k-1} u \rangle.
\end{aligned}$$

Now, the Schwarz inequality implies

$$\begin{aligned}
\|\varphi^{r+k} X^k u\|^2 &\leqslant C' M \|\varphi^{r+k} X^k u\| \cdot \|\varphi^{r+k-1} X^{k-1} u\| \\
&\quad + \|\varphi^{r+k+1} X^{k+1} u\| \cdot \|\varphi^{r+k-1} X^{k-1} u\| \\
&\leqslant \tfrac{1}{4} \|\varphi^{r+k} X^k u\|^2 + (2C'M)^2 \|\varphi^{r+k-1} X^{k-1} u\|^2 \\
&\quad + \frac{a}{M^2} \|\varphi^{r+k+1} X^{k+1} u\|^2 + \frac{M^2}{a} \|\varphi^{r+k-1} X^{k-1} u\|^2.
\end{aligned}$$

This immediately gives (2.1).

LEMMA 2. *Let $X \in \mathfrak{g}$, let $r, l, k, m$ be positive integers and $l < k < m$. For every $a > 0$ there is a constant $C$ depending only on $X, r, l, k, m$ and $a$ such that for every $M > 0$, $\varphi \in \mathcal{O}(M, 1)$ and $u \in C^\infty(B(1))$*

$$\|\varphi^{r+k} X^k u\|^2 \leqslant \frac{a}{M^{2(m-k)}} \|\varphi^{r+m} X^m u\|^2 + CM^{2(k-l)} \|\varphi^{r+l} X^l u\|^2. \tag{2.2}$$

Proof. We prove (2.2) by induction on $m-l$. If $m-l = 2$ it is just Lemma 1. Thus, suppose first that $m-k \geqslant 2$. Using the induction hypothesis for $k < m-1 < m$ we get

$$\|\varphi^{r+m-1} X^{m-1} u\|^2 \leqslant \frac{a}{M^2} \|\varphi^{r+m} X^m u\|^2 + C' M^{2(m-1-k)} \|\varphi^{r+k} X^k u\|^2. \tag{2.3}$$

Now, take $a' = \min(1/(2C'), 1/2)$ and by induction hypothesis for $l < k < m-1$ find a constant $C''$ such that

$$\begin{aligned}
\|\varphi^{r+k} X^k u\|^2 &\leqslant \frac{a'}{M^{2(m-1-k)}} \|\varphi^{r+m-1} X^{m-1} u\|^2 \\
&\quad + C'' M^{2(k-l)} \|\varphi^{r+l} X^l u\|^2.
\end{aligned} \tag{2.4}$$

Clearly, combining (2.3) and (2.4) gives (2.2). In the case when $k-l \geqslant 2$ we use a similar argument.

LEMMA 3. *Let $X \in \mathfrak{g}$ and let $k, r, q$ be positive integers. Then for every $a > 0$ there is a constant $C$ depending only on $k, r, q$ and $X$ such that for every $M > 0$, $\varphi \in \mathcal{O}(M, k)$ and $u, v \in C^\infty(B(1))$*

$$|\langle v, [X^k, \varphi^{k+r+q}] u \rangle| \leqslant \|\varphi^q v\| (a \|\varphi^{k+r} X^k u\|^2 + CM^{2k} \|\varphi^r u\|^2)^{1/2}. \tag{2.5}$$

Proof. By an easy induction

$$X^l \varphi^n = \sum_{j=1}^{l} \sum_{\nu} \alpha_\nu (X^{i_1}\varphi \ldots X^{i_j}\varphi) \varphi^{n-j}, \tag{2.6}$$

where the inner sum is taken over all multiindices $\nu = (i_1, \ldots, i_j)$ such that $i_1, \ldots, i_j \geqslant 1$, $i_1 + \ldots + i_j = l$. Moreover,

$$[X^k, \varphi^{k+r+q}] u = \sum_{l=1}^{k} \binom{k}{l} (X^l \varphi^{k+r+q})(X^{k-l} u).$$

Using (2.6), since $|X^{i_1}\varphi \ldots X^{i_j}\varphi| \leqslant C_1 M^l$ and $0 \leqslant \varphi \leqslant 1$, for any $l = 1, \ldots, k$ we have $|X^l \varphi^{k+r+q}| \leqslant C_2 M^l \varphi^{k+r+q-l}$. Therefore,

$$\begin{aligned}
|\langle v, X^l \varphi^{k+r+q} X^{k-l} u \rangle| &\leqslant \langle |v|, |X^l \varphi^{k+r+q} X^{k-l} u| \rangle \\
&\leqslant C_2 M^l \langle \varphi^q |v|, \varphi^{k+r-l} |X^{k-l} u| \rangle \\
&\leqslant C_2 M^l \|\varphi^q v\| \cdot \|\varphi^{k+r-l} X^{k-l} u\|.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\langle v, [X^k, \varphi^{k+r+q}] u \rangle| &\leqslant C_3 \|\varphi^q v\| \sum_{l=1}^{k} M^l \|\varphi^{k+r-l} X^{k-l} u\| \\
&\leqslant \|\varphi^q v\| \Big(C_4 \sum_{l=1}^{k} M^{2l} \|\varphi^{k+r-l} X^{k-l} u\|^2\Big)^{1/2}.
\end{aligned}$$

