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Continuity of derivations from radical convolution algebras

by

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Abstract. We study the question whether or not every derivation from a radical weighted convolution algebra $A = L^1(\boldsymbol{R}^+, \omega)$, or from certain related algebras, into a Banach A-module is necessarily continuous. We show that such a derivation must have striking continuity properties.

Introduction. This paper studies continuity properties of module derivations from radical convolution algebras defined on the non-negative real numbers and on its subsemigroups.

First we describe our algebras and recall some standard notions which may be found in the text of Bonsall and Duncan [6].

Let $\boldsymbol{R}^+ = [0, \infty)$. A *radical weight* on $\boldsymbol{R}^+$ is a continuous, positive, submultiplicative function ω for which $\omega(0) = 1$ and $\lim_{t\to\infty} \omega(t)^{1/t} = 0$. We denote by $L^1(\boldsymbol{R}^+, \omega)$ the set of all equivalence classes of Borel measurable functions on $\boldsymbol{R}^+$ for which $\|f\| = \int_{\boldsymbol{R}^+} |f(t)|\omega(t)\,dt < \infty$. The set of all Radon measures μ on the Borel sets of $\boldsymbol{R}^+$ for which $\|\mu\| = \int_{\boldsymbol{R}^+} \omega(t)|\mu|(dt) < \infty$ will be denoted by $M(\boldsymbol{R}^+, \omega)$. Under convolution multiplication, $M(\boldsymbol{R}^+, \omega)$ is a local Banach algebra having for its identity the unit point mass at zero, while $L^1(\boldsymbol{R}^+, \omega)$ is a radical Banach algebra which is a closed ideal in $M(\boldsymbol{R}^+, \omega)$. Let $\boldsymbol{S}$ be a subgroup of $\boldsymbol{R}$, and let $\boldsymbol{S}^+ = \boldsymbol{S} \cap \boldsymbol{R}^+$. Then $l^1(\boldsymbol{S}^+, \omega)$ is the subalgebra of $M(\boldsymbol{R}^+, \omega)$ consisting of those discrete measures $f = \sum\{f(s)\delta_s\colon s \in \boldsymbol{S}^+\}$ for which $\|f\| = \sum\{|f(s)|\omega(s)\colon s \in \boldsymbol{S}^+\} < \infty$.

Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be an $\mathfrak{A}$-module (so that the module operations satisfy $a\cdot x = x\cdot a$ $(a \in \mathfrak{A},\ x \in \mathfrak{M})$). Then $\mathfrak{M}$ is a *Banach $\mathfrak{A}$-module* if $\mathfrak{M}$ is a Banach space and if the module operation $(a, x) \mapsto a\cdot x$ satisfies

$$\|a\cdot x\| \leqslant \|a\|\,\|x\| \quad (a \in \mathfrak{A},\ x \in \mathfrak{M}).$$

For example, one can regard $\mathfrak{A}$ as a Banach $\mathfrak{A}$-module over itself, where, of course, the module operation is the product in $\mathfrak{A}$. Then the dual $\mathfrak{A}'$ of $\mathfrak{A}$ becomes the *dual module* with respect to the operation $(a, \lambda) \mapsto a\cdot\lambda$, where

$$(a\cdot\lambda)(b) = \lambda(ab) \quad (a, b \in \mathfrak{A},\ \lambda \in \mathfrak{A}').$$

Let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. A linear map $D\colon \mathfrak{A}\to\mathfrak{M}$ is a *derivation* if

$$D(ab) = a\cdot Db + Da\cdot b \qquad (a, b\in\mathfrak{A}).$$

The main question that we shall study in this paper is the following.

QUESTION. Let A be any of the algebras $L^1(\boldsymbol{R}^+, \omega)$, $M(\boldsymbol{R}^+, \omega)$, or $l^1(S^+, \omega)$. Is every derivation from A into a Banach A-module necessarily continuous?

The answer to this question is not known to us for any radical weight ω or any dense subgroup S of $\boldsymbol{R}$. However, we shall show that derivations from these algebras must have certain striking continuity properties.

In a companion paper [4], we have investigated derivations from the algebras $l^1(\boldsymbol{Z}^+, \omega)$ and, more generally, from Banach and Fréchet algebras of power series. We prove in [4] that every Fréchet algebra of power series admits a discontinuous module derivation which vanishes on the subalgebra $\boldsymbol{C}[X]$ of polynomials and has infinite-dimensional range. This theorem shows, in particular, that discontinuous module derivations abound from the algebras $l^1(\boldsymbol{Z}^+, \omega)$.

We now proceed to a more detailed description of the results of this paper. We shall use the following notation: $\boldsymbol{R}^+ = [0, \infty)$, $\boldsymbol{R}^- = (-\infty, 0]$, $\boldsymbol{Q}^+ = \boldsymbol{Q}\cap\boldsymbol{R}^+$, $\boldsymbol{Q}^- = \boldsymbol{Q}\cap\boldsymbol{R}^-$, $\boldsymbol{Z}^+ = \boldsymbol{Z}\cap\boldsymbol{R}^+$, $\boldsymbol{Z}^- = \boldsymbol{Z}\cap\boldsymbol{R}^-$, $\boldsymbol{N} = \boldsymbol{Z}^+\setminus\{0\}$.

Let ω be a radical weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. In Section 1 we represent the dual module A' as $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$, where $\check{\omega}(t) = \omega(-t)$ $(t\in\boldsymbol{R}^+)$. We show that the dual module action on A' is given by $a\cdot\varphi = (a*\varphi)\,|\,\boldsymbol{R}^-$ $(a\in A,\ \varphi\in A')$. By a theorem of Johnson, the question of the continuity of derivations from A into arbitrary Banach A-modules is reduced to the question of the continuity of derivations from A into its dual module A'. The dual module contains the separable submodule $\mathfrak{X} = C_0(\boldsymbol{R}^-, \check{\omega}^{-1})$, consisting of those continuous functions φ in $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ for which $\lim_{t\to-\infty}\varphi(t)/\omega(-t) = 0$. We prove that ω is regulated if and only if $a\cdot\varphi\in\mathfrak{X}$ for every $a\in A$ and $\varphi\in A'$. Thus for a regulated weight we need only discuss the continuity of derivations from A into $\mathfrak{X}$. We close this section with a discussion of the closed submodules of $\mathfrak{X}$ and their relation to the closed ideals in A.

In Section 2 we recall and strengthen theorems of Groenbaek [14] which yield a characterization of all continuous derivations from A into A'. Such a derivation has the form $D\colon f\mapsto(xf*\psi)\,|\,\boldsymbol{R}^-$, where $(xf)(t) = tf(t)$ $(t\in\boldsymbol{R}^+)$ and ψ is a certain measurable function on $\boldsymbol{R}^-$. A similar theorem holds for the algebras $l^1(S^+, \omega)$.

The main results of the paper concerning continuity properties of module derivations are found in Section 3. Again let ω be a radical weight function on $\boldsymbol{R}^+$. We begin by investigating the class W of all functions ψ which serve to

represent continuous derivations from A to A'. The space W is a Banach space for the norm

$$|||\psi||| = \sup\{\|(t\delta_t*\psi)\,|\,\boldsymbol{R}^-\|/\omega(t)\colon\ t\in\boldsymbol{R}^+\setminus\{0\}\}.$$

We prove that W coincides with the class of measurable functions ψ on $\boldsymbol{R}^-$ for which the function $x\psi\colon t\mapsto t\psi(t)$, $\boldsymbol{R}^-\to\boldsymbol{C}$, belongs to A'. In [12] Ghahramani characterized the class of all (necessarily continuous) derivations from A to itself. A map $D\colon A\to A$ is a continuous derivation if and only if $Df = xf*\mu$ $(f\in A)$, where μ is a Radon measure on $\boldsymbol{R}^+$ satisfying

$$\sup_{a>0}\frac{a}{\omega(a)}\int_{\boldsymbol{R}^+}\omega(t+a)\,|\mu|(dt) < \infty.$$

We prove that this class of measures coincides with the set of all Radon measures μ for which $\varphi*\mu\in A'$ for all $\varphi\in A'$.

Using results mentioned above, we prove, under mild conditions on ω, that if $D\colon A\to A'$ is any derivation and if $g\in A$, then D is continuous if and only if $g\cdot D$ is continuous (where $(g\cdot D)(f) = g\cdot D(f)$ $(f\in A)$). Moreover, an element g can be found so that $g\cdot D$ is continuous on a dense subalgebra of A. As a consequence, we obtain theorems of which the following is an example. Let ω satisfy

$$\sup_{a>0}\frac{a}{\omega(a)}\int_{\boldsymbol{R}^+}\omega(t+a)\,dt < \infty,$$

and let $g(t) = t^2$ $(t\in\boldsymbol{R}^+)$. Then, for any derivation $D\colon A\to A'$, the derivation $g\cdot D$ is continuous on the dense subalgebra of all polynomials in t.

Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. There is a standard way of constructing discontinuous derivations from $\mathfrak{A}$ into $\mathfrak{M}$ in the case where $\mathfrak{A}$ is an integral domain and $\mathfrak{M}$ contains a non-zero, $\mathfrak{A}$-divisible, torsion-free submodule. Since the algebras we are considering are all integral domains, one might hope that this method would work at least for the algebra $L^1(\boldsymbol{R}^+, \omega)$. However, in Section 4, we shall prove that this hope is futile, for we shall show that $L^1(\boldsymbol{R}^+, \omega)$ has no non-zero, $L^1(\boldsymbol{R}^+, \omega)$-divisible submodule in any Banach $L^1(\boldsymbol{R}^+, \omega)$-module.

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1. Weights, algebras, and modules. A *weight function* on $\boldsymbol{R}^+$ is a continuous function ω: $\boldsymbol{R}^+ \to \boldsymbol{R}^+ \setminus \{0\}$ satisfying

(1.1) $$\omega(0) = 1, \quad \omega(s+t) \leqslant \omega(s)\omega(t) \quad (s, t \in \boldsymbol{R}^+).$$

The weight function ω is *radical* if

(1.2) $$\lim_{t\to\infty} \omega(t)^{1/t} = 0.$$

For example, let $\omega(t) = e^{-t^2}$ $(t \in \boldsymbol{R}^+)$. Then ω is a radical weight function on $\boldsymbol{R}^+$.

Let S be a subgroup of the real numbers. We shall also consider radical weight functions on $S^+ = S \cap \boldsymbol{R}^+$: these are the restrictions to S^+ of radical weight functions on $\boldsymbol{R}^+$. Important cases are where $S = \boldsymbol{Q}$, $S = \boldsymbol{Z}$, and $S = \{m+n\alpha\colon m, n \in \boldsymbol{Z}\}$, where α is an irrational. In this paper emphasis is on the case where S is dense in $\boldsymbol{R}$. In a companion paper [4] we treat the case of radical weight functions on $\boldsymbol{Z}^+$.

We also considered radical weight functions on $\boldsymbol{R}^+$ in the paper [3]. Note, however, that, on that occasion, we only required that a weight function on $\boldsymbol{R}^+$ be measurable, rather than continuous, and so we allowed weight functions that were unbounded near the origin.

Let $L^1_{\rm loc}(\boldsymbol{R})$ be the class of locally integrable functions on $\boldsymbol{R}$. For $f \in L^1_{\rm loc}(\boldsymbol{R})$, write $\operatorname{supp} f$ for the support of f, and set

$$\alpha(f) = \inf \operatorname{supp} f \quad (f \in L^1_{\rm loc}(\boldsymbol{R}) \setminus \{0\}).$$

For convenience we set $\alpha(0) = \infty$. If $f, g \in L^1_{\rm loc}(\boldsymbol{R})$, the convolution $f*g$ is defined by the formula

$$(f*g)(t) = \int_{\boldsymbol{R}} f(t-s)g(s)\,ds$$

for those values of $t \in \boldsymbol{R}$ such that $\int_{\boldsymbol{R}} |f(t-s)g(s)|\,ds < \infty$. For example, if $\alpha(f) > -\infty$ and $\alpha(g) > -\infty$, then $(f*g)(t)$ exists for almost all $t \in \boldsymbol{R}$, and, by the Titchmarsh convolution theorem,

$$\alpha(f*g) = \alpha(f) + \alpha(g).$$

Hereafter a function defined on $\boldsymbol{R}^+$ or $\boldsymbol{R}^-$ will be regarded as being defined on the whole of $\boldsymbol{R}$, having the value zero on the complementary interval. If f is defined on $\boldsymbol{R}$, set

$$\check{f}(t) = f(-t) \quad (t \in \boldsymbol{R}).$$

For example, if $f(t)$ is defined for $t \in \boldsymbol{R}^+$, then $\check{f}(t) = f(-t)$ for $t \in \boldsymbol{R}^-$, and $\check{f}(t) = 0$ for $t \in \boldsymbol{R}^+ \setminus \{0\}$. If $f, g \in L^1_{\rm loc}(\boldsymbol{R}^+)$, then $f*g$ is given by

(1.3) $$(f*g)(t) = \int_0^t f(t-s)g(s)\,ds \quad (t \in \boldsymbol{R}),$$

and $f*g \in L^1_{\rm loc}(\boldsymbol{R}^+)$.

Before describing the Banach algebras we shall discuss, it is convenient to introduce the notion of a Radon measure on $\boldsymbol{R}^+$. Our reference for Radon measures will be [11, Ch. 4], although we modify some of the definitions there in inessential ways. Let $\mathfrak{K}(\boldsymbol{R}^-)$ be the linear space of all continuous, complex-valued functions φ for which $\operatorname{supp}\varphi$ is a compact subset of $\boldsymbol{R}^-$. Writing $\boldsymbol{R}^- = \bigcup_{n=1}^{\infty} S_n$, where $S_n = [-n, 0]$, we topologize $\mathfrak{K}(\boldsymbol{R}^-)$ as the strict inductive limit of the Banach spaces $\mathfrak{K}(S_n) = C([-n, 0])$ (see [11, p. 177 ff.]). A *Radon measure* on $\boldsymbol{R}^+$ is an element μ of the dual $\mathfrak{K}(\boldsymbol{R}^-)'$ of $\mathfrak{K}(\boldsymbol{R}^-)$. We denote by $M_{\rm loc}(\boldsymbol{R}^+)$ the set of all Radon measures on $\boldsymbol{R}^+$. The duality is given by the pairing

$$\langle \mu, \varphi\rangle = \int_{\operatorname{supp}\varphi} \check{\varphi}(t)\,\mu(dt) \quad (\mu \in M_{\rm loc}(\boldsymbol{R}^+),\ \varphi \in \mathfrak{K}(\boldsymbol{R}^-)).$$

If $\mu \in M_{\rm loc}(\boldsymbol{R}^+)$, then the restriction of μ to $\mathfrak{K}(S_n)$ induces, via the Riesz theorem, a measure $\dot{\mu}_n$ in the usual sense on the Borel subsets of $[0, n]$ via the pairing

(1.4) $$\langle \mu, \varphi\rangle = \int_{\operatorname{supp}\varphi} \check{\varphi}(t)\,\dot{\mu}(dt) \quad (\varphi \in \mathfrak{K}(S_n)).$$

Thus, by varying n, we obtain a set function $\dot{\mu}$ which is countably additive on the ring of all bounded Borel sets of $\boldsymbol{R}^+$. Conversely, any such "locally finite" measure $\dot{\mu}$ on the ring of bounded Borel sets determines a Radon measure μ by the pairing (1.4), where now φ ranges over $\mathfrak{K}(\boldsymbol{R}^-)$. Consequently, if $\mu \in M_{\rm loc}(\boldsymbol{R}^+)$, its *total variation* $|\mu|$ is defined as a Radon measure on $\boldsymbol{R}^+$ from the total variations $|\dot{\mu}_n|$ of the measures $\dot{\mu}_n$. Of course the corresponding positive set function $|\dot{\mu}|$ can be extended to a Borel measure on the σ-field of all Borel subsets of $\boldsymbol{R}^+$ (possibly taking infinite values). Hereafter we identify the functional μ and the set function $\dot{\mu}$, and assume, in the case of positive measures, that the extension has been made to all Borel sets.

Let $\mu, \nu \in M_{\rm loc}(\boldsymbol{R}^+)$. We define the convolution $\mu * \nu$ of μ and ν by the equation

(1.5) $$\langle \mu*\nu, \varphi\rangle = \int_{\boldsymbol{R}^+}\int_{\boldsymbol{R}^+} \check{\varphi}(s+t)\,\mu(ds)\,\nu(dt) \quad (\varphi \in \mathfrak{K}(\boldsymbol{R}^-)).$$

Since the function $\check{\phi}$: $(s, t) \mapsto \check{\phi}(s+t)$ has compact support on $\boldsymbol{R}^+ \times \boldsymbol{R}^+$, the integral exists and $\mu*\nu$ is a Radon measure on $\boldsymbol{R}^+$. Clearly $M_{\rm loc}(\boldsymbol{R}^+)$ is an algebra with respect to convolution multiplication.

So far we have not topologized $M_{\rm loc}(\boldsymbol{R}^+)$. For each $n \in \boldsymbol{N}$, define the seminorm $\varrho_n(\mu) = |\mu|([0, n])$ $(\mu \in M_{\rm loc}(\boldsymbol{R}^+))$. Then $M_{\rm loc}(\boldsymbol{R}^+)$ is a Fréchet space for the sequence (ϱ_n) of seminorms. Since $\varrho_n(\mu*\nu) \leqslant \varrho_n(\mu)\varrho_n(\nu)$ $(n \in \boldsymbol{N},$ $\mu, \nu \in M_{\rm loc}(\boldsymbol{R}^+))$, $M_{\rm loc}(\boldsymbol{R}^+)$ is also a Fréchet algebra.

If $f \in L^1_{\rm loc}(\boldsymbol{R}^+)$, define the measure μ_f by setting $\mu_f(dt) = f(t)\,dt$. Then $\mu_f \in M_{\rm loc}(\boldsymbol{R}^+)$, and the correspondence $f \mapsto \mu_f$ embeds $L^1_{\rm loc}(\boldsymbol{R}^+)$ as a closed ideal in

$M_{\rm loc}(\boldsymbol{R}^+)$. Moreover,

$$(f*\mu)(t) = \int_0^t f(t-s)\,\mu(ds) \qquad (f\in L^1_{\rm loc}(\boldsymbol{R}^+),\ \mu\in M_{\rm loc}(\boldsymbol{R}^+)).$$

If $\mu\in M_{\rm loc}(\boldsymbol{R}^+)$, we define $\operatorname{supp}\mu$ to be the support of the positive Borel measure $|\mu|$, and we define $\alpha(\mu) = \inf\operatorname{supp}\mu$. It follows from the Titchmarsh theorem for measures that $\alpha(\mu*\nu) = \alpha(\mu)+\alpha(\nu)$ $(\mu,\ \nu\in M_{\rm loc}(\boldsymbol{R}^+))$. Consequently $M_{\rm loc}(\boldsymbol{R}^+)$ is an integral domain. It has an identity δ_0, the unit point mass at zero.

In an exactly similar way, one can define a Radon measure on $\boldsymbol{R}$ as an element of $\mathfrak{K}(\boldsymbol{R})'$ via the pairing

$$\langle\mu,\varphi\rangle = \int_{\boldsymbol{R}} \check{\varphi}(t)\,\mu(dt) \qquad (\varphi\in\mathfrak{K}(\boldsymbol{R})).$$

The convolution of Radon measures on $\boldsymbol{R}$ is more complicated. If $\varphi\in\mathfrak{K}(\boldsymbol{R})\backslash\{0\}$, the function

$$\tilde{\varphi}\colon (s,t)\mapsto\check{\varphi}(s+t), \quad \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R},$$

does not necessarily have compact support in $\boldsymbol{R}\times\boldsymbol{R}$. For Radon measures $\mu,\ \nu\in M_{\rm loc}(\boldsymbol{R})$, the convolution $\mu*\nu$ exists as a Radon measure on $\boldsymbol{R}$ if and only if, for each $\varphi\in\mathfrak{K}(\boldsymbol{R})$, the function $\tilde{\varphi}$ is integrable for the product measure $|\mu|\times|\nu|$. In this case,

$$\langle\mu*\nu,\varphi\rangle = \int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \check{\varphi}(s+t)\,\mu(ds)\,\nu(dt) \qquad (\varphi\in\mathfrak{K}(\boldsymbol{R})).$$

This is the only case that we shall consider.

Finally, we shall need the existence of certain more general convolutions. Let φ be a Borel measurable function on $\boldsymbol{R}^-$, and let $\mu\in M_{\rm loc}(\boldsymbol{R}^+)$. We say the *convolution* $\varphi*\mu$ *exists on* $\boldsymbol{R}^-$ if

$$\int_{\boldsymbol{R}^+} |\varphi(t-s)|\,|\mu|(ds) < \infty$$

for almost all $t\in\boldsymbol{R}^-$. Then

$$(\varphi*\mu)(t) = \int_{\boldsymbol{R}^+} \varphi(t-s)\,\mu(ds) \qquad (t\in\boldsymbol{R}^-).$$

We now introduce the Banach algebras with which we shall be concerned. The first ones we discuss will all be subalgebras of $L^1_{\rm loc}(\boldsymbol{R}^+)$ or of $M_{\rm loc}(\boldsymbol{R}^+)$.

Let ω be a weight function on $\boldsymbol{R}^+$. Denote by $L^1(\boldsymbol{R}^+,\omega)$ the set of all equivalence classes of Lebesgue measurable, complex-valued functions f on $\boldsymbol{R}^+$ such that $\int_{\boldsymbol{R}^+}|f(t)|\omega(t)\,dt<\infty$. It is standard that $L^1(\boldsymbol{R}^+,\omega)$ is an algebra with respect to pointwise addition, scalar multiplication of functions, and the convolution product given by (1.3). Set $\|f\| = \int_{\boldsymbol{R}^+}|f(t)|\omega(t)\,dt$ $(f\in L^1(\boldsymbol{R}^+,\omega))$. Then $(L^1(\boldsymbol{R}^+,\omega),\ \|\cdot\|)$ is a Banach algebra. We shall be particularly concerned with radical weight functions ω: in this case $L^1(\boldsymbol{R}^+,\omega)$ is a radical Banach algebra.

Let $C_0(\boldsymbol{R}^-,\check{\omega}^{-1})$ denote the linear space of all continuous, complex-valued functions φ defined on $\boldsymbol{R}^-$ for which

$$\|\varphi\| = \sup\left\{\frac{|\check{\varphi}(t)|}{\omega(t)}\colon t\in\boldsymbol{R}^+\right\}<\infty \quad\text{and}\quad \frac{|\check{\varphi}(t)|}{\omega(t)}\to 0 \quad \text{as } t\to\infty.$$

Then $C_0(\boldsymbol{R}^-,\check{\omega}^{-1})$ is a Banach space. The space of measures $M(\boldsymbol{R}^+,\omega)$ is defined to be the dual space of $C_0(\boldsymbol{R}^-,\check{\omega}^{-1})$ with respect to the pairing

$$\langle\mu,\varphi\rangle = \int_{\boldsymbol{R}^+} |\check{\varphi}(t)|\,\mu(dt) \qquad (\mu\in M(\boldsymbol{R}^+,\omega),\ \varphi\in C_0(\boldsymbol{R}^-,\check{\omega}^{-1})).$$

It is clear that $M(\boldsymbol{R}^+,\omega)$ is a linear subspace of $M_{\rm loc}(\boldsymbol{R}^+)$, and that $M(\boldsymbol{R}^+,\omega)$ is a Banach space for the norm

$$\|\mu\| = \int_{\boldsymbol{R}^+} \omega(t)\,|\mu|(dt) \qquad (\mu\in M(\boldsymbol{R}^+,\omega)).$$

For example, if $s\in\boldsymbol{R}^+$, then δ_s, the unit point mass at s, belongs to $M(\boldsymbol{R}^+,\omega)$, and $\|\delta_s\| = \omega(s)$. If $\mu,\ \nu\in M(\boldsymbol{R}^+,\omega)$, the convolution $\mu*\nu$ belongs to $M(\boldsymbol{R}^+,\omega)$, and $M(\boldsymbol{R}^+,\omega)$ is a Banach algebra with respect to this product. The correspondence $f\mapsto\mu_f$, where $\mu_f(dt) = f(t)\,dt$, defines an isometric embedding of $L^1(\boldsymbol{R}^+,\omega)$ into $M(\boldsymbol{R}^+,\omega)$, and $L^1(\boldsymbol{R}^+,\omega)$ is a closed ideal in $M(\boldsymbol{R}^+,\omega)$.

Let $A = L^1(\boldsymbol{R}^+,\omega)$. We shall represent the dual space A' of A as the space of functions $L^\infty(\boldsymbol{R}^-,\check{\omega}^{-1})$ on $\boldsymbol{R}^-$. Thus A' is the linear space of equivalence classes of measurable functions φ on $\boldsymbol{R}^-$ for which

$$\|\varphi\| = \operatorname{ess\,sup}\left\{\frac{|\check{\varphi}(t)|}{\omega(t)}\colon t\in\boldsymbol{R}^+\right\}<\infty.$$

The pairing defining the duality is

$$\langle f,\varphi\rangle = \int_{\boldsymbol{R}^+} f(s)\check{\varphi}(s)\,ds \qquad (f\in A,\ \varphi\in A').$$

Let $f\in A$ and $\varphi\in A'$. Then the convolution product $f*\varphi$ is defined at least for $t\in\boldsymbol{R}^-$ by the formula

$$(f*\varphi)(t) = \int_{\boldsymbol{R}^+} \varphi(t-s)f(s)\,ds,$$

and $(f*\varphi)\,|\,\boldsymbol{R}^-$ belongs to A'. A straightforward calculation shows that

(1.6) $$\langle g,(f*\varphi)\,|\,\boldsymbol{R}^-\rangle = \langle f*g,\varphi\rangle \qquad (f,\ g\in A,\ \varphi\in A').$$

Let S be any subgroup of $\boldsymbol{R}$, and set $S^+ = S\cap\boldsymbol{R}^+$. We denote by $l^1(S^+,\omega)$ the subalgebra of $M(\boldsymbol{R}^+,\omega)$ consisting of the discrete measures $f = \sum\{f(s)\delta_s\colon s\in S^+\}$ for which $\|f\| = \sum\{|f(s)|\omega(s)\colon s\in S^+\}<\infty$. Since $l^1(S^+,\omega)$ is closed in $M(\boldsymbol{R}^+,\omega)$, it is a Banach algebra with respect to convolution multiplication. We shall be mainly concerned with the cases where $S = \boldsymbol{R}$, $\boldsymbol{Q}$, or $\boldsymbol{Z}$.

Note that, in contrast to the case of $L^1(\boldsymbol{R}^+, \omega)$, each of the algebras $M(\boldsymbol{R}^+, \omega)$ and $l^1(\boldsymbol{S}^+, \omega)$ has an identity element: it is the point mass δ_0 at zero.

We shall write $(\delta_t * f)(s) = f(s-t)$ $(s, t \in \boldsymbol{R})$ whenever f is a function defined on $\boldsymbol{R}$: the map $t \mapsto \delta_t * f$ is the *right shift operator* on f. Suppose that $f \in L^1(\boldsymbol{R}^+, \omega)$ and that $t \in \boldsymbol{R}^+$. Then we regard $\delta_t * f$ as being defined on $\boldsymbol{R}^+$:

$$(\delta_t * f)(s) = \begin{cases} 0 & (0 \leqslant s \leqslant t), \\ f(s-t) & (s > t), \end{cases}$$

and $\delta_t * f \in L^1(\boldsymbol{R}^+, \omega)$. Further, $\|\delta_t * f\| \leqslant \omega(t)\|f\|$ $(t \in \boldsymbol{R}^+)$.

The question we shall be concerned with in this paper is whether or not, in the case where ω is a radical weight function on $\boldsymbol{R}^+$, every derivation from any of the algebras $A = L^1(\boldsymbol{R}^+, \omega)$, $M(\boldsymbol{R}^+, \omega)$, or $l^1(\boldsymbol{S}^+, \omega)$ into a Banach A-module is necessarily continuous. Except for the algebra $l^1(\boldsymbol{Z}^+, \omega)$, the answer to this question is not known to us for any of the above algebras for any radical weight function ω. In the paper [4] we have constructed a variety of discontinuous derivations from the algebra $l^1(\boldsymbol{Z}^+, \omega)$.

We now begin our study of the continuity of derivations.

We first recall some standard notation. Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be a Banach space which is an $\mathfrak{A}$-bimodule with respect to the operations $(a, m) \mapsto a \cdot m$ and $(a, m) \mapsto m \cdot a$ from $\mathfrak{A} \times \mathfrak{M}$ into $\mathfrak{M}$ (see [6, p. 49]). Then we call $\mathfrak{M}$ a *Banach $\mathfrak{A}$-module* if $\mathfrak{A}$ is commutative (so that $a \cdot m = m \cdot a$ $(a \in \mathfrak{A},\ m \in \mathfrak{M})$) and if the above operations are continuous; in this case we always assume that $\|a \cdot m\| \leqslant \|a\|\,\|m\|$ $(a \in \mathfrak{A},\ m \in \mathfrak{M})$.

Throughout we write $\mathfrak{M}'$ for the dual of a Banach space $\mathfrak{M}$. Let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. Then $\mathfrak{M}'$ is a Banach $\mathfrak{A}$-module with respect to the operation defined by

$$\langle m, a \cdot \lambda \rangle = \langle a \cdot m, \lambda \rangle \quad (a \in \mathfrak{A},\ m \in \mathfrak{M},\ \lambda \in \mathfrak{M}'):$$

this module is the *dual module* to $\mathfrak{M}$ (see [6, p. 50]).

Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. A *derivation* from $\mathfrak{A}$ into $\mathfrak{M}$ is a linear map $D\colon \mathfrak{A} \to \mathfrak{M}$ such that

$$D(ab) = a \cdot Db + Da \cdot b \quad (a, b \in \mathfrak{A}).$$

Derivations of commutative Banach algebras into modules have been studied in [1], [17], and [18], for example.