Now, using Lemma 2 with $a$ replaced by $a/(kC_4)$ we get

$$C_4 \sum_{l=1}^{k} M^{2l} \|\varphi^{k+r-l} X^{k-l} u\|^2 \leqslant C_4 \sum_{l=1}^{k} M^{2l} \left(\frac{a}{kC_4 M^{2l}} \|\varphi^{r+k} X^k u\|^2 + CM^{2(k-l)} \|\varphi^r u\|^2\right) \leqslant a\|\varphi^{r+k} X^k u\|^2 + C' M^{2k} \|\varphi^r u\|^2.$$

This concludes the proof of Lemma 3.

LEMMA 4. (Gårding type inequality). *Let*

$$L = -\sum_{j=1}^{m} (-1)^{n_j} X_j^{2n_j},$$

$n = \max n_j$, $X_1, \ldots, X_m \in \mathfrak{g}$, *and let* $r$ *be a positive integer. For every* $a > 0$ *there is a constant* $C > 0$ *such that for any* $M \geqslant 1$, $\varphi \in \mathcal{O}(M, n)$ *and* $u \in C^\infty(B(1))$ *we have*

$$-\langle \varphi^{2(r+n)} Lu, u\rangle \geqslant (1-a) \sum_{j=1}^{m} \|\varphi^{r+n} X_j^{n_j} u\|^2 - CM^{2n} \|\varphi^r u\|^2. \tag{2.7}$$

Proof. The inequality (2.7) is an easy consequence of

$$(-1)^k \langle \varphi^{2(r+k)} X^{2k} u, u\rangle \geqslant (1-a)\|\varphi^{r+k} X^k u\|^2 - CM^{2k} \|\varphi^r u\|^2, \tag{2.8}$$

where the constant $C$ depends on $X$, $k$, $r$ and $a$ only. Indeed, using (2.8) we get

$$-\langle \varphi^{2(r+n)} Lu, u\rangle = \sum_{j=1}^{m} (-1)^{n_j} \langle \varphi^{2(r+n)} X_j^{2n_j} u, u\rangle \geqslant (1-a) \sum_{j=1}^{m} \|\varphi^{r+n} X_j^{n_j} u\|^2 - \sum_{j=1}^{m} C_j M^{2n_j} \|\varphi^{r+n-n_j} u\|^2 \geqslant (1-a) \sum_{j=1}^{m} \|\varphi^{r+n} X_j^{n_j} u\|^2 - CM^{2n} \|\varphi^r u\|^2.$$

Thus it suffices to prove (2.8). We have

$$(-1)^k \langle \varphi^{2(r+k)} X^{2k} u, u\rangle = \langle X^k u, X^k(\varphi^{2(r+k)} u)\rangle = \langle X^k u, \varphi^{2(r+k)} X^k u\rangle + \langle X^k u, [X^k, \varphi^{2(r+k)}] u\rangle \geqslant \|\varphi^{r+k} X^k u\|^2 - |\langle X^k u, [X^k, \varphi^{2(r+k)}] u\rangle|.$$



Now using Lemma 3 we estimate

$$\langle X^k u, [X^k, \varphi^{2(r+k)}] u\rangle| \leqslant \|\varphi^{k+r} X^k u\| (a^2 \|\varphi^{k+r} X^k u\|^2 + CM^{2k} \|\varphi^r u\|^2)^{1/2} \leqslant (a^{1/2} \|\varphi^{k+r} X^k u\|) \left(a\|\varphi^{k+r} X^k u\|^2 + \frac{CM^{2k}}{a} \|\varphi^r u\|^2\right)^{1/2} \leqslant \frac{1}{2}\left(a\|\varphi^{k+r} X^k u\|^2 + a\|\varphi^{k+r} X^k u\|^2 + \frac{CM^{2k}}{a} \|\varphi^r u\|^2\right) \leqslant a\|\varphi^{k+r} X^k u\|^2 + C' M^{2k} \|\varphi^r u\|^2.$$

Combining the estimates above gives (2.8).