Let E and F be Banach spaces, and let $T\colon E \to F$ be a linear map. The *separating space* of T is $\mathfrak{S}(T)$, where

$$\mathfrak{S}(T) = \{y \in F\colon \text{there exists } (x_n) \subset E \text{ such that } x_n \to 0 \text{ and } Tx_n \to y\}.$$

Certainly, $\mathfrak{S}(T)$ is a closed linear subspace of F, and T is continuous if and only if $\mathfrak{S}(T) = \{0\}$. If $\mathfrak{A}$ is a commutative Banach algebra, $\mathfrak{M}$ is a Banach $\mathfrak{A}$-module, and $D\colon \mathfrak{A} \to \mathfrak{M}$ is a derivation, then the *continuity ideal* of D is $\mathfrak{I}(D)$, where

$$\mathfrak{I}(D) = \{a \in \mathfrak{A}\colon a \cdot \mathfrak{S}(D) = \{0\}\}.$$

Then $\mathfrak{I}(D)$ is a closed ideal in $\mathfrak{A}$, and

$$\mathfrak{I}(D) = \{a \in \mathfrak{A}\colon \text{the map } b \mapsto D(ab),\ \mathfrak{A} \to \mathfrak{M}, \text{ is continuous}\}.$$

For these concepts from automatic continuity theory, see [20].

Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. A *bounded approximate identity* (*b.a.i.*) in $\mathfrak{A}$ is a bounded net $\{e_\nu\}$ such that, for each $a \in \mathfrak{A}$, $e_\nu a \to a$, and the net $\{e_\nu\}$ is a *b.a.i. for* $\mathfrak{M}$ if, further, $e_\nu \cdot x \to x$ for each $x \in \mathfrak{M}$ (see [6, Section 11.8]).

There are essentially only two known methods for constructing discontinuous derivations. These are the two methods which are exploited in [4].

The first of these methods is to use discontinuous point derivations. Let $\mathfrak{A}$ be a commutative Banach algebra, and let φ be a character on $\mathfrak{A}$. A *point derivation* at φ is a linear functional $d\colon \mathfrak{A} \to \boldsymbol{C}$ such that

$$d(ab) = d(a)\varphi(b) + d(b)\varphi(a) \quad (a, b \in \mathfrak{A}).$$

If $\mathfrak{A}$ has discontinuous point derivations, then there are discontinuous derivations from $\mathfrak{A}$. For $\boldsymbol{C}$ itself is a Banach $\mathfrak{A}$-module with respect to the map $(a, z) \mapsto \varphi(a)z$, $\mathfrak{A} \times \boldsymbol{C} \to \boldsymbol{C}$, and a point derivation is a derivation with respect to this module operation. Let $\mathfrak{A}$ be a commutative Banach algebra such that $\mathfrak{A}^2$ is not closed in $\mathfrak{A}$ or $\mathfrak{A}^2$ has infinite codimension in $\mathfrak{A}$. Then there exist discontinuous point derivations from $\mathfrak{A}^\#$, the algebra $\mathfrak{A}$ with identity adjoined. This method cannot work for $A = L^1(\boldsymbol{R}^+, \omega)$ because A has the b.a.i. $(e_n\colon n \in \boldsymbol{N})$, where $e_n = n\chi_{[0,1/n]}$, and so, by Cohen's factorization theorem (see [6, Section 11]), $A = A^2$. (We call (e_n) the *standard b.a.i.* for $L^1(\boldsymbol{R}^+, \omega)$.) We do not know whether or not the method works when $\mathfrak{A}$ is the unique maximal ideal in any of the algebras $M(\boldsymbol{R}^+, \omega)$, $l^1(\boldsymbol{R}^+, \omega)$, or $l^1(\boldsymbol{Q}^+, \omega)$ because we do not know whether or not $\mathfrak{A}^2 = \mathfrak{A}$. (We do know that $\mathfrak{A}$ does not have a b.a.i. in these cases.)

The second known way of constructing discontinuous derivations is through the use of $\mathfrak{A}$-divisible, torsion-free modules. This method is explained in Section 4. We will show in Section 4 that this method cannot work for the algebra $L^1(\boldsymbol{R}^+, \omega)$.

The first result shows that the problem of the continuity of derivations into modules can be reduced to the case where the module is the dual module. For similar results, see [2] and [22].

1.1. THEOREM. *Let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. For each $\lambda \in \mathfrak{M}'$, the map R_λ: $\mathfrak{M} \to \mathfrak{A}'$ defined by*

$$\langle a, R_\lambda(m)\rangle = \lambda(a\cdot m) \qquad (a \in \mathfrak{A},\ m \in \mathfrak{M})$$

is a continuous module homomorphism.

Let $\lambda \in \mathfrak{M}'$, let D: $\mathfrak{A} \to \mathfrak{M}$ be a derivation, and let $\tilde{D} = R_\lambda \circ D$. Then $\tilde{D}$ is a derivation from $\mathfrak{A}$ into $\mathfrak{A}'$. If D is discontinuous and if $\lambda(\mathfrak{A}\cdot\mathfrak{G}(D)) \neq \{0\}$, then $\tilde{D}$ is also discontinuous.

Proof. The properties of R_λ and of $\tilde{D}$ are straightforward to verify. The derivation $\tilde{D}$ will be discontinuous if and only if $R_\lambda(\mathfrak{G}(D)) \neq \{0\}$. Suppose that $\lambda(\mathfrak{A}\cdot\mathfrak{G}(D)) \neq \{0\}$. Then there exists $m_0 \in \mathfrak{G}(D)$ and $a_0 \in \mathfrak{A}$ with $\lambda(a_0\cdot m_0) \neq 0$. We have $\langle a_0, R_\lambda(m_0)\rangle = \lambda(a_0\cdot m_0) \neq 0$, and so $R_\lambda(m_0) \neq 0$ and $\tilde{D}$ is discontinuous. ∎

It follows from Theorem 1.1 that, if $\mathfrak{A}$ is a commutative Banach algebra with a b.a.i., and if there is a discontinuous derivation from $\mathfrak{A}$ into any Banach $\mathfrak{A}$-module, then there is a discontinuous derivation from $\mathfrak{A}$ into $\mathfrak{A}'$. For let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module, and let D: $\mathfrak{A} \to \mathfrak{M}$ be a discontinuous derivation. Then $\mathfrak{G}(D) \neq \{0\}$. Let $\{e_\nu\}$ be a b.a.i. for $\mathfrak{A}$ with bound K. We show, by an argument of Willis [22, Lemma 3.1], that $\{e_\nu\}$ is a b.a.i. for $\mathfrak{G}(D)$, and so $\mathfrak{A}\cdot\mathfrak{G}(D) = \mathfrak{G}(D) \neq \{0\}$. Take $(a_n) \subseteq \mathfrak{A}$ and $x \in \mathfrak{G}(D)\backslash\{0\}$ such that $a_n \to 0$ and $D(a_n) \to x$. Then, by the Cohen factorization theorem, there exists $b \in \mathfrak{A}$ and $c_n \to 0$ in $\mathfrak{A}$ such that $a_n = bc_n$ $(n \in N)$. We have $D(bc_n) = b\cdot D(c_n) + D(b)\cdot c_n$, and so $b\cdot D(c_n) \to x$. Take $\varepsilon > 0$ and choose N so that $\|x - b\cdot D(c_N)\| < \varepsilon$. Choose $\nu(N)$ so that $\|b - e_\nu b\| < \varepsilon\|D(c_N)\|^{-1}$ $(\nu \geqslant \nu(N))$. Then, if $\nu \geqslant \nu(N)$,

$$\|x - e_\nu\cdot x\| \leqslant \|x - b\cdot D(c_N)\| + \|(b - e_\nu b)\cdot D(c_N)\| + \|e_\nu\cdot(b\cdot D(c_N) - x)\| < (2+K)\varepsilon.$$

Hence $\{e_\nu\}$ is a b.a.i. for $\mathfrak{G}(D)$. Thus $\mathfrak{A}\cdot\mathfrak{G}(D) \neq \{0\}$, and there exists $\lambda \in \mathfrak{M}'$ with $\lambda(\mathfrak{A}\cdot\mathfrak{G}(D)) \neq \{0\}$.

Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. We now determine the module operations in the dual module A' of A (where A is considered as a module over itself). Since the dual module operation is defined by

$$\langle g, f\cdot\varphi\rangle = \langle f * g, \varphi\rangle \qquad (f, g \in A,\ \varphi \in A'),$$

it follows from (1.6) that

$$f\cdot\varphi = (f*\varphi)\,|\,\boldsymbol{R}^- \qquad (f \in A,\ \varphi \in A').$$

Now let $\boldsymbol{S}$ be any subgroup of $\boldsymbol{R}$ and let $A = l^1(\boldsymbol{S}^+, \omega)$, where ω is a weight function on $\boldsymbol{S}^+$. Then A is a local Banach algebra for convolution multiplication

$$(f*g)(s) = \sum\{f(s-r)g(r):\ r \in \boldsymbol{S}^+\} \qquad (f, g \in A).$$

Every $f \in A$ has countable support. We represent A' as $l^\infty(\boldsymbol{S}^-, \check{\omega}^{-1})$, the space of all bounded functions φ on $\boldsymbol{S}^-$ for which $\|\varphi\| = \sup\{|\check{\varphi}(s)|/\omega(s):\ s \in \boldsymbol{S}^+\} < \infty$. The pairing between A and A' is given by $\langle f, \varphi\rangle = \sum\{\check{\varphi}(s)f(s):\ s \in \boldsymbol{S}^+\}$ $(f \in A,\ \varphi \in A')$. A routine calculation shows that, for all $f, g \in A$ and $\varphi \in A'$, one has $\langle f*g, \varphi\rangle = \langle g, \psi\rangle$, where

$$\psi(r) = \sum\{\varphi(s)f(r-s):\ s \in \boldsymbol{S}^-\} = \sum\{\varphi(r-s)f(s):\ s \in \boldsymbol{S}^+\} \qquad (r \in \boldsymbol{S}^-).$$

Thus the dual module operation of A on A' is given by $f\cdot\varphi = (f*\varphi)\,|\,\boldsymbol{S}^-$ $(f \in A,\ \varphi \in A')$.

1.2. COROLLARY. *Let ω be a weight function on $\boldsymbol{R}^+$, and let A be any one of the algebras $L^1(\boldsymbol{R}^+, \omega)$, $M(\boldsymbol{R}^+, \omega)$, or $l^1(\boldsymbol{S}^+, \omega)$, where $\boldsymbol{S}$ is a subgroup of $\boldsymbol{R}$. Suppose that there exists a discontinuous derivation from A into some Banach A-module. Then there exists a discontinuous derivation from A into A'.*

Proof. Each of these algebras has either an identity or a b.a.i. ∎

Let $\mu \in M(\boldsymbol{R}^+, \omega)$ and $\varphi \in L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$. Then we set

$$\mu\cdot\varphi = (\mu*\varphi)\,|\,\boldsymbol{R}^-.$$

The space $C_0(\boldsymbol{R}^-, \check{\omega}^{-1})$ introduced above will now be denoted by $\mathfrak{X}$.

1.3. THEOREM. *Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$.*

(i) *The Banach A-module $A' = L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ is also a module over $M(\boldsymbol{R}^+, \omega)$, and*

$$\|\mu\cdot\varphi\| \leqslant \|\mu\|\,\|\varphi\| \qquad (\mu \in M(\boldsymbol{R}^+, \omega),\ \varphi \in A').$$

(ii) *The space $\mathfrak{X}$ is a closed $M(\boldsymbol{R}^+, \omega)$-submodule of A'.*

(iii) *We have*

$$\langle \nu, \mu\cdot\varphi\rangle = \langle \mu*\nu, \varphi\rangle \qquad (\mu, \nu \in M(\boldsymbol{R}^+, \omega),\ \varphi \in A').$$

(iv) *The standard b.a.i. $(e_n:\ n \in N)$ for A is also a b.a.i. for $\mathfrak{X}$.*

Proof. We have the estimates

$$\frac{|(\mu\cdot\varphi)^\vee(t)|}{\omega(t)} = \left|\int_{\boldsymbol{R}^+}\frac{\check{\varphi}(s+t)}{\omega(t)}\mu(ds)\right| \leqslant \int_{\boldsymbol{R}^+}\frac{|\check{\varphi}(s+t)|}{\omega(s+t)}\omega(s)\,|\mu|(ds) \leqslant \|\varphi\|\,\|\mu\| \qquad (\varphi \in A',\ \mu \in M(\boldsymbol{R}^+, \omega),\ t \in \boldsymbol{R}^+).$$

It follows that A' is a Banach $M(\boldsymbol{R}^+, \omega)$-module, proving (i).

Let $\varphi \in \mathfrak{X}$, and let $t_n \to t_0$ in $\boldsymbol{R}^-$. An application of the Lebesgue dominated convergence theorem shows that $(\mu\cdot\varphi)(t_n) \to (\mu\cdot\varphi)(t_0)$, so that $\mu\cdot\varphi$ is continuous on $\boldsymbol{R}^-$. Given $\varepsilon > 0$, there exists $N \in \boldsymbol{N}$ such that $|\check{\varphi}(t)|/\omega(t) < \varepsilon$ if $t \geqslant N$. But then

$$\frac{|(\mu\cdot\varphi)^\vee(t)|}{\omega(t)} \leqslant \varepsilon\|\mu\| \qquad (t \geqslant N).$$

Thus $|(\mu\cdot\varphi)^\vee(t)|/\omega(t) \to 0$ as $t \to \infty$, and so $\mu\cdot\varphi \in \mathfrak{X}$. It follows that $\mathfrak{X}$ is an $M(\boldsymbol{R}^+, \omega)$-submodule.

The proof of (iii) is straightforward.

To prove (iv), let $\varphi \in \mathfrak{X}$. Then

$$|(e_n\cdot\varphi)^\vee(t) - \check{\varphi}(t)| \leqslant n \int_0^{1/n} |\check{\varphi}(s+t) - \check{\varphi}(t)|\, ds \qquad (t \in \boldsymbol{R}^+).$$

Given $\varepsilon > 0$, choose T so large that $|\check{\varphi}(t)|/\omega(t) < \varepsilon$ if $t \geqslant T$, and then choose N so large that

$$\frac{|\check{\varphi}(s+t) - \check{\varphi}(t)|}{\omega(t)} < \varepsilon \qquad (t \in [0, T],\ s \in [0, 1/N]).$$

Let $K = \sup\{\omega(s)\colon 0 \leqslant s \leqslant 1\}$. Since

$$\frac{|\check{\varphi}(s+t) - \check{\varphi}(t)|}{\omega(t)} \leqslant \frac{|\check{\varphi}(s+t)|}{\omega(s+t)}\omega(s) + \frac{|\check{\varphi}(t)|}{\omega(t)} \qquad (s, t \in \boldsymbol{R}^+),$$

we have $|\check{\varphi}(s+t) - \check{\varphi}(t)|/\omega(t) < \varepsilon(K+1)$ if $t > T$ and $s \in [0, 1/N]$, and so $\|e_n\cdot\varphi - \varphi\| \leqslant \varepsilon(K+1)$ $(n \geqslant N)$. Thus (e_n) is a b.a.i. for $\mathfrak{X}$.

The following definition was given in [3, Definition 1.3].

1.4. Definition. A weight function ω on $\boldsymbol{R}^+$ is *regulated* if

$$\lim_{t\to\infty} \omega(s+t)/\omega(t) = 0$$

for every $s > 0$.

Certainly, a regulated weight is a radical weight [3, Lemma 1.2]. For example, the weight $\omega\colon t \mapsto e^{-t^2}$ is a regulated weight.

The next theorem shows that for a regulated weight we may as well take $\mathfrak{X}$ for our module instead of A', since, for such weights, $A\cdot A' \subseteq \mathfrak{X}$. Given $f \in A$, we write T_f for the map $g \mapsto g * f$, $A \to A$. A closed ideal $\mathfrak{I}$ in A is called a *standard ideal* if for some $\gamma \in [0, \infty]$, $\mathfrak{I}$ has the form

$$\mathfrak{I} = \{f \in A\colon \alpha(f) \geqslant \gamma\}.$$

1.5. Theorem. *Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Then the following statements are equivalent:*

(a) *ω is regulated;*

(b) *for every $f \in A$, $f\cdot A' \subseteq \mathfrak{X}$;*

(c) *for every $f \in A$, the map T_f is compact;*

(d) *for some $f \in A$ with $\alpha(f) = 0$, the map T_f is compact.*

Proof. To prove that (a)$\Rightarrow$(b), we first observe that for each $f \in A$, the right shift $S\colon t \to S_t(f) = \delta_t * f$, $\boldsymbol{R}^+ \to A$, is uniformly continuous: S is continuous on $\boldsymbol{R}^+$ by [3, Lemma 1.6], and $\|\delta_t * f\| \leqslant \omega(t)\|f\| \to 0$ as $t \to \infty$, and so S is uniformly continuous on $\boldsymbol{R}^+$. Now let $f \in A$ and $\varphi \in A'$. The estimate

$$|(f\cdot\varphi)^\vee(t_1) - (f\cdot\varphi)^\vee(t_2)| \leqslant \int_{\boldsymbol{R}^+} |f(s-t_1) - f(s-t_2)| \frac{|\check{\varphi}(s)|}{\omega(s)}\omega(s)\, ds$$
$$\leqslant \|S_{t_1}(f) - S_{t_2}(f)\|\,\|\varphi\| \qquad (t_1, t_2 \in \boldsymbol{R}^+)$$

shows that $f\cdot\varphi$ is uniformly continuous on $\boldsymbol{R}^-$. Moreover, as

$$\frac{|(f\cdot\varphi)^\vee(t)|}{\omega(t)} \leqslant \int_{\boldsymbol{R}^+} \frac{|\check{\varphi}(s+t)|}{\omega(s+t)} \frac{\omega(s+t)}{\omega(s)\omega(t)} |f(s)|\omega(s)\, ds$$
$$\leqslant \|\varphi\| \int_{\boldsymbol{R}^+} \frac{\omega(s+t)}{\omega(s)\omega(t)} |f(s)|\omega(s)\, ds \qquad (t \in \boldsymbol{R}^+),$$

the hypothesis that ω is regulated and the Lebesgue dominated convergence theorem show that $|(f\cdot\varphi)^\vee(t)|/\omega(t) \to 0$ as $t \to \infty$. Thus $f\cdot\varphi \in \mathfrak{X}$, proving (b).

The equivalence of (a) and (c) is proved in [3, Corollary 2.8], where it is also proved that T_f is compact if and only if it is weakly compact, and that the elements $f \in A$ for which T_f is compact form a standard ideal. It follows that, if (d) holds, then T_f is compact for every $f \in A$, and so (c) holds.

Thus it only remains to prove that (b)$\Rightarrow$(c). If $f \in A$, the adjoint T_f^* of T_f sends $\varphi \in A'$ to the element $f\cdot\varphi \in \mathfrak{X}$, so T_f^* maps an L^∞-space A' into the separable Banach space $\mathfrak{X}$. By a theorem of Grothendieck [8, p. 156], T_f^* is weakly compact. Thus T_f is weakly compact (see [10, Theorem VI.4.8]), and so T_f is compact. ∎

1.6. Corollary. *Let ω be a regulated weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$ and $\mathfrak{X} = C_0(\boldsymbol{R}^-, \check{\omega}^{-1})$. Suppose that there exists a discontinuous derivation from A into some Banach A-module. Then there exists a discontinuous derivation from A into $\mathfrak{X}$.*

Proof. By 1.2, we need only consider derivations from A into A'. Let $D\colon A \to A'$ be a derivation and take $f \in A$. Since A has a b.a.i., there exist $g, h \in A$ with $f = g * h$, and then $D(f) = g\cdot Dh + h\cdot Dg \in A\cdot A'$. Since ω is regulated, $A\cdot A' \subset \mathfrak{X}$ and so $D(A) \subset \mathfrak{X}$. ∎

We now discuss the closed submodules of A' and of $\mathfrak{X}$ and their relation to the closed ideals of A. The following notations will be convenient. If $\mathfrak{I} \subseteq A$, then $\mathfrak{I}^{\perp} = \{\varphi \in A': \langle \mathfrak{I}, \varphi \rangle = 0\}$, and, if $\mathfrak{M} \subseteq A'$, then $\mathfrak{M}^{\top} = \{f \in A: \langle f, \mathfrak{M} \rangle = 0\}$. Since $\mathfrak{X}' = M(\boldsymbol{R}^+, \omega)$ and A is a total subspace of $\mathfrak{X}'$, $\mathfrak{X}$ and A form a dual pair with the relevant $\sigma(\mathfrak{X}, A)$- and $\sigma(A, \mathfrak{X})$-topologies. If $\mathfrak{I} \subseteq A$, we write $\mathfrak{I}_{\perp} = \{\varphi \in \mathfrak{X}: \langle \mathfrak{I}, \varphi \rangle = 0\}$, and, if $\mathfrak{N} \subseteq \mathfrak{X}$, we let $\mathfrak{N}_{\top} = \{f \in A: \langle f, \mathfrak{N} \rangle = 0\}$. Suppose that γ is any number in the interval $[0, \infty]$. We define $A_{\gamma} = \{f \in A: \alpha(f) \geqslant \gamma\}$, $(A')_{\gamma} = \{\varphi \in A': \alpha(\varphi) \geqslant -\gamma\}$ and set $\mathfrak{X}_{\gamma} = (A')_{\gamma} \cap \mathfrak{X}$. As remarked earlier, we call the closed ideals A_{γ} the *standard ideals* of A. The closed submodules $(A')_{\gamma}$ and $\mathfrak{X}_{\gamma}$ are called the *standard submodules* of A' and $\mathfrak{X}$, respectively.

The proofs of the next two theorems are straightforward and will be omitted.

1.7. THEOREM. *Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$.*

(i) *Let $\mathfrak{I}$ be a norm-closed ideal in A. Then $\mathfrak{I}^{\perp}$ is a $\sigma(A', A)$-closed submodule of A', and $\mathfrak{I}^{\perp}$ is a $\sigma(\mathfrak{X}, A)$-closed submodule of $\mathfrak{X}$. Also $(\mathfrak{I}^{\perp})^{\top} = \mathfrak{I}$, and $(\mathfrak{I}_{\perp})_{\top}$ is the $\sigma(A, \mathfrak{X})$-closure of $\mathfrak{I}$.*

(ii) *Let $\mathfrak{M}$ be a norm-closed submodule of A'. Then $\mathfrak{M}^{\top}$ is a norm-closed ideal in A, and $(\mathfrak{M}^{\top})^{\perp}$ is the $\sigma(A', A)$-closure of $\mathfrak{M}$. Further, let $\mathfrak{N}$ be a norm-closed submodule of $\mathfrak{X}$. Then $\mathfrak{N}_{\top}$ is a $\sigma(A, \mathfrak{X})$-closed ideal in A and $(\mathfrak{N}_{\top})_{\perp}$ is the $\sigma(\mathfrak{X}, A)$-closure of $\mathfrak{N}$.* ■

1.8. THEOREM. *Let γ be any number in the interval $[0, \infty]$. Then $(A_{\gamma})^{\perp} = (A')_{\gamma}$ and $(A')_{\gamma}^{\top} = A_{\gamma}$. Also $(A_{\gamma})_{\perp} = \mathfrak{X}_{\gamma}$ and $(\mathfrak{X}_{\gamma})_{\top} = A_{\gamma}$.* ■

The next result is less straightforward. For its proof and for later uses, it is convenient to introduce the function $u = \chi_{[0,\infty)}$. The convolution powers of u are given by $u^{*(n)}(t) = t^{n-1}/(n-1)!$ $(n \in \boldsymbol{N})$, so the algebra $\mathfrak{P}(u)$ of all convolution polynomials in u is also the algebra of all ordinary polynomials in the variable t.

1.9. PROPOSITION. *Let ω be a regulated weight function, and let $A = L^1(\boldsymbol{R}^+, \omega)$.*

(i) *Let $\mathfrak{I}$ be a norm-closed ideal in A, and let $\mathfrak{K}$ be its $\sigma(A, \mathfrak{X})$-closure. If $\mathfrak{K}$ is a standard ideal, then $\mathfrak{I} = \mathfrak{K}$.*

(ii) *Let $\mathfrak{N}$ be a norm-closed submodule of $\mathfrak{X}$, and let $\mathfrak{M}$ be its $\sigma(\mathfrak{X}, A)$-closure. If $\mathfrak{M}$ is a standard submodule, then $\mathfrak{N} = \mathfrak{M}$.*

Proof. To prove (i), suppose first that $\mathfrak{K} = A_{\gamma}$, where $\gamma < \infty$, and that $\mathfrak{I}$ is properly contained in $\mathfrak{K}$. Then there exists $\varphi \in A'$ with $\langle \mathfrak{I}, \varphi \rangle = 0$, but $\langle A_{\gamma}, \varphi \rangle \neq 0$. Since $\varphi \notin (A_{\gamma})^{\perp} = (A')_{\gamma}$, it must be that $\alpha(\varphi) < -\gamma$. Let $\psi = u \cdot \varphi$. Since ω is regulated, it follows from Theorem 1.5 that $\psi \in \mathfrak{X}$. Also $\langle \mathfrak{I}, \psi \rangle = 0$. However, $\langle \mathfrak{K}, \psi \rangle = \langle A, \psi \rangle = 0$, so that $\psi \in (A_{\gamma})^{\perp} = (A')_{\gamma}$. It follows that $\alpha(\varphi) = \alpha(\psi) \geqslant -\gamma$. This contradiction completes the proof in the case where $\gamma < \infty$. If $\gamma = \infty$, then $\mathfrak{K} = \{0\}$, and so the result holds trivially in this case. The proof of (ii) is similar. ■

A weight function ω on $\boldsymbol{R}^+$ is *unicellular* if every closed ideal in $L^1(\boldsymbol{R}^+, \omega)$ is standard. A unicellular weight is necessarily a radical weight. The existence of unicellular weights on $\boldsymbol{R}^+$ has been proved by Domar ([9]). In particular, he proves that $\omega(t) = e^{-t^2}$ is unicellular. A proof that there exist non-unicellular radical weights on $\boldsymbol{R}^+$ may be found in Dales and McClure [7]. Their construction of such a weight is based on a deep example due to Thomas [21] of a non-unicellular radical weight on $\boldsymbol{Z}^+$.

1.10. THEOREM. *Let ω be a regulated weight on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Then the following statements are equivalent:*

(a) *every norm-closed ideal in A is standard;*
(b) *every $\sigma(A, \mathfrak{X})$-closed ideal in A is standard;*
(c) *every $\sigma(\mathfrak{X}, A)$-closed submodule of $\mathfrak{X}$ is standard;*
(d) *every norm-closed submodule of $\mathfrak{X}$ is standard.*

Proof. Clearly (a)$\Rightarrow$(b) and (d)$\Rightarrow$(c).

Suppose that (b) holds, and let $\mathfrak{M}$ be a $\sigma(\mathfrak{X}, A)$-closed submodule of $\mathfrak{X}$. Then $\mathfrak{M}_{\top}$ is a $\sigma(\mathfrak{X}, A)$-closed ideal, and hence $\mathfrak{M}_{\top} = A_{\gamma}$ for some γ. However, $\mathfrak{M} = (\mathfrak{M}_{\top})^{\perp}$, so that $\mathfrak{M} = (A_{\gamma})^{\perp} = (A')_{\gamma}$. Thus (b)$\Rightarrow$(c).

Now suppose that (c) holds. Let $\mathfrak{N}$ be a norm-closed submodule of $\mathfrak{X}$, and let $\mathfrak{M}$ be its $\sigma(\mathfrak{X}, A)$-closure. Then $\mathfrak{M} = \mathfrak{X}_{\gamma}$ for some γ, so that $\mathfrak{N} = \mathfrak{X}_{\gamma}$ by Proposition 1.9(ii). Thus (c)$\Rightarrow$(d).

Suppose that (c) holds, and let $\mathfrak{I}$ be a $\sigma(A, \mathfrak{X})$-closed ideal in A. Then $\mathfrak{I}_{\perp}$ is a $\sigma(\mathfrak{X}, A)$-closed submodule in $\mathfrak{X}$, so $\mathfrak{I}_{\perp} = \mathfrak{X}_{\gamma}$ for some γ. However, $\mathfrak{I} = (\mathfrak{I}_{\perp})_{\top} = (\mathfrak{X}_{\gamma})_{\top} = A_{\gamma}$. Thus (c)$\Rightarrow$(b).

Finally, we prove that (b)$\Rightarrow$(a). Let $\mathfrak{I}$ be a norm-closed ideal in A, and let $\mathfrak{I}$ be its $\sigma(A, \mathfrak{X})$-closure. Then $\mathfrak{I} = A_{\gamma}$ for some γ, so that $\mathfrak{I} = A_{\gamma}$ by Proposition 1.9(i). ■

We now describe the situation regarding analogous results for the algebras $l^1(\boldsymbol{S}^+, \omega)$.

In the case that $\boldsymbol{S}$ is a dense, discrete subgroup of $\boldsymbol{R}$ the notion of a standard ideal for $l^1(\boldsymbol{S}^+, \omega)$ is more complicated than in the above case. Moreover, N. Groenbaek has shown [13, Corollary 2.6] that in $l^1(\boldsymbol{S}^+, \omega)$ there are no non-zero elements f for which the map T_f: $g \mapsto f * g$ is compact or, equivalently, weakly compact. There is no analogue of the submodule $C_0(\boldsymbol{R}^-, \check{\omega}^{-1})$ of $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ in this case either.

When $\boldsymbol{S} = \boldsymbol{Z}$ and $A = l^1(\boldsymbol{Z}^+, \omega)$, then the dual module $l^\infty(\boldsymbol{Z}^-, \check{\omega}^{-1})$ has the closed submodule $\mathfrak{M} = C_0(\boldsymbol{Z}^-, \check{\omega}^{-1})$, which consists of those sequences φ on $\boldsymbol{Z}^-$ for which $\lim_{n\to\infty} \varphi(-n)/\omega(n) = 0$. However, in this case $\mathfrak{M}' = A$, so that the $\sigma(\mathfrak{M}, A)$-topology on $\mathfrak{M}$ is the ordinary weak topology, and the $\sigma(A, \mathfrak{M})$-topology on A is its weak-star topology. One defines ω to be *regulated* if $\lim_{n\to\infty} \omega(m+n)/\omega(n) = 0$ for each $m \in \boldsymbol{N}$. Then exact analogues of Theorems 1.7–1.10 hold for the algebra $l^1(\boldsymbol{Z}^+, \omega)$. The analogue of Proposition 1.9(i) was proved in [5, Theorem 5.3], while the analogue of 1.9(ii) is trivially true in this case. The analogue of Theorem 1.5 holds when A is the unique maximal ideal in $l^1(\boldsymbol{Z}^+, \omega)$.

2. Representation of derivations. This section is concerned with the concrete representation of derivations from the Banach algebra $L^1(\boldsymbol{R}^+, \omega)$ into either its dual module $A' = L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$, or into the submodule $\mathfrak{X} = C_0(\boldsymbol{R}^-, \check{\omega}^{-1})$. While we will find a partial representation for arbitrary (possibly discontinuous) derivations, it is particularly important for us to find a useful representation for continuous derivations, for this will be the key to the discussion of continuity properties of arbitrary derivations in Section 3.