LEMMA 5. *Let* $L$, $n$ *be as in Lemma 4 and let* $r$ *be a positive integer. There exists a constant* $C > 0$ *such that for any* $u$ *satisfying* $(\partial_t - L)u = 0$ *on* $B(1)$, $M \geqslant 1$ *and* $\varphi \in \mathcal{O}(M, n)$ *we have*

$$\|\varphi^{r+n} X_j^{n_j} u\| \leqslant CM^n \|\varphi^r u\|.$$

Proof. Clearly $\langle \partial_t v, v\rangle = 0$ for any $v \in C_0^\infty(\boldsymbol{R}^{\nu+1})$. Therefore, since $\varphi^{r+n} \leqslant \varphi^{r+n-1}$, by the Schwarz inequality we obtain

$$|\langle \varphi^{r+n} Lu, u\rangle| = |\langle \varphi^{r+n} \partial_t u, \varphi^{r+n} u\rangle| = |\langle (\partial_t \varphi^{r+n}) u, \varphi^{r+n} u\rangle| \leqslant M(r+n) \|\varphi^{r+n-1} u\|^2.$$

Finally, using Lemma 4 we get

$$(1-a) \sum_{j=1}^{m} \|\varphi^{r+n} X_j^{n_j} u\|^2 \leqslant |\langle \varphi^{r+n} Lu, \varphi^{r+n} u\rangle| + CM^{2n} \|\varphi^r u\|^2 \leqslant M(r+n) \|\varphi^{r+n-1} u\|^2 + CM^{2n} \|\varphi^r u\|^2 \leqslant C' M^{2n} \|\varphi^r u\|^2,$$

which completes the proof of Lemma 5.

Now we are ready to prove Proposition 1.

First, we show that the lemmas above imply

$$\|\varphi^{3n} \partial_t u\| \leqslant CM^{2n} \|\varphi^n u\| \tag{2.9}$$

with a constant $C > 0$ that depends only on $L$ (therefore also on $n$) but not on $M \geqslant 1$, $\varphi \in \mathcal{O}(M, n)$ and $u \in C^\infty(B(1))$ satisfying $(L - \partial_t)u = 0$. We have

$$\|\varphi^{3n} \partial_t u\|^2 = \langle \partial_t u, \varphi^{6n} \partial_t u\rangle = \langle Lu, \varphi^{6n} \partial_t u\rangle = -\sum_{j=1}^{m} \langle X_j^{n_j} u, X_j^{n_j}(\varphi^{6n} \partial_t u)\rangle \leqslant \sum_{j=1}^{m} |\langle X_j^{n_j} u, [X_j^{n_j} \varphi^{6n}] \partial_t u\rangle| + \sum_{j=1}^{m} |\langle X_j^{n_j} u, \varphi^{6n} X_j^{n_j} \partial_t u\rangle| \leqslant \sum_{j=1}^{m} |\langle X_j^{n_j} u, [X_j^{n_j} \varphi^{6n}] \partial_t u\rangle| + \sum_{j=1}^{m} |\langle \varphi^{3n} X_j^{n_j} u, (\partial_t \varphi^{3n}) X_j^{n_j} u\rangle|. \tag{2.10}$$

The last step is justified by the identity $\langle v, \partial_t v\rangle = 0$ with $v = \varphi^{3n} X_j^{n_j} u$. Next, applying Lemma 3 with $k = n_j$, $r = 4n - n_j$, $q = 2n$ we obtain

$$\langle X_j^{n_j} u, [X_j^{n_j} \varphi^{6n}]\partial_t u\rangle| \leqslant \|\varphi^{2n} X_j^{n_j} u\| (\|\varphi^{4n} X_j^{n_j} \partial_t u\|^2 + C_0 M^{2n} \|\varphi^{3n}\partial_t u\|^2)^{1/2}.$$

By Lemma 5, $\|\varphi^{2n} X_j^{n_j} u\| \leqslant CM^n \|\varphi^n u\|$ and $\|\varphi^{4n} X_j^{n_j} \partial_t u\| \leqslant CM^n \|\varphi^{3n}\partial_t u\|$. In all,

$$
\begin{aligned}
(2.11)\quad |\langle X_j^{n_j} u, [X_j^{n_j}, \varphi^{6n}]\partial_t u\rangle| &\leqslant CM^n \|\varphi^n u\| (C^2 M^{2n} + C_0 M^{2n})^{1/2} \|\varphi^{3n}\partial_t u\| \\
&\leqslant C_1 M^{2n} \|\varphi^n u\| \cdot \|\varphi^{3n}\partial_t u\| \\
&\leqslant \tfrac{1}{2}(mC_1^2 M^{4n} \|\varphi^n u\|^2 + m^{-1} \|\varphi^{3n}\partial_t u\|^2).
\end{aligned}
$$

On the other hand, the Schwarz inequality and Lemma 5 imply

$$
\begin{aligned}
(2.12)\quad |\langle \varphi^{3n} X_j^{n_j} u, (\partial_t \varphi^{3n}) X_j^{n_j} u\rangle| &\leqslant 3nM \|\varphi^{3n} X_j^{n_j} u\| \cdot \|\varphi^{3n-1} X_j^{n_j} u\| \\
&\leqslant 3nM \|\varphi^{2n} X_j^{n_j} u\|^2 \leqslant C_2 M^{2n+1} \|\varphi^n u\|^2 \leqslant C_2 M^{4n} \|\varphi^n u\|^2.
\end{aligned}
$$

Combining (2.10), (2.11) and (2.12) gives

$$\|\varphi^{3n}\partial_t u\|^2 \leqslant \tfrac{1}{2} m^2 C_1^2 M^{4n} \|\varphi^n u\|^2 + \tfrac{1}{2}\|\varphi^{3n}\partial_t u\|^2 + mC_2 M^{4n} \|\varphi^n u\|^2.$$

This clearly implies (2.9).