Representation theorems have been proved by N. Groenbaek [13], [14]. We first describe Groenbaek's results, and then in Theorem 2.5 we give a crucial strengthening of Groenbaek's representation theorem for continuous derivations of A into A': the theorem shows that the set of such derivations is in a natural correspondence with A'.

Groenbaek's method depends on the fact that every derivation D: $A \to A'$ can be extended to a derivation $\bar{D}$: $M(\boldsymbol{R}^+, \omega) \to A'$. The extension question also occurs in the work of Ghahramani [12] and Willis [22]. The next theorem is a modification of a result of Willis [22, Lemma 3.5], who works in the context of derivations into bimodules. See [14] for an abstract development of these ideas.

2.1. THEOREM. *Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Let D: $A \to A'$ be a derivation. Then there exists a unique derivation $\bar{D}$: $M(\boldsymbol{R}^+, \omega) \to A'$ which extends D. If D is continuous, then $\bar{D}$ is continuous and $\|\bar{D}\| = \|D\|$.*

Because of the importance of this result, we sketch the proof, omitting details which can be found in [22]. Let $\mu \in M(\boldsymbol{R}^+, \omega)$. For each $f \in A$, $\mu * f \in A$, and so $D(\mu * f)$ is defined in A'. We seek to define $\bar{D}(\mu)$ so that $D(\mu * f) = f \cdot \bar{D}(\mu) + \mu \cdot D(f)$. Define $S_\mu(f) = D(\mu * f) - \mu \cdot D(f)$ $(f \in A)$. Then S_μ: $A \to A'$ is a module homomorphism. Let $f_n \to 0$ in A. Since A has a b.a.i., there exist $g \in A$ and $h_n \to 0$ in A such that $f_n = g * h_n$ $(n \in \boldsymbol{N})$. Then $S_\mu(f_n) = h_n \cdot S_\mu(g) \to 0$, and so S_μ is continuous. The element $\bar{D}(\mu)$ is defined by $\bar{D}(\mu) = \lim_{n\to\infty} S_\mu(e_n)$ for the $\sigma(A', A)$-topology. Here $(e_n$: $n \in \boldsymbol{N})$ is the standard b.a.i. for A which was introduced in Section 1. The map $\mu \mapsto \bar{D}(\mu)$ can be shown to be a derivation with the required properties.

Before turning to the concrete representation of derivations, we introduce some notation that will be important throughout the rest of the paper. For a function f on $\boldsymbol{R}$ define the function xf on $\boldsymbol{R}$ by $(xf)(t) = tf(t)$. (If f is defined on $\boldsymbol{R}\backslash\{0\}$, we define $(xf)(0) = 0$.) If μ is a locally finite measure on $\boldsymbol{R}$, the locally finite measure $x\mu$ is defined for a bounded Borel set E by the equation

$$(x\mu)(E) = \int_E t\,\mu(dt).$$

The next theorem is taken from Groenbaek [14], who proves the analogous result in n dimensions.

2.2. THEOREM. *Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Let D: $A \to A'$ be a derivation, and let $\bar{D}$ be its unique extension to $M(\boldsymbol{R}^+, \omega)$. Then there exists a measurable, complex-valued function ψ on $\boldsymbol{R}^-\backslash\{0\}$ such that $(\delta_s * \psi)|\boldsymbol{R}^- \in A'$ for each $s > 0$ and such that*

$$\bar{D}(\delta_s) = (s\delta_s * \psi)|\boldsymbol{R}^- \quad (s \in \boldsymbol{Q}^+).$$

Suppose further that D is continuous. Then

$$(2.1) \qquad (Df)(t) = (xf * \psi)(t) \quad (t \in \boldsymbol{R}^-, f \in A)$$

and

$$(\bar{D}\mu)(t) = (x\mu * \psi)(t) \quad (t \in \boldsymbol{R}^-, \ \mu \in M(\boldsymbol{R}^+, \omega)).$$

Moreover,

$$(2.2) \qquad \|D\| = \|\bar{D}\| = \sup\{\|(t\delta_t * \psi)|\boldsymbol{R}^-\|/\omega(t)\colon\ t \in \boldsymbol{R}^+\backslash\{0\}\}.$$

Conversely, let ψ be any measurable function on $\boldsymbol{R}^-\backslash\{0\}$ such that

$$\sup\{\|(t\delta_t * \psi)|\boldsymbol{R}^-\|/\omega(t)\colon\ t \in \boldsymbol{R}^+\backslash\{0\}\} < \infty.$$

Then (2.1) defines a continuous derivation D: $A \to A'$. ∎

The function ψ which represents a derivation D: $A \to A'$ may be found as follows. Raise D to a derivation $\bar{D}$: $M(\boldsymbol{R}^+, \omega) \to A'$ and restrict $\bar{D}$ to the semigroup $\{\delta_s$: $s \in \boldsymbol{R}^+\}$ of point masses. Then ψ may be defined by the formula

$$\psi(t) = (L_s\bar{D}(\delta_s/s))(t) \quad (t \in (-\infty, -s])$$

for each $s \in \boldsymbol{Q}^+\backslash\{0\}$, where L_s is the "left shift"

$$(L_s\varphi)(t) = \begin{cases} \varphi(t-s) & (t \leqslant -s), \\ \varphi(0) & (-s < t < 0). \end{cases}$$

As in [14] we may identify the space $\mathfrak{D}(A, A')$ of all continuous derivations from A to A'. Clearly it is a closed linear subspace of the Banach space $\mathfrak{B}(A, A')$ of all continuous linear operators from A to A'. The correspondence $D \mapsto \psi$ given in (2.1) allows us to identify $\mathfrak{D}(A, A')$ with the space W of representing functions, defined below.

2.3. DEFINITION. Let ω be a weight function on $\boldsymbol{R}^+$. We denote by W the set of all equivalence classes ψ of measurable functions on $\boldsymbol{R}^-\backslash\{0\}$ for which

$$(2.3) \qquad |||\psi||| = \sup\{\|(t\delta_t * \psi)|\boldsymbol{R}^-\|/\omega(t)\colon\ t \in \boldsymbol{R}^+\backslash\{0\}\} < \infty.$$

It follows from Theorem 2.2 that W is a Banach space for the norm $|||\cdot|||$, and that $(W, |||\cdot|||)$ is isometrically isomorphic to $\mathfrak{D}(A, A')$. If $\psi \in W$, we say that ψ *represents* the derivation D which is defined by (2.1).

The next theorem generalizes Groenbaek's Theorem 2.2 and gives an alternative and useful characterization of the space W. We show that a measurable, complex-valued function ψ belongs to W if and only if it satisfies the equation $x\psi = \varphi$, where φ is an element of A', and we prove another characterization of W of "closed graph theorem" type.

To prove this second characterization and a later result we will need the following elementary fact concerning linear maps from a Banach space into a Fréchet space of measurable functions. Let S be a bounded interval of $\boldsymbol{R}$, and let $M(S)$ be the linear space of all equivalence classes of complex-valued, Borel measurable functions on S. It is well known that with respect to the metric ϱ given by

$$\varrho(f, g) = \int_S \frac{|f(s)-g(s)|\,ds}{1+|f(s)-g(s)|} \qquad (f, g \in M(S)),$$

$M(S)$ is a Fréchet space, and that a sequence (f_n) converges to f_0 in $M(S)$ if and only if (f_n) converges to f_0 in measure on S.

2.4. LEMMA. *Let $M(S)$ be as above, and let E be a Banach space. For each $n \in \boldsymbol{N}$, let T_n: $E \to M(S)$ be a continuous linear map. Suppose that for each $x \in E$, the limit*

$$T(x)(s) = \lim_{n\to\infty} T_n(x)(s)$$

exists and is finite for almost all $s \in S$. Then the linear map T: $E \to M(S)$ is continuous.

Proof. For each $x \in E$ we have $\lim \varrho(T_n(x), T(x)) = 0$, by the Lebesgue dominated convergence theorem. Hence T is continuous by the uniform boundedness theorem. ■

2.5. THEOREM. *Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Let ψ be a measurable function defined on $\boldsymbol{R}^-\backslash\{0\}$. Then the following conditions on ψ are equivalent:*

(a) $\psi \in W$;

(b) $x\psi \in A'$;

(c) *for each $f \in A$ the convolution $xf * \psi$ exists on $\boldsymbol{R}^-$, and $(xf*\psi)|\boldsymbol{R}^-$ belongs to A'.*

If any of (a)–(c) *hold, then the map D: $f \mapsto (xf*\psi)|\boldsymbol{R}^-$ is a continuous derivation of A into A', and $\|D\| = |||\psi||| = \|x\psi\|$.*

Proof. Clearly (a)⇒(c) follows from Theorem 2.2. We shall prove that (a) and (b) are equivalent, and that (c)⇒(a).

We first prove that (a)⇒(b). If $\psi \in W$, then ψ is integrable on each interval $(-\infty, c]$ with $c < 0$, and we may write (2.3) in the form

$$|||\psi||| = \sup_{t\in\boldsymbol{R}^+} \frac{t}{\omega(t)} \operatorname{ess\,sup}_{s\in\boldsymbol{R}^-} \frac{|\psi(s-t)|}{\omega(-s)}.$$

Let $m \in \boldsymbol{N}$ and let $A_m = (-\infty, -1/m]$. By the continuity of translation of $\psi\chi_{A_m}$ in $L^1(\boldsymbol{R})$, we have

$$\lim_{s\to 0^-} \psi(s+t) = \psi(t) \qquad (t \in A_m)$$

in measure on A_m. It follows that there exists a sequence (s_k) with $s_k < s_{k+1} < 0$ $(k \in \boldsymbol{N})$ and $s_k \to 0$, and such that, if $s_k < s < 0$, then $|\psi(s+t)-\psi(t)| \leqslant 1/2^k$ for all $t \in A_m \backslash M_{k,s}$, where $M_{k,s}$ is a subset of A_m having measure less than $1/2^k$. Fix $t \in A_m$, and suppose that $\varepsilon > 0$. Then there exists a Lebesgue null set $N_t \subseteq \boldsymbol{R}^-$ (where for convenience we also suppose that $0 \in N_t$) such that

$$\text{(2.4)} \qquad \frac{-t|\psi(s+t)|}{\omega(-t)\omega(-s)} \leqslant |||\psi|||+\varepsilon \qquad (s \in \boldsymbol{R}^-\backslash N_t).$$

Select an increasing sequence (u_k) such that $s_k < u_k < 0$ and $u_k \notin N_t$ $(k \in \boldsymbol{N})$. Then the set M_{k,u_k} has measure at most $1/2^k$. Following a standard argument, set $B_k = \bigcup_{p\geqslant k} M_{p,u_p}$ and $B_0 = \bigcap_{k=1}^{\infty} B_k$. Then the sets B_k are decreasing and $m(B_k) \leqslant 1/2^{k-1}$, and hence $m(B_0) = 0$. Moreover,

$$\lim_{k\to\infty} \psi(u_k+t) = \psi(t) \qquad (t \in A_m \backslash B_0).$$

Set $s = u_k$ in (2.4), and pass to the limit as $k \to \infty$. Since ω is continuous and $\omega(0) = 1$,

$$\frac{-t|\psi(t)|}{\omega(-t)} \leqslant |||\psi|||+\varepsilon \qquad (t \in A_m \backslash B_0).$$

Since m and ε are arbitrary, it follows that $-t|\psi(t)|/\omega(-t) \leqslant |||\psi|||$ a.e. on $\boldsymbol{R}^-$. Hence $x\psi \in A'$ and $\|x\psi\| \leqslant |||\psi|||$. This completes the proof that (a)⇒(b).

If (b) holds, then

$$\frac{|(xf*\psi)^{\vee}(t)|}{\omega(t)} \leqslant \int_{\boldsymbol{R}^+} \frac{|sf(s)\check{\psi}(s+t)|}{\omega(t)}\,ds$$
$$\leqslant \int_{\boldsymbol{R}^+} \left[\frac{(s+t)|\check{\psi}(s+t)|}{\omega(s+t)}\right]\left[\frac{\omega(s+t)}{\omega(s)\omega(t)}\right]|f(s)|\omega(s)\,ds$$
$$\leqslant \|x\psi\|\,\|f\| \qquad (t \in \boldsymbol{R}^+, f \in A),$$

so the map D: $f \mapsto (xf*\mu)|\boldsymbol{R}^-$, $A \to A'$, is continuous and $|||\psi||| \leqslant \|x\psi\|$. Thus (b)⇒(a). In particular, (b)⇒(c).

The proof that (c)⇒(a) will be given after we have proved that the map D: $f \mapsto xf*\psi$ is a continuous derivation.

Suppose that (c) holds. For $n \in \boldsymbol{N}$, let $C_n = \{t \in \boldsymbol{R}^-: t \in [-n, -1/n],\ |\psi(t)| \leqslant n\}$, and let $\psi_n = \psi\chi_{C_n}$. Since ψ_n is bounded and has compact support in $\boldsymbol{R}^-\backslash\{0\}$, $x\psi_n \in A'$, and so $\psi_n \in W$. Define D_n: $f \mapsto xf*\psi_n$, $A \to A'$. Then, by Theorem 2.2, D_n is a continuous derivation. By (c), for each $f \in A$,

$$\int_{\boldsymbol{R}^-} |(t-s)f(t-s)x(s)|\,ds < \infty$$

for almost all $t \in \boldsymbol{R}^-$. It follows that $\lim_{n\to\infty}(D_n f)(t) = (Df)(t)$ for almost all $t \in \boldsymbol{R}^-$ by the Lebesgue dominated convergence theorem. Let S be any bounded interval in $\boldsymbol{R}^-$. Then, for each n, the map $T_n\colon f \mapsto (D_n f)|S$, $A \to M(S)$, is continuous. Thus by Lemma 2.4, the map $T\colon f \mapsto (Df)|S$, $A \to M(S)$, is continuous.

We now prove D is continuous as a map from A into A' by using the closed graph theorem. Take $(f_k) \subseteq A$ with $f_k \to 0$ and $Df_k \to \theta$ in A'. It suffices to show that θ vanishes on each bounded interval S of $\boldsymbol{R}^-$. Since the map T above is continuous, Df_k converges to zero in measure on S. Hence $\theta|S = 0$, proving the continuity of D.

We next prove that D is a derivation. Since $x(f*g) = xf*g + xg*f$ and D is bounded, it suffices to prove that

$$[(f*xg)+(g*xf)]*\psi = f*(xg*\psi)+g*(xf*\psi), \tag{2.5}$$

as an equality in A', when f and g belong to the dense subspace $\mathfrak{K}(\boldsymbol{R}^+)$ of A. The difficulty in proving (2.5) lies in proving the necessary associativity of convolution, since ψ need not be locally integrable. However, (2.5) holds with ψ replaced by ψ_n, as a statement about Radon measures on $\boldsymbol{R}$, since then all elements have compact support (see [11, Corollary 4.19.9]). Using the Lebesgue dominated convergence theorem, as before, we obtain (2.5) by passing to the limit as $n \to \infty$. Thus D is a continuous derivation from A to A'.

We can now complete the proof by showing that (c)$\Rightarrow$(a).