Let $0 < \varepsilon < r < 1$. It is easy to choose a function $\varphi = \varphi_{r,\varepsilon} \in C_0^\infty(\boldsymbol{R}^{\nu+1})$ with the following properties: $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on $B(r-\varepsilon)$, $\varphi = 0$ outside $B(r)$ and $|\partial^\alpha \varphi| \leqslant D\varepsilon^{-|\alpha|} \leqslant (D/\varepsilon)^{|\alpha|}$ for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_{\nu+1})$ of length $\leqslant n$. Here the constant $D \geqslant 1$ does not depend on $r$ and $\varepsilon$. Indeed, fix a function $\varphi_0 \in C_0^\infty(\boldsymbol{R})$, $0 \leqslant \varphi_0 \leqslant 1$, $\varphi_0(y) = 1$ for $y \leqslant 0$ and $\varphi(y) = 0$ for $y \geqslant 1$ and put

$$\varphi_{r,\varepsilon}(t, x) = \varphi_0\left(\frac{\|(t, x)\| - r + \varepsilon}{\varepsilon}\right).$$

Thus applying (2.9) to the function $\varphi = \varphi_{r,\varepsilon}$ we obtain

$$\|\partial_t u\|_{B(r-\varepsilon)} \leqslant \|\varphi^{3n}\partial_t u\| \leqslant C(D/\varepsilon)^{2n} \|\varphi^n u\| \leqslant C_1 \varepsilon^{-2n} \|u\|_{B(r)}.$$

By induction on $k$

$$\|\partial_t^k u\|_{B(r-k\varepsilon)} \leqslant C_1^k \varepsilon^{-2nk} \|u\|_{B(r)}.$$

Setting $\varepsilon = 1/(2k)$, $r = 1$ and using the Stirling formula one has

$$\|\partial_t^k u\|_{B(1/2)} \leqslant (2^{2n} C_1)^k k^{2nk} \|u\|_{B(1)} \leqslant R^k (k!)^{2n} \|u\|_{B(1)},$$

with a positive constant $R$. This finishes the proof of Proposition 1.

**3. Proof of Theorem 2.** Recall that $Y_1, \ldots, Y_\nu$ is a basis in $\mathfrak{g}$ and we write $Y^\alpha = Y_1^{\alpha_1} \ldots Y_\alpha^{\alpha_\nu}$. As in [4] we use Proposition 1 to express the Gevrey regularity of the solution to $(\partial_t - L)u = 0$ in the $t$ variable in terms of the $L^\infty$-norm.



More precisely, we prove

PROPOSITION 2. *There is a constant $R_1$ such that for every $u$ satisfying $(\partial_t - L)u = 0$ in $B(1)$*

$$(3.1)\qquad \sup_{B(1/10)} |\partial_t^k \partial_t^s Y^\alpha u| \leqslant C_{s,\alpha} R_1^k (k!)^{2n} \|u\|_{L^2(B(1))},$$

*where the constant $C_{s,\alpha}$ depends only on $L$, $s$ and $\alpha$.*

Proof. Observe that (3.1) is a consequence of the following subelliptic estimate for the operator $L$:

$$(3.2)\qquad \|Y^\alpha v\|_{L^2(K)}^2 \leqslant C_{K,F,\alpha}\Big(\sum_{r=0}^{N} \|L^r v\|_{L^2(F)}^2\Big),$$

where $N = N(\alpha)$ is a positive integer, $K \subseteq \operatorname{Int} F \subseteq \boldsymbol{R}^\nu$, $K$ is compact, $F$ closed and the constant $C_{K,F,\alpha}$ does not depend on a smooth function $v$. Indeed, taking $K = \{x \in \boldsymbol{R}^\nu\colon \|x\| \leqslant 1/8\}$, $F = \{x \in \boldsymbol{R}^\nu\colon \|x\| \leqslant 1/4\}$ we have

$$
\begin{aligned}
\|\partial_t^{k+s} Y^\alpha u\|_{L^2(B(1/8))}^2 &\leqslant \int_{-1/8}^{1/8} \|Y^\alpha \partial_t^{k+s} u(t, \cdot)\|_{L^2(K)}^2\, dt \\
&\leqslant C \sum_{r=0}^{N} \int_{-1/8}^{1/8} \|L^r \partial_t^{k+s} u(t, \cdot)\|_{L^2(F)}^2\, dt \\
&\leqslant C \sum_{r=0}^{N} \|L^r \partial_t^{k+s} u\|_{L^2(B(1/2))}^2 \leqslant C \sum_{r=0}^{N} \|\partial_t^{k+s+r} u\|_{L^2(B(1/2))}^2 \\
&\leqslant C \sum_{r=0}^{N} (R^{k+s+r} [(k+s+r)!]^{2n} \|u\|_{B(1)})^2 \leqslant C_1^2 (R_1^k (k!)^{2n} \|u\|_{B(1)})^2.
\end{aligned}
$$

Clearly, the Sobolev lemma now gives (3.1).