By Theorem 2.2, we obtain a function $\psi_1 \in W$ such that

$$(Df)(t) = (xf*\psi_1)(t) \qquad (t \in \boldsymbol{R}^-, f \in A).$$

Let $\varphi = \psi - \psi_1$. Then for each $f \in A$, $(xf*\varphi)(t) = 0$ $(t \in \boldsymbol{R}^-)$, and

$$\int_{\boldsymbol{R}^-} |(xf)(t-s)\varphi(s)|\,ds < \infty$$

for almost all $t \in \boldsymbol{R}^-$. Given $\delta > 0$, take f so that $xf = \chi_{[\delta,\infty)}$. Then, for almost all $t \in \boldsymbol{R}^-$,

$$\int_{-\infty}^{t-\delta} |\varphi(s)|\,ds < \infty \quad \text{and} \quad \int_{-\infty}^{t-\delta} \varphi(s)\,ds = 0.$$

Since δ is arbitrary, it follows that $\varphi(s) = 0$ a.e., and so $\psi \in W$. ∎

As a consequence of Theorem 2.5, we have the following useful inequality, which shows that $(W, |||\cdot|||)$ is a Banach A-module. Let $\psi \in W$ and $f \in A$. Then $x\psi = \varphi \in A'$ and

$$|(x(f*\psi))(t)| \leqslant \int_{\boldsymbol{R}^+} \frac{t}{|t-s|}|\varphi(t-s)|\,|f(s)|\,ds$$
$$\leqslant (|f|*|\varphi|)(t) \qquad (t \in \boldsymbol{R}^-),$$

so

$$|||f*\psi||| = \|x(f*\psi)\| \leqslant \|\,|f|*|x\psi|\,\| \leqslant \|f\|\,\|x\psi\| = \|f\|\,|||\psi|||. \tag{2.6}$$

Hereafter we write $xf\cdot\psi$ for $(xf*\psi)|\boldsymbol{R}^-$ when $f \in A$ and $\psi \in W$.

It is now convenient to introduce the space

$$V = \{\varphi \in A'\colon \varphi|\boldsymbol{R}^- \setminus \{0\} \in W\}.$$

In the next section we shall prove that a derivation $D\colon A \to A'$ for which $D(A) \subseteq V$ must have important continuity properties.

We now prove a commutation property of continuous derivations into V. First we need the following lemma.

2.6. Lemma. *Let $D\colon A \to A'$ be a continuous derivation for which $D(A) \subseteq V$. Then, for each $g \in A$, the map $f \mapsto (xg*D(f))|\boldsymbol{R}^-$, $A \to A'$, is a continuous derivation.*

Proof. Take $g \in A$. Since $D(f) \in V$ for $f \in A$, the convolution $xg*D(f)$ exists on $\boldsymbol{R}^-$ and $xg\cdot D(f) = (xg*D(f))|\boldsymbol{R}^- \in A'$ for each $f \in A$. Set $E\colon f \mapsto xg\cdot D(f)$, $A \to A'$.

We first note that D is continuous as a map of A into W. For take $(f_n) \subseteq A$ with $f_n \to 0$ and $D(f_n) \to \theta \in W$. Since D is continuous as a map into A', $D(f_n)(s) \to 0$ a.e. on $\boldsymbol{R}^-$. However, by Theorem 2.5,

$$|||D(f_n) - \theta||| = \operatorname*{ess\,sup}_{s \in \boldsymbol{R}^-} \frac{|s[D(f_n)(s) - \theta(s)]|}{\omega(-s)},$$

so that $D(f_n)(s) \to \theta(s)$ a.e. on $\boldsymbol{R}^-$. Hence $\theta = 0$, and so $D\colon A \to W$ is continuous.

Now let $g \in A$ and consider the map E. Let $f_n \to 0$ in A, so that $|||D(f_n)||| \to 0$. Since $D(f_n) \in V$, $\|E(f_n)\| \leqslant \|g\|\,|||D(f_n)||| \to 0$ as $n \to \infty$ and so E is continuous.

Finally, we show that the map E is a derivation. The argument will not use the continuity of D. Let $g_n = g\chi_{[0,n]}$, so $xg_n \in A$ $(n \in N)$. Then, if $h, k \in A$,

$$xg_n\cdot D(h*k) = h\cdot(xg_n\cdot D(k)) + k\cdot(xg_n\cdot D(h)).$$

By Theorem 2.5, the map $f \mapsto xf\cdot\psi$ is continuous for each $\psi \in W$. But $D(h)$, $D(k)$, and $D(h*k) \in W$, and so

$$xg\cdot D(h*k) = h\cdot(xg\cdot D(k)) + k\cdot(xg\cdot D(h)) \qquad (h, k \in A).$$

Thus E is a derivation. ∎

2.7. Theorem. *Let $D\colon A \to A'$ be any continuous derivation with $D(A) \subseteq V$. Then*

$$xf\cdot D(g) = xg\cdot D(f) \qquad (f, g \in A). \tag{2.7}$$

Proof. By Theorem 2.5 both sides of this equation lie in A', and for each $g \in A$, the map $E\colon f \mapsto xf\cdot D(g)$ is continuous. Taking $f = u^{*(m)}$ and $g = u^{*(n)}$, we see that both the left- and right-hand sides of (2.7) equal $mnu^{*(m+n)}\cdot D(u)$. Hence (2.7) holds for f and g in the dense subalgebra of convolution polynomials in u. Since, by Lemma 2.6, the map $(f, g)\mapsto xf\cdot D(g)$ is continuous in each variable separately, we obtain (2.7) for all $f, g\in A$. ∎

We end this section by stating without proof the representation theorem for derivations from the discrete convolution algebras $l^1(S^+, \omega)$ for appropriate subgroups S of $\boldsymbol{R}$.

2.8. THEOREM. *Let S be a subgroup of $\boldsymbol{R}$ containing $\boldsymbol{Q}$, and let ω be a radical weight on S^+. If $D\colon l^1(S^+, \omega)\to l^\infty(S^-, \check{\omega}^{-1})$ is a derivation, then there exists a complex-valued function ψ defined on $S^-\setminus\{0\}$ such that $\delta_s * \psi$ belongs to $l^\infty(S^-, \check{\omega}^{-1})$ for every $s\in S^+\setminus\{0\}$ and*

$$D(\delta_s) = (s\delta_s * \psi)|S^- \quad (s\in \boldsymbol{Q}^+).$$

If D is continuous, then

$$D(f) = (xf * \psi)|S^- \quad (f\in l^1(S^+, \omega)), \tag{2.8}$$

and

$$\|D\| = \sup\{\|(s\delta_s * \psi)|S^-\|/\omega(s)\colon s\in S^+\}.$$

Conversely, let ψ be any function on $S^-\setminus\{0\}$ such that $\sup\{\|(s\delta_s * \psi)|S^-\|/\omega(s)\colon s\in S^+\} < \infty$. *Then* (2.8) *defines a continuous derivation* $D\colon l^1(S^+, \omega)\to l^\infty(S^+, \check{\omega}^{-1})$. ∎

3. Continuity properties of derivations. Let ω be a weight function on $\boldsymbol{R}^+$, and let $A = L^1(\boldsymbol{R}^+, \omega)$. In this section we show that certain classes of derivations from A into A' must have striking continuity properties. For example, we shall prove, under mild conditions on ω, that, if $D\colon A\to A'$ is any derivation and if $g\in A$, then D is continuous if and only if the pre-multiplied derivation $g\cdot D$ is continuous (where $(g\cdot D)(f) = g\cdot D(f)$ $(f\in A)$). Moreover, an element g can be found so that $g\cdot D$ is bounded on a dense subalgebra of A. The property of g that we require is that $g\cdot D$ shall have its range in V.

To begin this section we shall characterize those functions g in A such that $g\cdot A'\subseteq V$. More generally, we shall characterize those measures μ on $\boldsymbol{R}^+$ for which the convolution $\mu * \varphi$ is defined on $\boldsymbol{R}^-$, and $(\mu * \varphi)|(\boldsymbol{R}^-\setminus\{0\})$ belongs to W. It is interesting that the condition on μ is precisely that required by Ghahramani [12] to characterize the (necessarily continuous) derivations of A to itself.

3.1. THEOREM. *Let ω be a weight function, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Let μ be a Radon measure on $\boldsymbol{R}^+$. Then the following conditions on μ are equivalent:*

(a) *one has*

$$\sup_{a>0}\frac{a}{\omega(a)}\int_{\boldsymbol{R}^+}\omega(s+a)\,|\mu|(ds) < \infty; \tag{3.1}$$

(b) *for each $f\in A$, $xf * \mu\in A$, and the map*

$$D\colon f\mapsto xf * \mu, \quad A\to A, \tag{3.2}$$

defines a continuous derivation of A into itself;

(c) *for each $\varphi\in A'$, the convolution $\mu * \varphi$ is defined on $\boldsymbol{R}^-$, and the function*

$$x(\mu * \varphi)\colon t\mapsto t(\mu * \varphi)(t), \quad \boldsymbol{R}^-\to \boldsymbol{C},$$

belongs to A';

(d) *for each $\varphi\in A'$, the convolution $\mu * \varphi$ is defined on $\boldsymbol{R}^-$, and we have $(\mu * \varphi)|(\boldsymbol{R}^-\setminus\{0\})\in W$.*

If any of the conditions (a)–(d) *hold, then the map $J\colon \varphi\mapsto -x(\mu * \varphi)$, $A'\to A'$, is continuous, and $J = D^*$, the adjoint of D.*

Proof. It is proved in Ghahramani [12] that there is the following bijective correspondence between Radon measures μ for which (a) holds and continuous derivations on A. If μ satisfies (a), then the formula (3.2) defines a continuous derivation $D\colon A\to A$ whose norm is the supremum in (a). Conversely, to every continuous derivation $D\colon A\to A$ there corresponds a measure μ for which the supremum in (a) is finite, and D is given by the formula (3.2). Thus (a) and (b) are equivalent.

Suppose that (a) holds. For $\varphi\in A'$ and $t\in \boldsymbol{R}^-$, define

$$(J\varphi)(t) = -t(\mu * \varphi)(t) = -t\int_{\boldsymbol{R}^+}\varphi(t-s)\,\mu(ds).$$

Then

$$\frac{|(J\varphi)(t)|}{\omega(-t)} \leqslant -t\int_{\boldsymbol{R}^+}\frac{|\check{\varphi}(s-t)|\,\omega(s-t)}{\omega(s-t)\,\omega(-t)}|\mu|(ds)$$

$$\leqslant \|\varphi\|\sup_{a>0}\frac{a}{\omega(a)}\int_{\boldsymbol{R}^+}\omega(s+a)\,|\mu|(ds),$$

and so J is a bounded map of A' into itself. Thus (a)⇒(c).

Suppose now that (c) holds. We will prove (a). For each $n\in \boldsymbol{N}$ and $\varphi\in A'$, let $\varphi_n = \chi_{[-n,0]}\varphi$, and let $\mu_n = \chi_{[0,n]}\mu$. Define the map $J_n\colon \varphi\mapsto -x(\varphi_n * \mu_n)$, $A'\to A'$. Then $(J_n\varphi)(t) = 0$ for $t < -n$, and $\|J_n\varphi\| \leqslant n\|\mu_n\|\,\|\phi\|$, so J_n is continuous. Since for each $\varphi\in A'$ we have

$$\int_{\boldsymbol{R}^+}|\varphi(t-s)|\,|\mu|(ds) < \infty$$

for almost all $t\in \boldsymbol{R}^-$, it follows from the Lebesgue dominated convergence theorem that $\lim_{n\to\infty}(J_n\varphi)(t) = (J\varphi)(t)$ for almost all $t\in \boldsymbol{R}^-$. We now prove that $J\colon A'\to A'$ is continuous by using Lemma 2.4 exactly as in the proof of Theorem 2.5. If S is a bounded interval of $\boldsymbol{R}^-$, then J is continuous as a map of A' into $M(S)$, and one completes the proof as before, using the closed graph theorem.

Now, for $f \in A$ and $\varphi \in A'$, we have $\langle f, J\varphi\rangle = \langle J^*f, \varphi\rangle$, where certainly $J^*f \in A''$. We would like to know that $J^*f \in A$, but that is beyond our reach at this point. Suppose that $f \in \mathfrak{K}(\mathbf{R}^+)$ and $\varphi \in \mathfrak{K}(\mathbf{R}^-)$. Since f and φ vanish on $\mathbf{R}^-$ and $\mathbf{R}^+$, respectively, f, xf, and φ belong to $\mathfrak{K}(\mathbf{R})$. Then $\varphi * \mu$ and $xf * \mu$ exist as Radon measures on $\mathbf{R}$ and are, in fact, continuous functions on $\mathbf{R}$ ([11, p. 262]). Moreover, we have

$$(\varphi * \mu) * xf = \varphi * (xf * \mu),$$

since two of the three elements have compact support ([11, Corollary 4.19.9]). Hence

$$\begin{aligned}\langle f, J\varphi\rangle = \langle f, -x(\varphi*\mu)\rangle &= \int_{\mathbf{R}^-} (-sf(-s))(\varphi*\mu)(s)\,ds \\ &= (xf*(\varphi*\mu))(0) = (\varphi*(xf*\mu))(0) \\ &= \int_{\mathbf{R}^+} \check{\varphi}(s)(xf*\mu)(s)\,ds \qquad (f \in \mathfrak{K}(\mathbf{R}^+),\ \varphi \in \mathfrak{K}(\mathbf{R}^-)).\end{aligned}$$

Moreover, for each $a > 0$ and $f \in \mathfrak{K}(\mathbf{R}^+)$,

$$\begin{aligned}&\int_0^a |(xf*\mu)(t)|\omega(t)\,dt \\ &\qquad = \sup\{|\langle f, J\varphi\rangle|:\ \|\varphi\| = 1,\ \operatorname{supp}\varphi \subset [-a, 0]\} \leqslant \|J^*\|\,\|f\|,\end{aligned}$$

since $\langle f, J\varphi\rangle = \langle J^*f, \varphi\rangle$. Letting $a \to \infty$, we see that $xf*\mu \in A$ and $\|xf*\mu\| \leqslant \|J^*\|\,\|f\| = \|J\|\,\|f\|$, at least when $f \in \mathfrak{K}(\mathbf{R}^+)$.

Define $D(f) = xf*\mu$ $(f \in \mathfrak{K}(\mathbf{R}^+))$. Then D is a continuous operator mapping $\mathfrak{K}(\mathbf{R}^+)$ to A, and hence D has a unique extension to an operator $D\colon A \to A$. Since

$$\begin{aligned}D(f*g) = x(f*g)*\mu &= (f*xg + g*xf)*\mu \\ &= f*D(g) + D(f)*g \qquad (f, g \in \mathfrak{K}(\mathbf{R}^+)),\end{aligned}$$

the extension is a continuous derivation on A. By the remarks concerning Ghahramani's theorem at the beginning of the proof, there exists a Radon measure ν on $\mathbf{R}^+$ such that

$$\sup_{a>0} \frac{a}{\omega(a)} \int_{\mathbf{R}^+} \omega(t+a)\,|\nu|(dt) < \infty,$$

and $D(f) = xf*\nu$ $(f \in A)$. But then we have $xf*\mu = xf*\nu$ for all $f \in \mathfrak{K}(\mathbf{R}^+)$, so that $\mu = \nu$ by the Titchmarsh theorem. Hence we have proved that (c) ⇒ (a).