So, it remains to prove (3.2). First, note that the operator $Y^\alpha (I-L)^{-N}$ for $N \geqslant 1 + (Q + |\alpha|)/(2n)$ is the operator of convolution with a bounded function smooth outside zero that decreases rapidly at infinity together with all its derivatives. Indeed, denoting by the same symbol an operator and the corresponding kernel, we have

$$(3.3)\qquad Y^\alpha (I-L)^{-N}(x) = \frac{1}{\Gamma(N)} \int_0^\infty t^{N-1} e^{-t} Y^\alpha p_t(x)\, dt.$$

Now, by a recent result of Dziubański and Hulanicki [1] (cf. (1.4))

$$|Y^\alpha p_t(x)| \leqslant C_\alpha t^{-(Q+|\alpha|)/(2n)} e^{-|x|t^{-1/(2n)}}.$$

Therefore

$$|Y^\alpha (I-L)^{-N}(x)| \leqslant C_\alpha \cdot \Gamma(N)^{-1} \int_0^\infty t^{N-1-(Q+|\alpha|)/(2n)} e^{-t} e^{-|x|t^{-1/(2n)}}\, dt.$$

Since for $N \geqslant 1+(Q+|\alpha|)/(2n)$ the last integral is bounded uniformly in $x$, so is the kernel $Y^\alpha(I-L)^{-N}(x)$.

Further, replacing $\alpha$ by $\alpha+\beta$ in (3.3) and using the fact that for $|x| > \varepsilon$ and $t \leqslant |x|$

$$t^{N-1-(Q+|\alpha+\beta|)/(2n)} e^{-|x|t^{-1/(2n)}} \leqslant C_\varepsilon \exp(-\tfrac{1}{2}|x|t^{-1/(2n)}),$$

we obtain

$$|Y^\beta Y^\alpha (I-L)^{-N}(x)| \leqslant C \int_0^{|x|} e^{-t} \exp(-\tfrac{1}{2}|x|t^{-1/(2n)})\, dt + |x|^{N_1} \int_{|x|}^\infty e^{-t}\, dt$$
$$\leqslant C(e^{-|x|^{1/2}/2} + |x|^{N_1} e^{-|x|}),$$

where $N_1 = N-1-(Q+|\alpha+\beta|)/(2n)$. This proves that all derivatives of $Y^\alpha(I-L)^{-N}(x)$ rapidly decrease at infinity.

Now, take $K \subseteq \operatorname{Int} F \subseteq \boldsymbol{R}^\nu$, $K$ compact, $F$ closed. Choose a symmetric relatively compact neighborhood $U$ of zero such that $K \cdot \bar{U} \subseteq \operatorname{Int} F$. Also choose $C_c^\infty$-functions $\varphi_1$, $\varphi_2$ on $\boldsymbol{R}^\nu$ with the properties: $0 \leqslant \varphi_i \leqslant 1$, $\operatorname{supp} \varphi_1 \subseteq U$, $\varphi_1 = 1$ in a neighborhood of zero, $\varphi_2 = 1$ on $K \cdot \bar{U}$, $\operatorname{supp} \varphi_2 \subseteq \operatorname{Int} F$. We define

$$\tilde{R}(x) = \varphi_1(x) Y^\alpha (I-L)^{-N}(x), \qquad W(x) = (1-\varphi_1(x)) Y^\alpha (I-L)^{-N}(x).$$

Denoting, as before, by $\tilde{R}$ and $W$ the convolution operators with the corresponding kernels, we can say that $\tilde{R}$ is the operator of convolution with a function supported in $U$ and that the kernel of the operator $W$ is smooth. Also the operator $W_1 = W(I-L)^N$ has a smooth kernel.

Considering $v$ as a distribution on $\boldsymbol{R}^\nu$ ($v = 0$ outside $F$) we have on $K$

$$Y^\alpha v = Y^\alpha (I-L)^{-N}(I-L)^N v = \tilde{R}(I-L)^N v + W_1 v = \tilde{R}(\varphi_2 (I-L)^N v) + W_1 v.$$

The last identity follows from the symmetry of $U$ and the fact that $\varphi_2 = 1$ on $K \cdot U$. Consequently,

$$\| Y^\alpha v\|_{L^2(K)} \leqslant \|\tilde{R}(\varphi_2(I-L)^N v)\|_{L^2(K)} + \|W_1 v\|_{L^2(K)}$$
$$\leqslant C_1(\|(I-L)^N v\|_{L^2(F)} + \|v\|_{L^2(F)}) \leqslant C_2 \sum_{r=0}^N \|L^r v\|_{L^2(F)}.$$

This proves (3.2) and thus concludes the proof of Proposition 2.

Remark. At the end of the paper we propose a different proof of the estimate (3.2) that does not depend on the result of Dziubański and Hulanicki.



LEMMA 6. *Let $f \in C^\infty(-1, 1)$ and $f(t) = 0$ for $-1 < t \leqslant 0$. If, for a $\gamma > 1$, $|f^{(k)}(t)| \leqslant R^k (k!)^\gamma$, $0 \leqslant t < 1$, $k = 0, 1, 2, \ldots$, then*

$$|f(t)| \leqslant e^{\gamma-1} \exp\{-(\gamma-1)e^{-1}(Rt)^{-1/(\gamma-1)}\}.$$

Proof.

$$|f(t)| \leqslant \int_0^t \frac{(t-s)^k}{k!} |f^{(k+1)}(s)|\, ds \leqslant R^{k+1}((k+1)!)^\gamma \frac{t^{k+1}}{(k+1)!}$$
$$= R^{k+1}((k+1)!)^{\gamma-1} t^{k+1} \leqslant [(Rt)^{1/(\gamma-1)}(k+1)]^{(\gamma-1)(k+1)}.$$

Choose $k$ so that $k+1 \leqslant e^{-1}(Rt)^{-1/(\gamma-1)} < k+2$. Then $(Rt)^{1/(\gamma-1)} \leqslant 1/(e(k+1))$ and so

$$|f(t)| \leqslant [(Rt)^{1/(\gamma-1)}(k+1)]^{(\gamma-1)(k+1)} \leqslant e^{-(\gamma-1)(k+1)}$$
$$\leqslant \exp\{-(\gamma-1)(e^{-1}(Rt)^{-1/(\gamma-1)} - 1)\},$$

which proves the lemma.

We are now ready to prove Theorem 1. Since $\partial_t^s p = L^s p$ it suffices to estimate only the space derivatives $Y^\alpha p(t, x)$ of $p(t, x)$, for any multiindex $\alpha$.

By applying Proposition 2 to the function $p(t, x)$, for any $(t_0, x_0) \neq (0, 0)$ in some neighborhood $U_{(t_0, x_0)}$ of $(t_0, x_0)$ we have

$$\sup_{U_{(t_0,x_0)}} |\partial_t^k Y^\alpha p(t, x)| \leqslant C_0 R_0^k (k!)^{2n}, \tag{3.4}$$

where the constant $C_0$ depends on $(t_0, x_0)$ and $\alpha$, whereas $R_0$ depends on $(t_0, x_0)$ only.

The set $K = \{(t, x) \in \boldsymbol{R} \times G\colon -1 \leqslant t \leqslant 1, |x| = 1\}$ is compact and does not contain $(0, 0)$ so we cover it by a finite family of neighborhoods $U_{(t_i, x_i)}$ as above, and using (3.4) we get

$$\sup_K |\partial_t^k Y^\alpha p(t, x)| \leqslant CR^k (k!)^{2n},$$

with an absolute constant $R$ and a $C = C(\alpha)$.

Since $p(t, x) = 0$ for $t \leqslant 0$, using Lemma 6 we estimate

$$|Y^\alpha p(t, x)| \leqslant C_{\alpha, K} e^{-a/t^{1/(2n-1)}}$$

uniformly in $(t, x) \in K$, with a constant $a > 0$. As we mentioned before, one has

$$(Y^\alpha p_t)(\delta_{s^{-1}} x) = s^{|\alpha|+Q} Y^\alpha p_{s^{2n}t}(x).$$

Consequently, for $|x| \geqslant 1$

$$|Y^\alpha p_1(x)| = |Y^\alpha p_1(\delta_{|x|}\delta_{|x|^{-1}} x)| = |x|^{|\alpha|+Q} |Y^\alpha p_{|x|^{-2n}}(\delta_{|x|^{-1}} x)|$$
$$\leqslant C_{\alpha, K} |x|^{|\alpha|+Q} e^{-a|x|^{2n/(2n-1)}}.$$

Take an arbitrary $0 < c_1 < a$. Since $|Y^\alpha p_1(x)|$ is bounded on $|x| \leqslant 1$, therefore, for a sufficiently large constant $C_\alpha$

$$|Y^\alpha p_1(x)| \leqslant C_\alpha e^{-c_1|x|^{2n/(2n-1)}}.$$

Finally, by dilation

$$|Y^\alpha p_t(x)| = |Y^\alpha p_t(\delta_{t^{1/(2n)}}\delta_{t^{-1/(2n)}}x)| \leqslant t^{-(|\alpha|+Q)/(2n)}|Y^\alpha p_1(\delta_{t^{-1/(2n)}}x)|$$
$$\leqslant C_\alpha t^{-(|\alpha|+Q)/(2n)}\exp\{-c_1(|x|^{2n}/t)^{1/(2n-1)}\}.$$

This concludes the proof of Theorem 1.