The equivalence of (c) and (d) follows from Theorem 2.5.

It remains to show that $J = D^*$ under the hypothesis that (a)–(d) hold. Let μ satisfy (a)–(d), let $\varphi \in A'$, and let $f \in A$. Then $|\varphi| \in A'$, $|f| \in A$, and $|\mu|$ satisfies (b).

So $x|f|*|\mu| \in A$ and $\langle x|f|*|\mu|, |\varphi|\rangle < \infty$, i.e.

$$\int_{\mathbf{R}^+} |\check{\varphi}(t)| \int_0^t (t-s)|f(t-s)|\,|\mu|(ds)\,dt < \infty.$$

This shows that Fubini's theorem allows us to interchange the order of integration in the obvious double integral to obtain $\langle Df, \varphi\rangle = \langle f, J\varphi\rangle$. Thus $J = D^*$ and $J^*f \in A$. ∎

Ghahramani [12] has proved that there exists a non-zero measure μ satisfying (3.1), and hence that there exists a non-zero, continuous derivation $D\colon A \to A$ if and only if there exists a number $b > 0$ such that

$$\sup_{t\in\mathbf{R}^+} t\omega(t+b)/\omega(t) < \infty.$$

3.2. Corollary. *Let $g \in A$. Then $g\cdot A' \subseteq V$ if and only if*

$$\sup_{a>0} \frac{a}{\omega(a)} \int_{\mathbf{R}^+} |g(t)|\omega(t+a)\,dt < \infty. \ \blacksquare \tag{3.3}$$

It follows from Corollary 3.2 that, if g satisfies (3.3), then $g\cdot D$ has its range in V for every derivation $D\colon A \to A'$.

We now examine the question of how the continuity of a derivation D is related to that of $g\cdot D$. We first need information concerning the separating space and continuity ideal of a discontinuous derivation.

3.3. Theorem. *Let ω be a weight function on $\mathbf{R}^+$, and let $A = L^1(\mathbf{R}^+, \omega)$. Let $D\colon A \to A'$ be a discontinuous derivation. Then $\mathfrak{I}(D) = \mathfrak{S}(D)^{\top}$, and $\mathfrak{S}(D)$ contains elements φ for which $\alpha(\varphi) = -\infty$. Either $\mathfrak{I}(D) = \{0\}$, or $\mathfrak{I}(D)$ is a non-standard closed ideal in A. If ω is regulated and unicellular, then $\mathfrak{S}(D) = \mathfrak{X}$ and $\mathfrak{I}(D) = \{0\}$.*

Proof. Let $f \in A$. If $f \in \mathfrak{I}(D)$, then $f\cdot\varphi = 0$ $(\varphi \in \mathfrak{S}(D))$, and so $f \in \mathfrak{S}(D)^{\top}$. Conversely, if $f \in \mathfrak{S}(D)^{\top}$, then $\delta_t * f \in \mathfrak{S}(D)^{\top}$ $(t \in \mathbf{R}^+)$, so that $(f*\varphi)(-t) = \langle \delta_t * f, \varphi\rangle = 0$ $(t \in \mathbf{R}^+,\ \varphi \in \mathfrak{S}(D))$. Thus $(f*\varphi)|\mathbf{R}^- = 0$ and $f \in \mathfrak{I}(D)$.

We show that $\mathfrak{S}(D)$ must contain elements φ with $\alpha(\varphi) = -\infty$. For suppose that

$$\inf\{\alpha(\varphi):\ \varphi \in \mathfrak{S}(D)\} = -\beta > -\infty.$$

Since D is discontinuous, $\mathfrak{S}(D) \neq \{0\}$, and so $\beta > 0$. Choose a sequence (f_n) in A with $\alpha(f_n) > 0$ $(n \in \mathbf{N})$ and $\sum_{n=1}^{\infty} \alpha(f_n) < \beta$. By the stability lemma [20, Lemma 1.6] there is an integer N such that

$$\overline{(f_1 * \ldots * f_n)\cdot\mathfrak{S}(D)} = \overline{(f_1 * \ldots * f_N)\cdot\mathfrak{S}(D)}$$

for all $n \geqslant N$. It follows by the Titchmarsh convolution theorem that $\sum_{j=1}^{n} \alpha(f_j) - \beta = \sum_{j=1}^{N} \alpha(f_j) - \beta$ $(n \geqslant N)$, a contradiction. Hence we can form a series of elements in $\mathfrak{G}(D)$ whose sum φ lies in $\mathfrak{G}(D)$ and such that $\alpha(\varphi) = -\infty$.

Suppose that $\mathfrak{J}(D) \neq \{0\}$. If $\mathfrak{J}(D)$ is one of the standard ideals A_γ, where $0 < \gamma < \infty$, then by Theorem 1.7, $A_\gamma^\perp = \mathfrak{G}(D)^{\top\perp} = (A')_\gamma \supseteq \mathfrak{G}(D)$, so we must have the contradiction that $\alpha(\varphi) \geqslant -\gamma$ for all $\varphi \in \mathfrak{G}(D)$. If $\mathfrak{J}(D) = A_0 = A$, then D is continuous, since A has a b.a.i. Thus if $\mathfrak{J}(D) \neq \{0\}$, $\mathfrak{J}(D)$ must be a non-standard ideal.

Finally, suppose that ω is regulated and unicellular. Since ω is unicellular, $\mathfrak{J}(D) = \{0\}$, and, since ω is regulated, $\mathfrak{G}(D) \subseteq \mathfrak{X}$. Suppose that $\mathfrak{G}(D) \neq \mathfrak{X}$. Then there exists $\mu \in \mathfrak{X}' = M(\boldsymbol{R}^+, \omega)$ such that $\mu \neq 0$ and $\langle \mu, \varphi \rangle = 0$ $(\varphi \in \mathfrak{G}(D))$. Then $u * \mu \in A$ and $\langle u * \mu, \varphi \rangle = \langle \mu, u * \varphi \rangle = 0$ $(\varphi \in \mathfrak{G}(D))$, and so $u * \mu \in \mathfrak{J}(D) \setminus \{0\}$, a contradiction. Thus $\mathfrak{G}(D) = \mathfrak{X}$. ∎

3.4. Theorem. *Let ω be a regulated and unicellular weight function, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Let D be a derivation of A into $\mathfrak{X}$, and let $g \in A \setminus \{0\}$. Then D is continuous if and only if $g \cdot D$ is continuous.*

Proof. If D is continuous, then $g \cdot D$ is continuous. Conversely, suppose that $g \cdot D$ is continuous and D is discontinuous. Then $g \in \mathfrak{J}(D)$, and so, by Theorem 3.3, $g = 0$, a contradiction. Thus, if $g \cdot D$ is continuous, then D is continuous. ∎

For special elements g the above result is true with fewer hypotheses on ω.

3.5. Theorem. *Let ω be a radical weight on $\boldsymbol{R}^+$, and let $D\colon A \to A'$ be a derivation. Then the derivation $u \cdot D$ is continuous if and only if D is continuous.*

Proof. Suppose that $u \cdot D$ is continuous but that D is discontinuous. Then $\mathfrak{G}(D) \neq \{0\}$, $u \in \mathfrak{J}(D)$, and $u \cdot \mathfrak{G}(D) = \{0\}$. However, if $\varphi \in A'$ and $(u \cdot \varphi)(t) = \int_{-\infty}^{t} \varphi(s)\, ds \equiv 0$, then $\varphi = 0$. This contradiction shows that D must be continuous. ∎

For $v \in A$, we denote by $\mathfrak{P}(v)$ the subalgebra of all convolution polynomials in v.

3.6. Theorem. *Let $D\colon A \to A'$ be a derivation with $D(A) \subseteq V$. Let $v \in A$. Then the derivation*

$$\tilde{D}\colon f \mapsto xv \cdot Df, \quad A \to A',$$

agrees with the continuous derivation

$$\tilde{E}\colon f \mapsto xf \cdot Dv, \quad A \to A',$$

on the subalgebra $\mathfrak{P}(v)$ of A.

Proof. Since D has range in V, the element $xv \cdot Df$ lies in A'. One shows, as in the proof of Lemma 2.6, that $\tilde{D}$ is a derivation. Again, as $D(v) \in V$, $\tilde{E}$ is a continuous derivation. Since $\tilde{D}(v) = \tilde{E}(v)$, $\tilde{D}$ and $\tilde{E}$ must agree on $\mathfrak{P}(v)$. ∎

We consider now the case where $v = u$. The condition (3.3) that $u \cdot A' \subseteq V$ is the condition

$$\sup_{a>0} \frac{a}{\omega(a)} \int_0^\infty \omega(t+a)\, dt < \infty. \tag{3.4}$$

It follows from [3, Lemma 1.4] that a weight which satisfies (3.4) is regulated, and hence is a radical weight. Recall that if ω is regulated, then by Theorem 1.5, every derivation $D\colon A \to A'$ has its range in $\mathfrak{X}$.

3.7. Corollary. *Let ω be a weight satisfying condition (3.4), and let $D\colon A \to \mathfrak{X}$ be a derivation. Then the derivation $u^{*(3)} \cdot D$ is continuous on the dense subalgebra $\mathfrak{P}(u)$.*

Proof. The derivation $\tilde{D} = xu \cdot D = u^{*(3)} \cdot D$ is continuous on $\mathfrak{P}(u)$ by Theorem 3.6. ∎

3.8. Definition. We say that a derivation $D\colon A \to \mathfrak{X}$ is *splittable* over a subalgebra $\mathfrak{P}$ of A if $D = E + F$, where E is a continuous derivation and F is a derivation which vanishes on $\mathfrak{P}$. We call F *singular over* $\mathfrak{P}$.

3.9. Theorem. *Let $D\colon A \to \mathfrak{X}$ be a derivation. Then D is splittable over the dense subalgebra $\mathfrak{P}(u)$ if and only if the following conditions hold:*

(a) *$D(u)$ is continuously differentiable on $\boldsymbol{R}^-$;*
(b) *$D(u)'$ is absolutely continuous on bounded intervals of $\boldsymbol{R}^-$;*
(c) *$D(u)'' \in W$.*

Then the continuous part E of D is given by

$$E(f) = xf \cdot D(u)'' \quad (f \in A).$$

Proof. First suppose that $D(u)$ has the stated properties, and let $E(f) = xf \cdot \psi$ $(f \in A)$, where $\psi = D(u)''$. Then E is a continuous derivation by Theorem 2.5, and

$$E(u)(t) = (u^{*(2)} * \psi)(t) = \int_{-\infty}^{t} \int_{-\infty}^{s} D(u)''(r)\, dr\, ds \quad (t \in \boldsymbol{R}^-).$$

Thus $E(u)'$ and $D(u)'$ differ by a constant and $E(u)$ and $D(u)$ differ by a linear term. Since both vanish at $-\infty$, we have $E(u) = D(u)$. Hence $D = E$ on $\mathfrak{P}(u)$. Thus D is splittable over $\mathfrak{P}(u)$.

Conversely, suppose that $D = E + F$ is splittable over $\mathfrak{P}(u)$. Then E is the unique continuous extension of $D | \mathfrak{P}(u)$ to A. Hence $E(f) = xf \cdot \psi$ $(f \in A)$ for some $\psi \in W$. Since $D(u) = E(u) = u^{*(2)} * \psi$, it follows that $D(u)$ has the required properties. ∎

3.10. THEOREM. *Let ω be a weight function satisfying condition (3.4). Let $A = L^1(\mathbf{R}^+, \omega)$ and $\mathfrak{X} = C_0(\mathbf{R}^-, \check{\omega}^{-1})$. Assume that every derivation $D\colon A \to \mathfrak{X}$ which is zero on $\mathfrak{P}(u)$ is actually zero. Then every derivation from A into any Banach A-module is automatically continuous.*

Proof. Let $E\colon A \to \mathfrak{X}$ be a derivation. By Theorem 3.5, E is continuous if and only if $u^{*(3)}\cdot E$ is continuous. By Corollary 3.7, $u^{*(3)}\cdot E$ is continuous on $\mathfrak{P}(u)$. Let F be the continuous extension of $u^{*(3)}\cdot E$ to A. Then $u^{*(3)}\cdot E - F$ is zero on $\mathfrak{P}(u)$. By the special assumption, it is zero on A, so $u^{*(3)}\cdot E$ is continuous, and hence E is continuous. ∎

Groenbaek has also proved partial continuity results which are closely related to our theorem. We quote the following theorem, which is a specialization to our situation of [14, Corollary 3.6].

3.11. THEOREM. *Let ω be a weight function such that*

$$\sup_{t \in \mathbf{R}^+} \frac{t\omega(t+b)}{\omega(t)} < \infty$$

for some $b > 0$. Let $D\colon A \to A'$ be a derivation, and let $\bar{D}\colon M(\mathbf{R}^+, \omega) \to A'$ be its unique extension. Then there is a continuous derivation $E\colon A \to A'$ such that $\bar{E}$ and $\delta_b \cdot \bar{D}$ coincide on the subalgebra of all finitely supported measures in $M(\mathbf{R}^+, \omega)$. ∎

4. Non-existence of divisible submodules. Let $\mathfrak{A}$ be a commutative algebra, and let $\mathfrak{M}$ be an $\mathfrak{A}$-module. Suppose that $a \in \mathfrak{A}$. Then $\mathfrak{M}$ is *a-divisible* if, for each $\varphi \in \mathfrak{M}$, there exists $\psi \in \mathfrak{M}$ such that $a\cdot\psi = \varphi$. The module $\mathfrak{M}$ is *$\mathfrak{A}$-divisible* if it is a-divisible for each $a \in \mathfrak{A}\backslash\{0\}$; $\mathfrak{M}$ is *torsion-free* if $a\cdot\psi \neq 0$ whenever $a \in \mathfrak{A}\backslash\{0\}$ and $\psi \in \mathfrak{M}\backslash\{0\}$.

Now let $\mathfrak{A}$ be a commutative Banach algebra, and let $\mathfrak{M}$ be a Banach $\mathfrak{A}$-module. Then there is a standard way of constructing discontinuous derivations from $\mathfrak{A}$ into $\mathfrak{M}$ in the case where $\mathfrak{A}$ is an integral domain and $\mathfrak{M}$ contains a non-zero, $\mathfrak{A}$-divisible, torsion-free submodule. The construction depends on the following algebraic fact, proved in [16, Section 8]. Let $\mathfrak{A}$ be an integral domain, and let $\mathfrak{Y}$ be a non-zero, $\mathfrak{A}$-divisible, torsion-free $\mathfrak{A}$-module. Suppose that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, that $D\colon \mathfrak{B} \to \mathfrak{Y}$ is a derivation, and that $a \in \mathfrak{A}\backslash\mathfrak{B}$. Then there is an extension of D to a derivation $\bar{D}$ from $\mathfrak{B}[a]$, the subalgebra of A generated by $\mathfrak{B} \cup \{a\}$, into $\mathfrak{Y}$: if a is algebraic with respect to $\mathfrak{B}$, then $\bar{D}$ is uniquely determined on $\mathfrak{B}[a]$, but if a is transcendental with respect to $\mathfrak{B}$, then $\bar{D}(a)$ can be chosen to be any element in $\mathfrak{Y}$. It is the freedom of choice of $\bar{D}(a)$ in this construction that enables discontinuous derivations to be built. Thus now suppose that $\mathfrak{A}$ is also a Banach algebra and that $\mathfrak{M}$ is a Banach $\mathfrak{A}$-module containing $\mathfrak{Y}$ as a submodule. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ and suppose that $a \in \mathfrak{A}$ is such that $a \in \bar{\mathfrak{B}}$ and a is transcendental with respect to $\mathfrak{B}$. Define $D(b) = 0$ $(b \in \mathfrak{B})$, and extend D to a derivation $\bar{D}$ on $\mathfrak{B}[a]$ by taking $\bar{D}(a)$ to be any non-zero element of $\mathfrak{Y}$. Then, using the above algebraic fact and Zorn's lemma, extend $\bar{D}$ to be a derivation from $\mathfrak{A}$ into $\mathfrak{Y}$: clearly we obtain a discontinuous derivation from $\mathfrak{A}$ into $\mathfrak{M}$.