**4. Another proof of (3.2).** The following proof of (3.2), independent of the estimate (1.4), is a modification of a proof of a theorem in Hörmander's book [3], p. 353.

As usual, by $H(s)$ we will denote the ordinary Sobolev space of order $s$ on $\boldsymbol{R}^\nu$ and $\|u\|_{H(s)} = (\int_{\boldsymbol{R}^\nu}|\hat{u}(\xi)|^2(1+|\xi|^2)^s d\xi)^{1/2}$ denotes the norm of $u \in H(s)$. We also write $\|f\|$ to denote the $L^2$-norm of a function $f$ on $\boldsymbol{R}^\nu$.

We write (3.2) in an equivalent form

$$\|\varphi_1 u\|_{H(s)} \leqslant C\sum_{r=0}^{N}\|\varphi_2 L^r u\|, \tag{4.1}$$

where $\varphi_1$, $\varphi_2$ are compactly supported $C^\infty$-functions, $\varphi_2 = 1$ on a neighborhood of the support of $\varphi_1$, $N = N(s)$, $C = C(s, \varphi_1, \varphi_2)$. When $n = 1$, (4.1) is a subelliptic estimate used in the proof of Hörmander's hypoellipticity theorem (cf. [4]). It is easy to observe that Lemmas 3 and 4 give

$$\sum_{j=1}^{m}\|X_j(\varphi_1 u)\| \leqslant C(\|\varphi_2 Lu\| + \|\varphi_2 u\|).$$

Set $Q_1 = \operatorname{lin}\{X_i: i = 1, \ldots, m\} \subseteq \mathfrak{g}$, $Q_{k+1} = [Q_k, Q_1]$, $k = 1, 2, \ldots$, and $l = \min\{i: \mathfrak{g} = \operatorname{lin}\{\bigcup_{k=1}^i Q_k\}\}$.

The following lemma is due to J. J. Kohn (cf. e.g. [5]).

LEMMA 7. *If $q_k \in Q_k$ and $\varepsilon \leqslant 2^{1-k}$ then*

$$\|q_k(\varphi_1 u)\|_{H(\varepsilon-1)} \leqslant C(\|\varphi_1 u\| + \sum_{i=1}^{m}\|X_i(\varphi_1 u)\|) \tag{4.2}$$

*with a constant $C$ independent of $u$.*

COROLLARY 1.

$$\|\varphi_1 u\|_{H(1/2^{l-1})} \leqslant C(\|\varphi_2 Lu\| + \|\varphi_2 u\|).$$

In fact, if $q_1, \ldots, q_\nu$ generate $\mathfrak{g}$ then for $\varepsilon \leqslant 1/2^{l-1}$

$$\|\varphi_1 u\|_{H(\varepsilon)} \leqslant C\sum_{i=1}^{\nu}\|q_i\varphi_1 u\|_{H(\varepsilon-1)} \leqslant C'(\|\varphi_1 u\| + \sum_{i=1}^{m}\|X_i(\varphi_1 u)\|).$$



In order to get (4.1) we wil need operators identifying different $H(s)$ and commuting with $L$. To do this we use powers of a right-invariant Laplacian on $G$. Denote it by $\Delta$.

LEMMA 8. *For any number of the type $s = k/2^l$, $k$, $l$ being integers, $l > 0$, the operator $(I-\Delta)^{s/2}$ is continuous from $H_{m,\mathrm{comp}}$ to $H_{m-s,\mathrm{loc}}$, $m \in \boldsymbol{R}$.*

Proof. For $s = 2$ or $s = -2$ the claim is well known (Friedrichs' inequality if $s = -2$). So, by induction, it suffices to prove that continuity for $s$ implies continuity for $s/2$. Using the induction hypothesis, we obtain

$$\|(I-\Delta)^{s/2}u\| = \langle(I-\Delta)^s u, u\rangle$$
$$\leqslant \|(I-\Delta)^s u\|_{H(-s)}\|u\|_{H(s)} \leqslant C\|u\|_{H(s)}^2.$$

Next, we estimate the norm $\|(I-\Delta)^{s/2}u\|_{H(2m)}$, $m$ being an integer:

$$\|(I-\Delta)^m(I-\Delta)^{s/2}u\| = \|(I-\Delta)^{s/2}(I-\Delta)^m u\|$$
$$\leqslant C\|(I-\Delta)^m u\|_{H(s)} \leqslant C_1\|u\|_{H(2m+s)}.$$

The result for any $m$ is now obtained by interpolation.