In the paper [4] we used this method to construct discontinuous derivations from Banach algebras of power series into modules. The purpose of the present section is to show that this method of constructing discontinuous derivations fails for the algebra $L^1(\mathbf{R}^+, \omega)$: we prove that there is no non-zero, $L^1(\mathbf{R}^+, \omega)$-divisible submodule in any Banach $L^1(\mathbf{R}^+, \omega)$-module.

The following simple remark shows that in looking for $\mathfrak{A}$-divisible submodules it is sufficient to consider $\mathfrak{A}$-divisible submodules of the dual module $\mathfrak{A}'$.

4.1. LEMMA. *If a Banach algebra $\mathfrak{A}$ has a non-zero, $\mathfrak{A}$-divisible submodule in a Banach $\mathfrak{A}$-module $\mathfrak{M}$, then there is a non-zero, $\mathfrak{A}$-divisible submodule in the dual module $\mathfrak{A}'$.*

Proof. Let $\mathfrak{N}$ be a non-zero, $\mathfrak{A}$-divisible submodule in $\mathfrak{M}$, and choose $\lambda \in \mathfrak{M}'$ with $\lambda(\mathfrak{N}) \neq \{0\}$. Then the module homomorphism R_λ defined in Theorem 1.1 carries $\mathfrak{N}$ onto a non-zero, $\mathfrak{A}$-divisible submodule of $\mathfrak{A}'$. ∎

The proofs of the next theorems will involve analytic functions. We adopt the notation $\Pi = \{z \in \mathbf{C}\colon \operatorname{Re} z > 0\}$.

Let ω be a radical weight function on $\mathbf{R}^+$. Then $L^\infty(\mathbf{R}^-, \check{\omega}^{-1})$ is a Banach $L^1(\mathbf{R}^+)$-module.

4.2. THEOREM. *Let ω be a radical weight function. Then the $L^1(\mathbf{R}^+)$-module $L^\infty(\mathbf{R}^-, \check{\omega}^{-1})$ has no non-zero, $L^1(\mathbf{R}^+)$-divisible submodule.*

Proof. Let $A = L^1(\mathbf{R}^+, \omega)$, so that $L^\infty(\mathbf{R}^-, \check{\omega}^{-1}) = A'$. To obtain a contradiction, suppose that $\mathfrak{Y}$ is a non-zero, $L^1(\mathbf{R}^+)$-divisible submodule of A'.

We show first that $\alpha(\varphi) = -\infty$ for every non-zero element of $\mathfrak{Y}$. For suppose that $\varphi \in \mathfrak{Y}$ and that $\alpha(\varphi) = -c$, where $c > 0$. Choose a positive function $f \in L^1(\mathbf{R}^+)$ with f increasing on $[0, c]$ and a sequence $(t_n) \subseteq \mathbf{R}^+\backslash\{0\}$ with $0 < t_{n+1} < t_n < c$, such that $t_n \to 0$, $|\varphi(-c+t_n)| > 0$ and $0 < f(t_n) < |\varphi(-c+t_n)|$ $(n \in \mathbf{N})$. Since $\mathfrak{Y}$ is f-divisible, there exists $\psi \in \mathfrak{Y}$ with $f\cdot\psi = \varphi$. Since $\alpha(f) = 0$, $\alpha(\psi) = -c$. Let $\psi_1 = \delta_c * \psi$ and $\varphi_1 = \delta_c * \varphi$. Since $(f*\psi)(t) = \varphi(t)$ on $[-c, 0]$, $(f*\psi_1)(t) = \varphi_1(t)$ on $[0, c]$. But then

$$|\varphi_1(t)| \leqslant \int_0^t f(t-s)|\psi_1(s)|\,ds \leqslant f(t)\int_0^t |\psi_1(s)|\,ds \qquad (t \in [0, c]),$$

and so for large n we must have

$$|\varphi(-c+t_n)| = |\varphi_1(t_n)| < f(t_n),$$

a contradiction. Thus $\alpha(\varphi) = -\infty$.

Now fix $\varphi \in \mathfrak{Y}$ with $\alpha(\varphi) = -\infty$. Since φ vanishes on $\boldsymbol{R}^+$ and $|\varphi(t)| = O(\check{\omega}(t))$ as $t \to -\infty$, so that $|\varphi(t)|e^{-xt} \in L^1(\boldsymbol{R}^-)$ for each $x \in \boldsymbol{R}$, φ has a Laplace transform

$$\Phi(z) = \int_{-\infty}^{0} \varphi(t)e^{-zt}\,dt \qquad (z \in \boldsymbol{C})$$

which is an entire function. Clearly Φ is bounded on each left-hand half-plane. Since $\alpha(\varphi) = -\infty$, there is a sequence $(z_n) \subseteq \boldsymbol{C}$ such that $x_n = \operatorname{Re} z_n \to \infty$ and $|\Phi(z_n)| \to \infty$ as $n \to \infty$. By passing to a subsequence, we may suppose that $\sum_{n=1}^{\infty} x_n/(1+|z_n|^2) < \infty$. By a standard result [15] there is a Blaschke product B with $B(z_n) = 0$ $(n \in \boldsymbol{N})$ and $|B(z)| \leqslant 1$ on $\{z\colon \operatorname{Re} z \geqslant -1\}$.

Let $F(z) = B(z)\exp(-(z+1)^{1/2})$. Then F is analytic on $\{z\colon \operatorname{Re} z > -1\}$, $F(z) \to 0$ as $z \to \infty$ in $\bar{\Pi}$, and F is integrable on the imaginary axis. Let $f = \mathfrak{L}^{-1}F$ be the inverse Laplace transform of F. Since F is analytic near $\bar{\Pi}$, $f|\boldsymbol{R}^- = 0$ and $f|\boldsymbol{R}^+ \in L^1(\boldsymbol{R}^+)$.

Suppose, if possible, that there exists $\psi \in \mathfrak{Y}$ such that $f\cdot\psi = \varphi$. Since $f, \psi \in L^1(\boldsymbol{R})$, $f * \psi \in L^1(\boldsymbol{R})$. Define $g = f*\psi - \varphi$. Then $g \in L^1(\boldsymbol{R})$, and g vanishes on $\boldsymbol{R}^-$ because $\varphi = f\cdot\psi = (f*\psi)|\boldsymbol{R}^-$. Let G and Ψ be the Laplace transforms of g and ψ, respectively. Then we have the equation

$$G(z) = F(z)\Psi(z) - \Phi(z) \qquad (z \in \Pi).$$

Since $F(z_n) = 0$, $G(z_n) = -\Phi(z_n)$ $(n \in \boldsymbol{N})$. However, $|G(z_n)| \to 0$ and $|\Phi(z_n)| \to \infty$ as $n \to \infty$, giving the required contradiction. Thus, A' has no non-zero, $L^1(\boldsymbol{R}^+)$-divisible submodule. ∎

4.3. Corollary. *Let ω be a radical weight function, and let $A = L^1(\boldsymbol{R}^+, \omega)$. Then A has no non-zero, A-divisible submodule in any Banach A-module.*

Proof. If A has a non-zero, A-divisible submodule in some Banach A-module, then by Lemma 4.1, A has a non-zero, A-divisible submodule, say $\mathfrak{N}$, in A'. But then $\mathfrak{N}$ would be $L^1(\boldsymbol{R}^+)$-divisible, contradicting Theorem 4.2. ∎

Recall from Theorem 1.3(i) that $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ is a module over $M(\boldsymbol{R}^+, \omega)$. Let ω be a radical weight on $\boldsymbol{R}^+$, and let S be any dense subgroup of $\boldsymbol{R}$. For example, we might have $S = \boldsymbol{Q}$ or $\boldsymbol{R}$ or even $\{m+n\alpha\colon m, n \in \boldsymbol{Z}\}$, where α is irrational. Then $l^1(S^+) \subseteq l^1(S^+, \omega) \subseteq M(\boldsymbol{R}^+, \omega)$, so $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ is an $l^1(S^+)$-module. For our final result, we shall prove that $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ has no non-zero $l^1(S^+)$-divisible submodule. We shall require the following lemma.

4.4. Lemma. *Let (Δ_n) be a sequence of disjoint open disks in $\boldsymbol{C}$ with $\inf\{\operatorname{Re} z\colon z \in \Delta_n\} \to \infty$ as $n \to \infty$. Let (t_n) be a sequence with $0 < t_{n+1} < t_n$, and $t_n \to 0$. Then there exists a sequence (α_n) in $\boldsymbol{C}$ with $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ such that the function*

$$F\colon z \mapsto \sum_{n=1}^{\infty} \alpha_n e^{-t_n z}$$

has a zero in Δ_n for infinitely many values of n.

Proof. We may suppose that the disks Δ_n have disjoint closures and that $\operatorname{radius}(\Delta_n) \to 0$. Let $n_1 = 1$ and define $F_1(z) = \alpha_1 e^{-t_1 z}$, where $0 < |\alpha_1| < 1/2$. Now suppose that we have constructed positive integers $n_1 < n_2 < \ldots < n_k$, coefficients $\alpha_1, \ldots, \alpha_k$, and positive numbers $\varepsilon_1, \ldots, \varepsilon_k$, with $\varepsilon_1 = 1$, satisfying the following conditions:

(a) $0 < |\alpha_j| < 2^{-j}$ $(1 \leqslant j \leqslant k)$;

(b) $F_j(z) = \sum_{m=1}^{j} \alpha_m e^{-t_m z}$ has a zero in Δ_{n_j}, but F_j has no zero on $\bigcup_{m=1}^{k} \partial\Delta_{n_m}$ for $j = 1, \ldots, k$;

(c) $\varepsilon_k < \min\{2^{-k}, \varepsilon_{k-1}/2, \gamma_k/4\}$, where

$$\gamma_k = \inf\{|F_{k-1}(z)|\colon z \in \bigcup_{m=1}^{k} \partial\Delta_{n_m}\}.$$

Now choose ε_{k+1} so small that (c) is satisfied for $k+1$. Consider

$$F_{k+1}(z) = \sum_{j=1}^{k+1} \alpha_j e^{-t_j z} = e^{-t_{k+1} z}(G_{k+1}(z) + \alpha_{k+1}),$$

where α_{k+1} is to be chosen. Since $|G_{k+1}(z)| \to 0$ as $\operatorname{Re} z \to \infty$, we can choose $n_{k+1} > n_k$ so that

$$\sup\{|G_{k+1}(z)|\colon z \in \Delta_{n_{k+1}}\} < \varepsilon_{k+1}.$$

Set $\alpha_{k+1} = -G_{k+1}(z_{k+1})$, where z_{k+1} is the centre of $\Delta_{n_{k+1}}$. Thus F_{k+1} has a zero in $\Delta_{n_{k+1}}$. By slightly shrinking the radius of $\Delta_{n_{k+1}}$, we may suppose that none of the functions $F_1, \ldots, F_{k+1}$ has a zero on $\partial\Delta_{n_{k+1}}$. Then

$$\begin{aligned} |F_{k+1}(z) - F_k(z)| &= |\alpha_{k+1} e^{-t_{k+1} z}| < |\alpha_{k+1}| \\ &= |G_{k+1}(z_{k+1})| < \varepsilon_{k+1} \\ &< \gamma_{k+1}/2 < |F_k(z)| \qquad (z \in \bigcup_{m=1}^{k} \partial\Delta_{n_m}), \end{aligned}$$

so that F_{k+1} has no zero on $\bigcup_{m=1}^{k+1} \partial\Delta_{n_m}$, thus completing the inductive choice.

If k is fixed and $p > k$, then

$$|F_p(z)-F_k(z)| \leqslant \sum_{j=k}^{p-1} |F_{j+1}(z)-F_j(z)| \leqslant \sum_{j=k}^{p-1} \varepsilon_j \leqslant 2\varepsilon_{k+1} < \gamma_{k+1}/2 < |F_k(z)| \quad (z \in \Delta_{n_k}).$$

Define $F(z) = \sum_{k=1}^{\infty} \alpha_k e^{-t_k z}$. Then, for each $k \in N$ and $z \in \Delta_{n_k}$,

$$|F(z)-F_k(z)| \leqslant \gamma_{k+1}/2 < |F_k(z)|,$$

so F has a zero in Δ_{n_k}. This completes the proof. ∎

4.5. THEOREM. *Let ω be a radical weight on $\boldsymbol{R}^+$, and let $\boldsymbol{S}$ be a dense subgroup of $\boldsymbol{R}$. Then $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$ contains no non-zero, $l^1(S^+)$-divisible submodule.*

Proof. To obtain a contradiction, suppose that $\mathfrak{N}$ is an $l^1(S^+)$-divisible submodule of $L^\infty(\boldsymbol{R}^-, \check{\omega}^{-1})$, that $\varphi \in \mathfrak{N}$, and that $\varphi \neq 0$. Then Φ, the Laplace transform of φ, is an entire function. Clearly, Φ is bounded on each left-hand half-plane, and Φ is not bounded on $\boldsymbol{C}$, and so there is a sequence $(z_n) \subseteq \boldsymbol{C}$ with $\operatorname{Re} z_n \to \infty$ such that $|\Phi(z_n)| \to \infty$. Let (Δ_n) be a sequence of open disks, with Δ_n having centre z_n, and let (t_n) be a sequence in S^+ with $t_n \to 0$. By Lemma 4.4, there is a sequence $(\alpha_k) \subseteq \boldsymbol{C}$ with $\sum |\alpha_k| < \infty$ so that

$$F(z) = \sum_{k=1}^{\infty} \alpha_k e^{-t_k z}$$

has a zero in Δ_n for infinitely many values of n. Therefore, by passing to a subsequence (ζ_n) of (z_n), we may suppose that F has a sequence of zeros (ζ_n) for which $|\Phi(\zeta_n)| \to \infty$.

Let $f = \sum_{n=1}^{\infty} \alpha_n \delta_{t_n}$. Then $f \in l^1(S^+)$. Since $\mathfrak{N}$ is $l^1(S^+)$-divisible, there exists $\psi \in \mathfrak{N}$ with $f \cdot \psi = \varphi$. As in the proof of Theorem 4.2, the function $g = f * \psi - \varphi$ lies in $L^1(\boldsymbol{R}^+)$ and vanishes on $\boldsymbol{R}^-$. Writing $G = \mathfrak{L}(g)$, $F = \mathfrak{L}(f)$, and $\Psi = \mathfrak{L}(\psi)$, we have $G(z) = F(z)\Psi(z) - \Phi(z)$ $(z \in \Pi)$, so that $G(\zeta_n) = -\Phi(\zeta_n)$, but $|G(\zeta_n)| \to 0$, while $|\Phi(\zeta_n)| \to \infty$. This contradiction completes the proof. ∎

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