LEMMA 9. *The operator $(I-\Delta)^s$, $s \in \boldsymbol{R}$, is the operator of convolution with a distribution smooth outside zero. In addition, all derivatives of the kernel are integrable outside any neighborhood of zero.*

Proof. Since $\Delta$ is the image of a homogeneous right-invariant sublaplacian on a free nilpotent group it suffices to prove the lemma for such an operator.

In fact, if $G = F/N$ then

$$\int_{G\setminus U}|Y^\alpha\Delta^s(aN)|\,d(aN) = \int_{F/N\setminus U}\Big|\int_N Y^\alpha\Delta_F^s(an)\,dn\Big|\,d(aN)$$
$$\leqslant \int_{F\setminus\pi^{-1}(U)}|Y^\alpha\Delta_F^s(a)|\,da.$$

Next, the smoothness follows from the integrability of derivatives.

Now, considering a nilpotent free group, we can apply the estimate of Theorem 1 for $L = \Delta_F$ and then (3.3) gives the integrability of $Y^\alpha(I-\Delta_F)^s$ outside any neighborhood of zero. This concludes the proof of Lemma 9.

Equipped with the above lemmas we can start the proof of (4.1). First, given neighborhoods $U \subset V$ of $e$, choose a $C^\infty$-function $\varphi$ supported on $V$ with $\varphi = 1$ on $U$. For any $s \in \boldsymbol{R}$ define compactly supported distributions $A_s = \varphi(I-\Delta)^{-s/2}$, $B_s = \varphi(I-\Delta)^{s/2}$ (now $(I-\Delta)^{-s/2}$, $(I-\Delta)^{s/2}$ are considered as distributions).

From Lemma 9 it follows that $W_1 = (I-\Delta)^{s/2} - B_s$ and $W_2 = (I-\Delta)^{-s/2} - A_s$ are integrable functions and, moreover, all their derivatives are also integrable. Since

$$A_s * B_s = I - A_s * W_1 - W_2 * B_s + W_2 * W_1,$$

$A_s * B_s - I$ has a smooth kernel, and similarly for $B_s * A_s - I$.

It is easy to observe that, by induction, (4.1) follows from

$$\|\varphi_1 u\|_{H(s+\varepsilon)} \leqslant C(\|\varphi_2' Lu\|_{H(s)} + \|\varphi_2' u\|_{H(s)}), \tag{4.3}$$

where $\varphi_1$, $\varphi_2'$ are compactly supported $C^\infty$-functions, $\varphi_2' = 1$ on a neighborhood of $\operatorname{supp}\varphi_1$. Now, define $K_1 = \operatorname{supp}\varphi_1$, $K_4 = \{x\colon \varphi_2'(x) = 1\}$ and choose $V$, a symmetric neighborhood of zero, and compact sets $K_2$, $K_3$ satisfying $K_i \cdot V^2 \subseteq K_{i+1}$, $i = 1, 2, 3$. Choose also distributions $A_s$, $B_s$ as above and $C^\infty$-functions $\psi_i$, $i = 1, \ldots, 4$, $\psi_1 = \varphi_1$, $\operatorname{supp}\psi_i \subseteq K_i$, $\psi_i = 1$ on $K_{i-1} \cdot V$, $i = 2, 3, 4$.

Now, since $\|u\|_{H(m)} \leqslant C(\|B_s u\|_{H(m-s)} + \|u\|_{H(m-s)})$ and $\|B_s u\|_{H(m)} \leqslant C\|u\|_{H(m+s)}$, we have

$$\begin{aligned}\|\psi_1 u\|_{H(s+\varepsilon)} &\leqslant C(\|\psi_2 B_s u\|_{H(\varepsilon)} + \|\psi_2 u\|_{H(s)}) \\ &\leqslant C_2(\|\psi_3 LB_s u\| + \|\psi_3 B_s u\| + \|\psi_2 u\|_{H(s)}) \\ &\leqslant C(\|\psi_4 Lu\|_{H(s)} + \|\psi_4 u\|_{H(s)}),\end{aligned}$$

which clearly gives (4.3) and thus concludes the proof of (4.1).

## References


[1] J. Dziubański and A. Hulanicki, *On semigroups generated by left-invariant positive differential operators on nilpotent Lie groups*, Studia Math. 94 (1989), 81–95.

[2] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, 1982.

[3] L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer, 1985.

[4] D. S. Jerison and A. Sánchez-Calle, *Estimates for the heat kernel for a sum of squares of vector fields*, Indiana Univ. Math. J. 35 (1986), 835–854.

[5] J. J. Kohn, *Pseudo-differential operators and non-elliptic problems*, in Pseudo-differential Operators, CIME Conference, Stresa 1968, Edizioni Cremonese, Roma 1969, 157–165.



INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland


*Received October 24, 1988* (2500)

# STUDIA MATHEMATICA

*Contents of forthcoming issues*

### Volume XCV, number 2



### Volume XCV, number 3



### Volume XCVI, number 1